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WHAT IS Forcing?

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What is *forcing*? Forcing is a remarkably powerful technique for the construction of models of set theory. It was invented in 1963 by Paul Cohen who used it to solve the Continuum Hypothesis. He constructed a model of set theory in which the continuum hypothesis (CH) fails, thus showing that CH is not provable from the axioms of set theory.

What is the *Continuum Hypothesis*? In 1873 Georg Cantor proved that the continuum is uncountable; that there exists no mapping of the set $\mathbf{N}$ of all integers onto the set $\mathbf{R}$ of all real numbers. Since $\mathbf{R}$ contains $\mathbf{N}$, we have $2^{\aleph_0} > \aleph_0$, where $2^{\aleph_0}$ and $\aleph_0$ are the cardinalities of $\mathbf{R}$ and $\mathbf{N}$, respectively. A question arises whether $2^{\aleph_0}$ is equal to the cardinal $\aleph_1$, the immediate successor of $\aleph_0$. Cantor's conjecture that $2^{\aleph_0} = \aleph_1$ is the celebrated continuum hypothesis made famous by David Hilbert who put it on the top of his list of major open problems in the year 1900. Cohen's solution of Cantor's problem does not prove $2^{\aleph_0} = \aleph_1$, nor does it prove $2^{\aleph_0} \neq \aleph_1$. The answer is that CH is undecidable.

What does it mean that a conjecture is *undecidable*? Whenever a new mathematical result is established it is obtained by proving a theorem. A proof is a sequence of logical steps using self-evident true facts. There is however no reason why for every well formulated mathematical statement such a sequence should exist either proving the statement or disproving it. To make this vague discussion more precise we need first to analyze the concepts of theorem and proof.

What are *theorems* and *proofs*? It is a useful fact that every mathematical statement can be expressed in the language of set theory. All mathematical objects can be regarded as sets, and relations between them can be reduced to expressions that use only the relation $\in$. It is not essential how it is done, but it can be done: For instance, integers are certain finite sets, rational numbers are pairs of integers, real numbers are identified with Dedekind cuts in the rationals, functions are some sets of pairs etc etc. Moreover all "self-evident true facts" used in proofs can be formally derived from the axioms of set theory. The accepted system of axioms of set theory is ZFC, the Zermelo-Fraenkel axioms plus the axiom of choice. As a consequence every mathematical theorem can be formulated and proved from the axioms of ZFC.

When we consider a well formulated mathematical statement (say, the Riemann Hypothesis) there is a priori no guarantee that there exists a proof of the statement or a proof of its negation. Does ZFC decide every statement? Is ZFC complete? It turns out that not only ZFC is not complete but it cannot be replaced by a complete system of axioms. This was Gödel's 1931 discovery known as the Incompleteness Theorem.

What is *Gödel's Incompleteness Theorem*? In short, no system of axioms that is (i) recursive and (ii) sufficiently expressive is incomplete. "Sufficiently expressive" means that it includes the "self-evident truths" about integers, and "recursive" means roughly that a computer program can decide whether a statement is an axiom or not. (ZFC is one such system, Peano's system of axioms for arithmetic is another such system etc.)

It is of course one thing to know that, by Gödel's theorem, undecidable statements exist, and another to show that a particular conjecture is undecidable. One way to show that some statement is unprovable from given axioms is to find a model.

What is a *model*? A model of a theory interprets the objects of the theory, as well as the language of the theory, in such a way that the axioms of the theory are true in the model. Then all theorems of the theory are true in the model, and if a given statement is false in the model, then it cannot be proved from the axioms of the theory. A well known example is a model of non Euclidean geometry.

A model of set theory is a collection M with the property that under the interpretation that "sets" are all the sets belonging to M and only those sets, then the axioms of ZFC are satisfied. That is to say that M satisfies ZFC. If, for instance, M also satisfies the negation of CH, then CH cannot be provable in ZFC.

Here we mention another result of Gödel, from 1938: the consistency of CH. Gödel constructed a model of ZFC, the constructible universe L, that satisfies CH. The model L is basically the minimal possible collection of sets that satisfies the axioms of ZFC. Since CH is true in L, it follows that CH cannot be refuted in ZFC. In other words, CH is consistent.

Cohen's accomplishment was that he found a method how to construct other models of ZFC. The idea is to start with a given model M (the *ground model*) and extend it by adjooining an object G, a sort of imaginary set. The resulting model $M[G]$ is more or less a minimal possible collection of sets that includes M, contains G, and most importantly, also satisfies ZFC. Cohen showed the way how to find (or imagine) the set G so that CH fails in $M[G]$. Thus CH is unprovable in ZFC, and because it is also consistent, CH is *independent*, or *undecidable*.

It should be pointed out that one consequence of Gödel's theorem about L is that it cannot be proved that there exists a set outside the minimal model L so we have to pull G out of thin air. The genius of Cohen was to introduce so called *forcing conditions* that give a partial information about G and then assume that G is a *generic* set. A generic set decides which forcing conditions are considered true. With Cohen's definition of forcing and generic sets it is consistent that a generic set exists, and if G is generic, then $M[G]$ is a model of set theory.

To illustrate the method of forcing, let us consider the simplest possible example, and assume that G should be a set of integers. As forcing conditions we consider finite sets of expressions $a \in G$ and $a \notin G$ where a ranges over the set of all integers. (Thus $\{1 \in G, 2 \notin G, 3 \in G, 4 \in G\}$ is a condition that forces $G \cap \{1,2,3,4\} = \{1,3,4\}$.) The genericity of G guarantees that the conditions in G are mutually compatible, and, more importantly, that every statement of the forcing language is decided one way or the other. We shall not define here what "generic" means precisely, but let me point out one important feature. Let us identify the above generic set of integers G with the set G of forcing conditions that describe the initial segments of G. Genericity implies that if A is a statement of the forcing language then if every condition can be extended to a condition that forces A then some condition p in G forces A (and so A is true in $M[G]$). For instance, if S is a set of integers and S is in the ground model M then every condition p of the kind described above can be extended to a condition that forces $G \neq S$: simply add the expression "$a \in G$" for some $a \notin S$ (or "$a \notin G$" for some $a \in S$) where a is an integer not mentioned in p. It follows that the resulting set of integers G is not in M, no matter what the generic set is.

Cohen showed how to construct the set of forcing conditions so that CH fails in the resulting generic extension $M[G]$. Soon after Cohen's discovery his method was applied to other statements of set theory. The method is extremely versatile: every partially ordered set P can be taken as the set of forcing conditions, and when $G \subset P$ is a generic set then the model $M[G]$ is a model of ZFC. Moreover, properties of the model $M[G]$ can be deduced from the structure of P. In practice, the forcing P can be constructed with the independence result in mind; forcing conditions usually "approximate" the desired generic object G.

In the 45 years since Cohen, literally hundreds of applications of forcing have been discovered, giving a better picture of the universe of sets by producing examples of statements that are undecidable from the axioms of mathematics.

A word of caution: If after reading this you entertain the idea that perhaps the Riemann Hypothesis could be solved by forcing, forget it. That conjecture belongs to a class of statements that, by virtue of their logical structure, are *absolute* for forcing extension: such a statement is true in the generic extension $M[G]$ if and only if it is true in the ground model M. This is the content of Shoenfield's Absoluteness Theorem.

And what is *Shoenfield's Absoluteness Theorem*? Well, that is for someone else to explain, some other time.