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Some recollections on early work with Jan Pelant

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.

Abstract

In this note we consider three questions which can be traced to our early collaboration with Jan "Honza" Pelant. We present them from the contemporary perspective, sometimes complementing our earlier work. The questions relate to a Ramsey problem, uniform spaces and tournaments.

1 Introduction

In this note we discuss some of Jan's mathematical interactions with the authors that date back to early 70ies. Jan Pelant was a remarkable man whose influence on his contemporaries transcendented in Prague mathematical life. He had a gift to understand and to solve problems, he was an excellent mathematician. However, Jan Pelant was not just an expert in his own field. His interests and talents were broad and he could have been successful in other areas. His passing away is a great loss to all of us.

Here we deal with his work related to 3 problems: Ramsey topological spaces, characters of uniformities and tournaments algebras.

2 Ramsey topological spaces

Ramsey theory was developing very rapidly during 70ies. One of the most significant changes was the fact that the original set theory (and graph theory) setting of Ramsey theory was generalized to other structures. In this context the notions of Ramsey class and Ramsey property were defined and understood. This development is nicely described in the first monograph devoted to Ramsey theory [6]. Motivated by our work on Ramsey graphs [25] we discussed the Ramsey problems intensively, and so it was only natural that soon we translated the Ramsey theory also to a topological setting.

Definition 1
A topological space Y is said to be point Ramsey for the space X if for every (set) partition $Y = Y_1 \cup Y_2$ one of the classes Y_i contains a subspace which is homeomorphic to X.

In the classical Erdös-Rado notation this is denoted by $Y \to (X)^1_2$. If α parts are allowed then we write $Y \to (X)^1_\alpha$. We say that a class $\mathcal{T}$ of topological spaces is point Ramsey if for every $X \in \mathcal{T}$ and every cardinal α there exists $Y \in \mathcal{T}$ such that

$$Y \to (X)^1_\alpha.$$

Jointly with J. Pelant we proved [26]:

Theorem 1

1. The class $\mathcal{T}_1$ of all T_1-topological spaces is point Ramsey.
2. The class $\mathcal{T}_0$ of all T_0-topological spaces is point Ramsey.

This is an easy result which is obtained by the lexicographic (nested) product.

It is not known whether the class $\mathcal{T}_2$ of Hausdorf topological spaces is point Ramsey. Particularly, the following problem concerning the unit interval I popularized the study of Ramsey topological spaces.

Problem 1
Is it true that for every α there exists β such that $I^\beta \to (I)^1_\alpha$?

This is clearly equivalent to the question whether the class of completely regular spaces is point Ramsey. The above is contained in the conference volume of TOPOSYM'76 [26].

We were pleased to learn that this note was quickly followed by research by W. Weiss, V. I. Malyhin, S. Todorčević and others [13, 40, 39]. A survey article by W. Weiss about this research appeared in [41]. In fact the TOPOSYM paper [26] contains only a sketch of the proof of Theorem 1 and, in the hindsight, it proves more. Thus after 30 years we take the liberty to include here the following mild strengthening of [26]:

Theorem 2
For every topological space X, every linear ordering $\leq$ of its points and every cardinal α there exists a linearly ordered topological space $(Y, \leq_Y)$ such that $(Y, \leq_Y) \to (X, \leq_X)^1_\alpha$.

Proof. Let $(X, \leq_X)$ and α be given. We define the base set of Y as X^α. Let $\leq_Y$ be the lexicographic ordering of sequences $(x_\iota; \iota < \alpha)$. The topology of Y will be defined by the subbase neighbourhods $U(x^0, U, \gamma)$: For $\gamma < \alpha$, $x^0 = (x^0_\iota; \iota < \alpha)$ and a neighbourhood U of x_γ (in X) we put $(x_\iota; \iota < \alpha) \in U(x^0, U, \gamma)$ iff $x_\iota = x^0_\iota$ for $\iota < \gamma$ and $x_\gamma \in U$.

We prove $(Y, \leq) \to (X, \leq)^1_\alpha$. Thus let $c : Y \to \alpha$ be a coloring of points of Y. We construct by the transfinite induction points $x_\lambda \in X$ such that $c(u) \neq \lambda$ whenever $u \in Y$, $u_\gamma = x_\gamma (\gamma \leq \lambda)$. Suppose that $\lambda < \alpha$ and $x_\gamma \in X (\gamma < \lambda)$ have already been constructed. Suppose on the contrary that there is no x_λ with the required property. This means that for each $v \in X$ there exists $y^v \in Y$ satisfying $y^v_\gamma = x_\gamma$ $(\gamma < \lambda)$, $y^v_\lambda = v$ and $c(y^v) = \lambda$. Then the set $\{y^v : v \in X\}$ induces an ordered subspace of Y monotone homeomorphic to $(X, \leq)$. Clearly the set is homogeneous for the coloring c, a contradiction. Hence we can construct the elements $x_\lambda (\lambda < \alpha)$ with the required property. Then the sequence $x = (x_\lambda)_{\lambda < \alpha} \in Y$ satisfies $c(x) \neq \lambda$ for each $\lambda < \alpha$, a contradiction.

□

Remark 1
One should note that the above definition of (point) Ramsey property deals with partitions of points only. If we consider the partitions of arbitrary subspaces then, in the positive case, we speak about Ramsey classes.

One has to stress that no non-trivial Ramsey class of topological spaces is known. This perhaps is not even a good question (as also indicated by examples given in [26]). (However, an interesting graph-theory proof of $\kappa \not\to (\omega)_2^\omega$ [28] found other applications [38].) The more fruitful area was developed here in the context of topological restricted colorings which led to the intensive development [5, 3, 12].

On the other hand for finite topological spaces the full characterization of Ramsey classes is given in [22, 23]. Ramsey classes of finite structures are related to ultrahomogeneous structures [21, 22, 11] and this connection found recently a spectacular application in the context of topological dynamics [11].

Remark 2
Ramsey problems depend very much on the underlying category. The more restrictive maps lead to fewer subspaces and thus we can expect a richer spectrum of results. Examples of this phenomenon are Euclidean and geometric Ramsey theorems [15] and also metric Ramsey theorems [2], [16] (which should be distinguished from Ramsey theorem for finite metric spaces [24]). However, these questions were studied much later.

3 A point character of $\ell_p(\kappa)$

Let (X, ρ) be a metric space. An open covering $\mathcal{U}$ of (X, ρ) is a family of open subsets of X with $X = \bigcup \mathcal{U}$. We say that $\mathcal{U}$ is bounded if there exists $b > 0$ with the property that $diam\, U < b$ for all $U \in \mathcal{U}$. The covering $\mathcal{U}$ is called uniform if there exists $\varepsilon > 0$ such that for every $x \in X$ there is a $U \in \mathcal{U}$ which contains the ε-ball $B_\varepsilon(x) = \{y; \rho(x, y) < \varepsilon\}$. By the well-known theorem of A. H. Stone [36], each metric space is paracompact and hence each open covering $\mathcal{U}$ of (X, ρ) has an open locally finite refinement $\mathcal{V}$ - i.e., there is an open covering $\mathcal{V}$ with the following two properties:

1. for each $x \in X$ there is a neighborhood of x which meets only finitely members of $\mathcal{V}$
2. for every $V \in \mathcal{V}$ there is a $U \in \mathcal{U}$ with $V \subset U$.

The question whether in Stone's theorem the open coverings may be replaced by the uniform ones (i.e., whether every uniform covering has a locally finite uniform refinement) was originally formulated by A.H. Stone [37] and is also mentioned in Isbell's book [8]. The answer to this question is clearly positive for any Euclidean or more generally separable space. However, it was shown independently by Pelant [29] and Shepin [35] that the space $\ell_\infty(\kappa)$ for κ sufficiently large does not have the property. Subsequently in [34] and in [32] we proved that the space $\ell_p(\kappa)$, $1 \le p < \infty$ (κ large) does not have the property, either. Here we describe the result from [32] which is related to a paper from this volume [1].

Definition 2
Let (X,ρ) be a metric space. A point character $pc(X,\rho)$ of (X,ρ) is the least infinite cardinal β with the property that for each uniform cover $\mathcal{U}$ and each cardinal $\lambda < \beta$ there exists $x \in X$ which is in at least λ members of $\mathcal{U}$.

A space with $pc(X,\rho) \leq \aleph_0$ is also called point finite. For any Euclidean space E_n clearly $pc(E_n) = n+2$. So the point character provides a suitable generalization of the notion of dimension for the "infinite dimensional case".

The Stone question is equivalent to the problem whether every metric space is point finite.

For an infinite cardinal κ and $p \geq 1$ recall that $\ell_p(\kappa)$ is the Banach space whose elements are the real functions on κ such that $\sum_{i<\kappa} |f(i)|^p$ converges, the operations are pointwise and the norm is defined by

$$\|f\| = \left(\sum_{i<\kappa} |f(i)|^p \right)^{1/p} .$$

The main objection of this paragraph is to prove the following.

Theorem 3
Let α be a limit ordinal number. Then $pc\ \ell_1(\omega_\alpha) \geq \omega_\alpha$.

For the proof we shall need the following lemma. Let X be a set. By a symbol $[X]^n$ we denote the system of all n-element subsets of X.

Lemma 1
Let n be a positive integer, $n \geq 2$ and γ an ordinal number. Then for every mapping $f : [\omega_{\gamma+n-1}]^n \to \omega_{\gamma+n-1}$ satisfying $x, y \in [\omega_{\gamma+n-1}]^n$, $x \cap y = \emptyset$ implies $f(x) \neq f(y)$ there exists $C \subset [\omega_{\gamma+n-1}]^n$ with the following properties:

1. $|C| = \omega_\gamma$,
2. $x_1 \neq x_2$, $x_1, x_2 \in C \to f(x_1) \neq f(x_2)$,
3. $|\cap C| = n-1$.

For the proof see [1].

Proof. (Theorem) We will prove that $pc\ \ell_1(\omega_\alpha) \geq \omega_\alpha$.

Let $\mathcal{U}$ be an arbitrary bounded uniform covering of $\ell_1(\omega_\alpha)$. Without loss of generality we may assume that $\mathrm{diam} U \leq 1$ for any $U \in \mathcal{U}$.

Let us consider the topological subspace of $\ell_1(\omega_\alpha)$ on the set

$$\{f | f : \omega_\alpha \to \langle 0,1 \rangle, |cozf| < \omega_0 \text{ and } f(x) = 1/|cozf|, \text{ for } x \in cozf\}$$

where $cozf = \{m | f(m) \neq 0\}$. This subspace we denote by $F(\omega_\alpha)$. Let $\mathcal{U}$ be a uniform cover of $F(\omega_\alpha)$ such that $\mathcal{U}$ is a refinement of $\mathcal{L}(1) = \{B(x,1) : x \in F(\omega_\alpha)\}$ (for short $\mathcal{U} \prec \mathcal{L}(1)$).

As $\mathcal{U}$ is a uniform covering, there exists $\varepsilon > 0$ such that for every $x \in F(\omega_\alpha)$ there is a $U \in \mathcal{U}$ with $B(x,\varepsilon) \subset U$. Let us take n so large that $1/n < \varepsilon/2$. Consider

$$F^n(\omega_\alpha) = \{f | f \in F(\omega_\alpha) \text{ and } |cozf| = n\}$$

Let us define the mapping $g : [\omega_\alpha]^n \to \mathcal{U}$ so that if $f \in F^n(\omega_\alpha)$ with $cozf = M$ then $B(f, \varepsilon) \subset g(M)$. (Recall that for each $M \in [\omega_\alpha]^n$ there exists the only mapping $f \in F^n(\omega_\alpha)$ such that $cozf = M$; denote this mapping by f_M). The mapping g satisfies the assumption of Lemma 1 as for x,y disjoint we have $\mathrm{dist}(f_x, f_y) = 2$. Since $\mathcal{U} \prec \mathcal{L}(1)$, $g(x)$ and $g(y)$ must be different elements of $\mathcal{U}$.

Let now $\gamma < \alpha$. As α is a limit ordinal we have also $\omega_{\gamma+n-1} < \omega_\alpha$ and thus, by Lemma 1, there is a family $C \subset F^n(\omega_\alpha)$ with the properties

1. $|C| = \omega_\gamma$,
2. $c_1, c_2 \in C$, $c_1 \neq c_2 \Rightarrow g(c_1) \neq g(c_2)$,
3. $|\cap C| = n - 1$. Fix $c \in C$. For each $c' \in C$ we have $\rho(f_c, f_{c'}) = \frac{2}{n} < \varepsilon$, and so $f_c \in B(f_{c'}, \varepsilon) \subset g(c')$. Hence c is containned in ω_γ elements of $\mathcal{U}$.

Since $\omega_\gamma < \omega_\alpha$ and $\mathcal{U}$ were arbitrary, we have $pcF^n(\omega_\alpha) \geq \omega_\alpha$, and consequently, $pc\ell_1(\omega_\alpha) \geq \omega_\alpha$.

□

Finally, let us note that the proof for $p > 1$ is analogous.

For more details see [29, 30, 33].

4 Tournaments and algebras

The first two papers [31, 17] of Jan Pelant deal with relations: [31] can be traced to a dimension question of M. Katětov while [17] is an abstract of the main activity of the combinatorial seminars in 1970 – 1971. It deals with the following notion:

Definition 3
A tournament (X, R) is a reflexive relation which is complete and antisymmetric. Explicitly, R satisfies

$$R \cup R^{-1} = X^2, R \cap R^{-1} = \Delta_X$$

In [17, 18, 19] we studied tournaments from the algebraic point of view:

Every tournament $T = (X, R)$ corresponds uniquely to the binary tournament algebra $(X, \cdot_T)$ defined by $x \cdot_T y = z$ if $(x, y) \in R$ and $x = z$.

Clearly tournament algebras are just quasitrivial ($x \cdot y \in \{x, y\}$), commutative and idempotent algebras. Note also that $f : (X, R) \to (X', R')$ is a (relational) homomorphism iff $f : (X, \cdot_T) \to (X', \cdot_{T'})$ is an (algebraic) homomorphism.

This connection led us to investigate the tournament algebras thoroughly. This resulted in papers [18, 19] where we (among others) characterized the congruence lattices of tournaments algebras. It also led to new notions such as the simple tournament.

Definition 4
A tournament $T = (X, R)$ is simple if every non-constant homomorphism $f : T \to T$ is an automorphism. (In today terminology these are just core tournaments [7].)

Inspired by the characterization of the groups automorphism of tournaments we proved that every such group can be represented by a simple tournament. We also characterized scores of simple tournaments and scores for which every tournament is simple (these are just scores $(1, 1, 1)$, $(2, 2, 2, 2, 2)$,$(3, 3, 3, 3, 3, 3, 3)$). It came then as a surprise that the same notion was studied independently at the same time by P. Erdös, E. Milner and Moon [4, 20]. This was a great encouragement to our work.

Tournament algebras proved to be useful. Denote by $\mathcal{V}_T$ the variety generated by the finite tournament algebras. In [18] we isolated infinitely many irreducible equations valid in $\mathcal{V}_T$ and posed as a problem whether $\mathcal{V}_T$ is finitely axiomatizable. This problem was solved by J. Ježek, M. Márosi and R. McKenzie [9] (there is no finite axiomatization). It appeared that tournament algebras form an important class, see e.g. [14]. They played a role in Ramsey theory as well. Let us finish this paper by stating explicitely this connection.

Let $\mathcal{K}$ be a class of idempotent algebras (by this we mean that every single element subset induces a subalgebra). The notation $B \to (A)^1_k$ has the analogous meaning as above in Section 2 (for topological spaces). More generally given algebras A, B we also write $C \to (B)^A_K$ if the following statement holds:

For every partition of the set $\binom{C}{A}$ of all subalgebras of C which are isomorphic to A into k classes there exists a subalgebra B' of C, $B' \simeq B$, such that $\binom{B'}{A}$ is a subset of one of the classes of the partition. We say that $\mathcal{K}$ has the A-Ramsey property if for every positive k and every $A, B \in \mathcal{K}$ there exists C such that $C \to (B)^A_k$.

In [10] we proved:

Theorem 4

1. Every variety $\mathcal{V}$ of idempotent algebras has point Ramsey property.
2. The variety $\mathcal{V}_T$ generated by the tournament algebras has A-Ramsey property iff A is a singleton.

In [27] we investigated varieties of partially ordered sets and lattices. Particularly we characterized those lattices A for which the class of all finite distributive lattices has A-Ramsey property and for which the class of all lattices have A-Ramsey property. However, for the class Mod of all finite modular lattices the situation is not clear and still presents an open problem:

Problem 2
Characterize those modular lattices A for which the class Mod has A Ramsey property.

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