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BOUNDEDNESS OF THE FRACTIONAL MAXIMAL OPERATOR IN LOCAL MORREY-TYPE SPACES

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Abstract. The problem of the boundedness of the fractional maximal operator M_α, $0 \le \alpha < n$ in local Morrey-type spaces is reduced to the problem of the boundedness of the Hardy operator in weighted L_p-spaces on the cone of non-negative non-increasing functions. This allows obtaining sharp sufficient conditions for the boundedness for all admissible values of the parameters.

1. Introduction

If E is a nonempty measurable subset on $\mathbb{R}^n$ and f is a measurable function on E, then we put

$$\|g\|_{L_p(E)} := \left(\int_E |f(y)|^p dy\right)^{\frac{1}{p}}, \ 0 < p < +\infty,$$

$$\|f\|_{L_\infty(E)} := \sup\{\alpha : |\{y \in E : |f(y)| \ge \alpha\}| > 0\}.$$

If I a nonempty measurable subset on $(0, +\infty)$ and g is a measurable function on I, then we define $\|g\|_{L_p(I)}$ and $\|g\|_{L_\infty(I)}$ correspondingly.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r)$ denote the open ball centered at x of radius r and ${}^{\complement}B(x,r)$ denote the set $\mathbb{R}^n \backslash B(x,r)$.

Let $f \in L_1^{\rm loc}(\mathbb{R}^n)$. The fractional maximal operator M_α and the Riesz potential I_α is defined by

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\frac{\alpha}{n}} \int_{B(x,t)} |f(y)| dy, \ 0 \le \alpha < n,$$

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$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \; 0 < \alpha < n,$$

where $0 \le \alpha < n$ and $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$. If $\alpha = 0$, then $M \equiv M_0$ is the Hardy-Littlewood maximal operator.

The operators $M \equiv M_0$, M_α and I_α play an important role in real and harmonic analysis. (see, for example [15] and [16])

In the theory of partial differential equations, together with weighted $L_{p,w}$ spaces, Morrey spaces $\mathcal{M}_{p,\lambda}$ play an important role. They were introduced by C. Morrey in 1938 [19] and defined as follows: *For* $\lambda \ge 0$, $1 \le p \le \infty$, $f \in \mathcal{M}_{p,\lambda}$ *if* $f \in L_p^{\rm loc}(\mathbb{R}^n)$ *and*

$$\|f\|_{\mathcal{M}_{p,\lambda}} \equiv \|f\|_{\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, \, r>0} r^{-\lambda/p} \|f\|_{L_p(B(x,r))} < \infty$$

holds .

These spaces appeared to be quite useful in the study of local behavior of the solutions of elliptic partial differential equations.

Also by $W\mathcal{M}_{p,\lambda}$ we denote the weak Morrey space of all functions $f \in {\rm WL}_p^{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{W\mathcal{M}_{p,\lambda}} \equiv \|f\|_{W\mathcal{M}_{p,\lambda}(\mathbb{R}^n)} = \sup_{x \in \mathbb{R}^n, \, r>0} r^{-\lambda/p} \|f\|_{WL_p(B(x,r))} < \infty,$$

where ${\rm WL}_p$ denotes the weak L_p-space.

Spanne (see [22]) and Adams [1] studied the boundedness of the fractional maximal operator M_α for $0 < \alpha < n$ in Morrey spaces $\mathcal{M}_{p,\lambda}$. Later on Chiarenza and Frasca [11] studied the boundedness of the maximal operator M in these spaces. Their results can be summarized as follows:

Theorem 1.1. *(1) Let* $0 \le \alpha < n$, $1 < p_1 < n/\alpha$, $0 < \lambda < n - \alpha p_1$ *and* $1/p_1 - 1/p_2 = \alpha/n - \lambda$. *Then* M_α *is bounded from* $\mathcal{M}_{p_1,\lambda}$ *to* $\mathcal{M}_{p_2,\lambda}$.

(2) Let $0 \le \alpha < n$, $0 < \lambda < n - \alpha$ *and* $1 - 1/p_2 = \alpha/(n - \lambda)$. *Then* M_α *is bounded from* $\mathcal{M}_{1,\lambda}$ *to* $W\mathcal{M}_{p_2,\lambda}$.

If in the place of the power function $r^{-\lambda/p}$ in the definition of $\mathcal{M}_{p,\lambda}$ we consider any positive weight function w defined on $(0,\infty)$, then it becomes the Morrey-type space $\mathcal{M}_{p,w}$. T. Mizuhara [17] and E. Nakai [20] extended the above results to these spaces and obtained the following sufficient conditions on a weight w ensuring the boundedness of the maximal operator M and the fractional maximal operator M_α.

Theorem 1.2. *Let* w *be a positive decreasing function satisfying the following condition: there exists* $1 \le c_1 < 2^{n/p}$, *such that*

$$w(r) \le c_1 w(2r)$$

for all $r > 0$.

For $1 < p < \infty$ M *is bounded from* $\mathcal{M}_{p,w}$ *to* $\mathcal{M}_{p,w}$, *and for* $p = 1$ M *is bounded from* $\mathcal{M}_{1,w}$ *to* $W\mathcal{M}_{1,w}$

Theorem 1.3. *Let w be a positive decreasing function satisfying the following condition: there exists $c_2 > 0$, such that*

$$0 < r \le t \le 2r \Rightarrow w(r) \le c_2^{-1} w(t) \le w(r) \le c_2 w(t). \tag{1.1}$$

Moreover, let $\alpha = n(1/p_1 - 1/p_2)$ and let for some $c_3 > 0$ for all $r > 0$

$$\int_r^\infty \frac{dt}{w^{p_1}(t)t^{n+1-\alpha p_1}} \le \frac{c_3}{w^{p_1}(r)r^{np_1/p_2}}. \tag{1.2}$$

(1) For $1 < p_1 = p_2 < \infty$ M_α is bounded from $\mathcal{M}_{p_1,w}$ to $\mathcal{M}_{p_1,w}$, and for $p = 1$ M is bounded from $\mathcal{M}_{1,w}$ to $W\mathcal{M}_{1,w}$.

(2) For $1 < p_1 < p_2 < \infty$ M_α is bounded from $\mathcal{M}_{p_1,w}$ to $\mathcal{M}_{p_2,w}$, and for $p_1 = 1$ M_α is bounded from $\mathcal{M}_{1,w}$ to $W\mathcal{M}_{p_2,w}$.

Theorem 1.2 was proved by Mizuhara [17] and Theorem 1.3 by Nakai. Note that Theorem 3 implies Theorem 2.

In [2] D.R.Adams introduced a variant of Morrey-type spaces as follows: For $0 \le \lambda \le n$, $1 \le p, \theta \le \infty$, $f \in \mathcal{M}_{p\theta,\lambda}$ if $f \in L_p^{\text{loc}}(\mathbb{R}^n)$ and

$$\|f\|_{\mathcal{M}_{p\theta,\lambda}} \equiv \|f\|_{\mathcal{M}_{p\theta,\lambda}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \|r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))}\|_{L_\theta(0,\infty)} < \infty.$$

(If $\theta = \infty$, then $\mathcal{M}_{p\theta,\lambda} = \mathcal{M}_{p,\lambda}$.)

In [5]-[8] the boundedness of maximal and fractional maximal operators from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ and from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$ have been investigated. Moreover, for some values of the parameters necessary and sufficient conditions for the operators Mf and $M_\alpha f$ to be bounded from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ were obtained.

Theorem 1.4. *Let $1 < p_1 < \infty$, $0 < p_2 < \infty$, $n(1/p_1 - 1/p_2)_+ \le \alpha < n$, $0 < \theta_2 \le \infty$, $\omega_2 \in \Omega_{\theta_2}$.*

1. For $\alpha < n/p_1$, let $\omega_1 \in \Omega_{\theta_1}$ and

$$\|\omega_2(r) r^{n/p_2} \|\omega_1^{-1}(t) t^{\alpha - n/p_1 - 1/\min\{p_1,\theta_1\}}\|_{L_s(r,\infty)}\|_{L_{\theta_2}(0,\infty)} < \infty, \tag{1.3}$$

where $s = p_1\theta_1/(\theta_1 - p_1)_+$. (If $\theta_1 \le p_1$, then $s = \infty$) Then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

2. For $\alpha = n/p_1$, let

$$\omega_2(r) r^{\alpha - n(1/p_1 - 1/p_2)} \in L_{\theta_2}(0,\infty). \tag{1.4}$$

Then M_α is bounded from L_{p_1} to $LM_{p_2,\theta_2,\omega_2}$.

Theorem 1.5. *1. If $0 \le p_1 \le \infty$, $0 < p_2 \le \infty$, $0 \le \alpha < n$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$, then the condition*

$$t^{\alpha - n/p_1 + \min\{n-\alpha, n/p_2\}} \left\| \omega_2(r) \frac{r^{n/p_2}}{(t+r)^{\min\{n-\alpha, n/p_2\}}} \right\|_{L_{\theta_2}(0,\infty)} \le c\|\omega_1\|_{L_{\theta_1}(t,\infty)} \tag{1.5}$$

for all $t > 0$, where $c > 0$ is independent of t, is necessary for the boundedness of M_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

2. If $1 < p_1 < \infty$, $0 < p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le p_1\ n(1/p_1 - 1/p_2)_+ \le \alpha < n/p_1$, $\omega_1 \in \Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| \omega_2(r) \frac{r^{n/p_2}}{(t+r)^{n/p_1-\alpha}} \right\|_{L_{\theta_2}(0,\infty)} \le c\|\omega_1\|_{L_{\theta_1}(t,\infty)} \tag{1.6}$$

for all $t > 0$, *where* $c > 0$ *is independent of* t, *is sufficient for the boundedness of* M_α *from* $LM_{p_1,\theta_1,\omega_1}$ *to* $LM_{p_2,\theta_2,\omega_2}$ *and from* $GM_{p_1\theta_1,\omega_1}$ *to* $GM_{p_2\theta_2,\omega_2}$. *(In the latter case we assume that* $\omega_1 \in \Omega_{p_1,\theta_1}$, $\omega_2 \in \Omega_{p_2,\theta_2}$*)*

3. In particular, if $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\theta_1 \le p_1$, $\alpha = n(1/p_1 - 1/p_2)$, $\omega_1 \in \Omega_{\theta_1}$ *and* $\omega_2 \in \Omega_{\theta_2}$, *then the condition*

$$\left\| \omega_2(r) \left(\frac{r}{t+r} \right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} \le c\|\omega_1\|_{L_{\theta_1}(t,\infty)} \tag{1.7}$$

for all $t > 0$, *where* $c > 0$ *is independent of* t, *is necessary and sufficient for the boundedness of* M_α *from* $LM_{p_1,\theta_1,\omega_1}$ *to* $LM_{p_2,\theta_2,\omega_2}$.

Theorem 1.4 and Theorem 1.5 were proved in [8].

In this paper we improve the estimate of L_p norm of the farctional maximal operator over balls obtained in [8], and find sufficient conditions for the boundedness of M_α from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ for all admissible values of parameters. It is evident that these conditions are sufficient for the boundedness of M_α from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$ too.

2. Definitions and basic properties of Morrey-type spaces

Definition 2.1. Let $0 < p, \theta \le \infty$ and let w be a non-negative measurable function on $(0,\infty)$. We denote by $LM_{p_1,\theta_1,\omega_1}$, $GM_{p,\theta,\omega}$, the local Morrey-type spaces, the global Morrey-type spaces respectively, the spaces of all functions $f \in L_p^{\rm loc}(\mathbb{R}^n)$ with finite quasinorms

$$\|f\|_{LM_{p_1,\theta_1,\omega_1}} \equiv \|f\|_{LM_{p_1,\theta_1,\omega_1}(\mathbb{R}^n)} = \left\| w(r)\|f\|_{L_p(B(0,r))} \right\|_{L_\theta(0,\infty)},$$

$$\|f\|_{GM_{p,\theta,\omega}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p_1,\theta_1,\omega_1}}$$

respectively.

Note that

$$\|f\|_{LM_{p\infty,1}} = \|f\|_{GM_{p\infty,1}} = \|f\|_{L_p}.$$

Furthermore, $GM_{p\infty,r^{-\lambda/p}} \equiv \mathcal{M}_{p,\lambda}$, $0 < \lambda < n$. The interpolation properties of the spaces $GM_{p\infty,w}$ were studied by S. Spanne in [22]. The spaces $GM_{p\theta,r^{-\lambda}}$ were used by G. Lu [21] for studying the embedding theorems for vector fields of Hörmander type. The boundedness of various integral operators in the spaces $GM_{p\infty,w}$ was studied by T. Mizuhara [17] and E. Nakai [20]. In [5, 6] the boundedness of the maximal operator M from $LM_{p_1\theta_1,w_1}$ to $LM_{p_2\theta_2,w_2}$ and from $GM_{p_1\theta_1,w_1}$ to $GM_{p_2\theta_2,w_2}$ was investigated.

In [6] the following statement was proved.

Lemma 2.2. *Let $0 < p, \theta \leq \infty$ and let w be a non-negative measurable function on $(0, \infty)$.*

1. If for all $t > 0$
$$\|w(r)\|_{L_\theta(t,\infty)} = \infty, \tag{2.1}$$
then $LM_{p_1,\theta_1,\omega_1} = GM_{p,\theta,\omega} = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

2. If for all $t > 0$
$$\|w(r) r^{n/p}\|_{L_\theta(0,t)} = \infty, \tag{2.2}$$
then, for all functions $f \in LM_{p_1,\theta_1,\omega_1}$, continuous at 0, $f(0) = 0$, and for $0 < p < \infty$ $GM_{p,\theta,\omega} = \Theta$.

Definition 2.3. Let $0 < p, \theta \leq \infty$. We denote by Ω_θ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t > 0$
$$\|w(r)\|_{L_\theta(t,\infty)} < \infty. \tag{2.3}$$
Moreover, we denote by $\Omega_{p,\theta}$ the set of all functions w which are non-negative, measurable on $(0, \infty)$, not equivalent to 0 and such that for some $t_1, t_2 > 0$
$$\|w(r)\|_{L_\theta(t_1,\infty)} < \infty, \qquad \|w(r) r^{n/p}\|_{L_\theta(0,t_2)} < \infty. \tag{2.4}$$

In the sequel, keeping in mind Lemma 2.2, we always assume that either $w \in \Omega_\theta$ or $w \in \Omega_{p,\theta}$.

In [9] the following statements were proved.

Lemma 2.4. *Let $1 < p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 \leq \alpha < n$, $0 < \theta_1, \theta_2 \leq \infty$, $\omega_1 \in \Omega_{\theta_1}$, and $\omega_2 \in \Omega_{\theta_2}$. Then the condition*
$$\alpha \leq \frac{n}{p_1}$$
is necessary for the boundedness of M_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Lemma 2.5. *Let $1 \leq p_1 \leq \infty$, $0 < p_2 \leq \infty$, $0 \leq \alpha < n$, $0 < \theta_1, \theta_2 \leq \infty$, $\omega_1 \in \Omega_{\theta_1}$, and $\omega_2 \in \Omega_{\theta_2}$. Moreover, let $\omega_1 \in L_{\theta_1}(0, \infty)$. Then the condition* [1]
$$\alpha \geq n \left(\frac{n}{p_1} - \frac{n}{p_2} \right)_+ \tag{2.5}$$
is necessary for the boundedness of M_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Remark 2.6. If $\omega_1 \in \Omega_{\theta_1}$ but $\omega_1 \notin L_{\theta_1}(0, \infty)$, then condition (2.5) is not necessary for the boundedness of M_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Throughout this paper $a \lesssim b$, $(b \gtrsim a)$, means that $a \leq \lambda b$, where $\lambda > 0$ depends on inessential parameters. If $b \lesssim a \lesssim b$, then we write $a \approx b$.

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[1] Here and in the sequel $t_+ = t$ if $t \geq 0$ and $t_+ = 0$ if $t < 0$ and $t_- = -t$ if $t \leq 0$ and $t_- = 0$ if $t > 0$.

3. L_p-estimates of Fractional maximal function over balls

The following Theorem is true.

Theorem 3.1. *Let $1 < p < \infty$, and $f \in L_p^{\rm loc}(\mathbb{R}^n)$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$*

$$\|M_\alpha f\|_{L_p(B)} \lesssim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}} \left(\sup_{t\geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right) \tag{3.1}$$

Proof. It is obvious that for any ball $B = B(x,r)$

$$\|M_\alpha f\|_{L_p(B)} \leq \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + M_\alpha(f\chi_{\mathbb{R}^n\setminus(2B)})\|_{L_p(B)}.$$

Let y be an arbitrary point from B. If $B(y,t)\cap\{\mathbb{R}^n\setminus(2B)\} \neq \emptyset$, then $t > r$. Indeed, if $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then $t \geq |z-y| \geq |z-x| - |x-y| > 2r - r = r$.

On the other hand $B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\} \subset B(x,2t)$. Indeed, $z \in B(y,t) \cap \{\mathbb{R}^n\setminus(2B)\}$, then we get $|z-x| \leq |z-y| + |y-x| \leq t + r \leq 2t$.

Hence

$$M_\alpha(f\chi_{\mathbb{R}^n\setminus(2B)})(y) = \sup_{t>0} \frac{1}{|B(y,t)|^{1-\frac{\alpha}{n}}} \int_{B(y,t)\cap\{\mathbb{R}^n\setminus(2B)\}} |f|$$

$$\lesssim \sup_{t\geq r} \frac{1}{|B(x,2t)|^{1-\frac{\alpha}{n}}} \int_{B(x,2t)} |f| = \sup_{t\geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|.$$

Thus

$$\|M_\alpha f\|_{L_p(B)} \lesssim \|M_\alpha(f\chi_{(2B)})\|_{L_p(B)} + |B|^{\frac{1}{p}} \left(\sup_{t\geq 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right)$$

□

Theorem 3.2. *Let $1 < p < \infty$, and $f \in L_p^{\rm loc}(\mathbb{R}^n)$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$*

$$\|M_\alpha f\|_{L_p(B)} \gtrsim |B|^{\frac{1}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right) \tag{3.2}$$

Proof. Since $B(x,\frac{t}{2}) \subset B(y,t),\ t > 2r$, then

$$M_\alpha f(y) \gtrsim \sup_{t>2r} \frac{1}{|B(x,\frac{t}{2})|^{1-\frac{\alpha}{n}}} \int_{B(x,\frac{t}{2})} = \sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|,$$

thus

$$\|M_\alpha f\|_{L_p(B)} \gtrsim |B|^{\frac{1}{p}} \left(\sup_{t>r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right). \tag{3.3}$$

□

The following Lemma is true

Lemma 3.3. *Let* $0 \le \alpha < n$, $0 < p_2 < \infty$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then*

$$\|M_\alpha(f\chi_{B(0,2r)})\|_{L_{p_2}(B(0,r))} \lesssim r^{\alpha - n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(B(0,2r))}, \tag{3.4}$$

for all $r > 0$ *and* $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$.

Proof. Since

$$M_\alpha f(x) \lesssim I_\alpha(|f|)(x), \tag{3.5}$$

then statement immediately follows from Lemma 3.1 in [4]. □

From Theorem 3.1 and Lemma 3.3 follows next statement.

Lemma 3.4. *Let* $0 \le \alpha < n$, $0 < p_2 < \infty$, $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then for any ball* $B = B(x,r) \subset \mathbb{R}^n$

$$\begin{aligned}&\|M_\alpha f\|_{L_{p_2}(B)}\\ &\le c|B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right) + c|B|^{\frac{\alpha}{n}-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2B)},\end{aligned} \tag{3.6}$$

where constant c *does not depend on* $|B|$.

The following Lemma is true.

Lemma 3.5. *Let* $0 < \alpha < n$, $0 < p_2 < \infty$, $f \in L^{\mathrm{loc}}_{p_1}(\mathbb{R}^n)$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$. *Then for any ball* $B = B(x,r) \subset \mathbb{R}^n$

$$\|M_\alpha f\|_{L_{p_2}(B)} \le c|B|^{\frac{1}{p_2}} \left(\sup_{t\ge r} \frac{1}{|B(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(x,t)} |f|^{p_1}\right)^{\frac{1}{p_1}}\right), \tag{3.7}$$

where constant c *does not depend on* $|B|$.

Proof. Denote by

$$M_1 := |B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f|\right),\ M_2 := |B|^{\frac{\alpha}{n}-\left(\frac{1}{p_1}-\frac{1}{p_2}\right)} \|f\|_{L_{p_1}(2B)}.$$

Applying Hölder's inequality, we get

$$M_1 \lesssim |B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(x,t)} |f|^{p_1}\right)^{\frac{1}{p_1}}\right).$$

On the other hand

$$|B|^{\frac{1}{p_2}} \left(\sup_{t\ge 2r} \frac{1}{|B(x,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(x,t)} |f|^{p_1}\right)^{\frac{1}{p_1}}\right)$$

$$\gtrsim |B|^{\frac{1}{p_2}} \left(\sup_{t \geq 2r} |B(x,t)|^{\frac{\alpha}{n}-\frac{1}{p_1}} \right) \|f\|_{L_{p_1}(2B)} \approx M_2.$$

Since by Lemma 3.4

$$\|M_\alpha f\|_{L_{p_2}(B)} \leq M_1 + M_2,$$

we arrive at (3.7). □

Remark 3.6. Inequality (3.7) improves the inequality (22) in [8]

$$\|M_\alpha f\|_{L_{p_2}(B(0,r))} \leq c r^{\frac{n}{p_2}} \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-\alpha p_1+1}} \right)^{\frac{1}{p_1}}.$$

This follows since

$$\sup_{t \geq r} \frac{1}{|B(0,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(0,t)} |f|^{p_1} \right)^{\frac{1}{p_1}} \leq \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-\alpha p_1+1}} \right)^{\frac{1}{p_1}}$$

Indeed, by easy calculation and the Fubini theorem, we get

$$\sup_{t \geq r} \frac{1}{|B(0,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(0,t)} |f|^{p_1} \right)^{\frac{1}{p_1}}$$

$$\leq \sup_{t \geq r} \frac{1}{|B(0,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(0,r)} |f|^{p_1} \right)^{\frac{1}{p_1}} + \sup_{t \geq r} \frac{1}{|B(0,t)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(0,t) \setminus B(0,r)} |f|^{p_1} \right)^{\frac{1}{p_1}}$$

$$\leq \frac{1}{|B(0,r)|^{\frac{1}{p_1}-\frac{\alpha}{n}}} \left(\int_{B(0,r)} |f|^{p_1} \right)^{\frac{1}{p_1}} + \left(\int_{\mathbb{R}^n \setminus B(0,r)} \frac{|f(x)|^{p_1}}{|x|^{n-\alpha p_1}} dx \right)^{\frac{1}{p_1}}$$

$$\lesssim \left(\int_r^\infty \left(\int_{B(0,r)} |f|^{p_1} \right) \frac{dt}{t^{n-\alpha p_1+1}} \right)^{\frac{1}{p_1}}$$

$$+ \left(\int_r^\infty \left(\int_{B(0,t) \setminus B(0,r)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-\alpha p_1+1}} \right)^{\frac{1}{p_1}}$$

$$\lesssim \left(\int_r^\infty \left(\int_{B(0,t)} |f(x)|^{p_1} dx \right) \frac{dt}{t^{n-\alpha p_1+1}} \right)^{\frac{1}{p_1}}.$$

The following Theorem is true.

Theorem 3.7. *Let $0 \leq \alpha < n$, $0 < p < \infty$, $\frac{pn}{n+\alpha p} < 1$, and $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$. Then for any ball $B = B(x,r) \subset \mathbb{R}^n$*

$$\|M_\alpha f\|_{L_p(B)} \approx r^{\frac{n}{p}} \left(\sup_{t > r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f| \right). \tag{3.8}$$

Proof. In view of the Theorem 3.2 we need only to prove that

$$\|M_\alpha f\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \left(\sup_{t > r} \frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}} \int_{B(x,t)} |f| \right).$$

By Lemma 3.4, we have

$$\|M_\alpha f\|_{L_p(B)} \lesssim |B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)}\|f\|_{L_1(2B)} + |B|^{\frac{1}{p}}\left(\sup_{t\geq 2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}|f|\right).$$

But

$$\begin{aligned}|B|^{\frac{\alpha}{n}-\left(1-\frac{1}{p}\right)}\|f\|_{L_1(2B)} &\approx |B|^{\frac{1}{p}}\frac{1}{|2B|^{1-\frac{\alpha}{n}}}\int_{2B}|f(y)|dy\\ &\leq |B|^{\frac{1}{p}}\left(\sup_{t\geq 2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}|f|\right).\end{aligned}$$

Hence

$$\begin{aligned}\|M_\alpha f\|_{L_p(B)} &\lesssim r^{\frac{n}{p}}\left(\sup_{t>2r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}|f|\right)\\ &\leq r^{\frac{n}{p}}\left(\sup_{t>r}\frac{1}{|B(x,t)|^{1-\frac{\alpha}{n}}}\int_{B(x,t)}|f|\right).\end{aligned}$$

□

4. Fractional maximal operator and Hardy-type operator involving suprema

Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{f \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} f(t) = 0\right\}.$$

Let u be a continous weight on $(0,\infty)$. We define the Hardy-type operator involving suprema H_u on $g \in \mathfrak{M}^+(0,\infty)$ by

$$(H_u g)(t); = \sup_{t\leq r<\infty} u(r)g(r),\ t \in (0,\infty).$$

The following Lemma is true.

Lemma 4.1. *Let* $0 \leq \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_2 \leq \infty$ *and* $\omega_2 \in \Omega_{\theta_2}$. *Moreover, let* $1 < \frac{p_2 n}{n+\alpha p_2} \leq p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} < 1 \leq p_1 < \infty$, *or* $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1 < \infty$.

Then

$$\|M_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|H_u g\|_{L_{\theta_2,\upsilon_2}(0,\infty)} \tag{4.1}$$

for all $f \in L_p^{\rm loc}(\mathbb{R}^n)$, *where*

$$g(t) = \|f\|_{L_{p_1}(B(0,t))}, \tag{4.2}$$

$$u(r) = r^{\alpha-\frac{n}{p_1}} \tag{4.3}$$

and

$$\upsilon_2(r) = \omega_2^{\theta_2}(r) r^{\theta_2\frac{n}{p_2}}. \tag{4.4}$$

Proof. By Lemma 3.5 we have

$$\begin{aligned} \|M_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim & \left\| \omega_2(t) t^{\frac{n}{p_2}} \sup_{t \le r < \infty} r^{\alpha - \frac{n}{p_1}} \|f\|_{L_{p_1}(B(0,r))} \right\|_{L_{\theta_2}(0,\infty)} \\ = & \|H_u g\|_{L_{\theta_2,\upsilon_2}(0,\infty)}. \end{aligned} \tag{4.5}$$

□

Theorem 4.2. *Let $0 \le \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Moreover, let $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1$, or $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1$, or $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1$.*

Assume that the operator H_u is bounded from $L_{\theta_1,\upsilon_1}(0,\infty)$ to $L_{\theta_2,\upsilon_2}(0,\infty)$ on $\mathbb{A}$, that is,

$$\|H_u g\|_{L_{\theta_2,\upsilon_2}(0,\infty)} \lesssim \|g\|_{L_{\theta_1,\upsilon_1}(0,\infty)}, \tag{4.6}$$

where

$$\upsilon_1(r) = \omega_1^{\theta_1}(r) \tag{4.7}$$

and

$$\upsilon_2(r) = \omega_2^{\theta_2}(r) r^{\theta_2 \frac{n}{p_2}}. \tag{4.8}$$

Then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Proof. Since g is non-negative and non-decreasing function on $(0,\infty)$ and H_u is bounded from L_{θ_1,υ_1} to L_{θ_2,υ_2} on the cone of functions containing g, by Lemma 4.1 we have

$$\|M_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|g\|_{L_{\theta_1,\upsilon_1}(0,\infty)} = \left(\int_0^\infty \upsilon_1(r) (g(r))^{\theta_1} dr \right)^{\frac{1}{\theta_1}}.$$

Hence

$$\|M_\alpha f\|_{LM_{p_2,\theta_2,\omega_2}} \lesssim \|\omega_1(r) \|f\|_{L_{p_1}(B(0,r))}\|_{L_{\theta_1}(0,\infty)} = \|f\|_{LM_{p_1,\theta_1,\omega_1}}.$$

□

5. Weighted inequalities for Hardy-type operators involving suprema

Note that the inequality

$$\|H_u \varphi\|_{L_{\theta_2,w_2}} \lesssim \|\varphi\|_{L_{\theta_1,w_1}}, \ \varphi \in \mathbb{A}, \tag{5.1}$$

that is

$$\left(\int_0^\infty w_2(t) \left(\operatorname*{ess\,sup}_{t \le r < \infty} u(r) \varphi(r) \right)^{\theta_2} dt \right)^{\frac{1}{\theta_2}} \lesssim \left(\int_0^\infty w_1(t) \left(\varphi(t) \right)^{\theta_1} dt \right)^{\frac{1}{\theta_1}} \tag{5.2}$$

is equivalent to the inequality

$$\left(\int_0^\infty w_2(t) \left(\operatorname*{ess\,sup}_{t \le r < \infty} (u(r)^{\theta_1} \varphi(r) \right)^{\frac{\theta_2}{\theta_1}} dt \right)^{\frac{\theta_1}{\theta_2}} \lesssim \int_0^\infty w_1(t) \varphi(t) dt, \ \varphi \in \mathbb{A}. \tag{5.3}$$

Given $\varphi \in \mathbb{A}$, there is a sequence $\{h_n\}$ of positive functions such that

$$\int_0^t h_n(s)ds \nearrow \varphi, \; t \in (0,\infty). \tag{5.4}$$

By the Fatou lemma, we see that (5.1) holds if and only if the inequality

$$\begin{aligned}\left(\int_0^\infty w_2(t)\left(\sup_{t\le r<\infty}(u(r))^{\theta_1}\int_0^r h(s)ds\right)^{\frac{\theta_2}{\theta_1}}dt\right)^{\frac{\theta_1}{\theta_2}} \\ \lesssim \int_0^\infty w_1(t)\left(\int_0^t h(s)ds\right)dt\end{aligned} \tag{5.5}$$

are satisfied for all $h \in \mathfrak{M}^+(0,\infty)$. Summarizing, By Fubini theorem, we obtain that (5.1) holds if and only if the inequality

$$\begin{aligned}\left(\int_0^\infty w_2(t)\left(\sup_{t\le r<\infty}(u(r))^{\theta_1}\int_0^r h(s)ds\right)^{\frac{\theta_2}{\theta_1}}dt\right)^{\frac{\theta_1}{\theta_2}} \\ \lesssim \int_0^\infty h(s)\left(\int_s^\infty w_1(t)dt\right)ds\end{aligned} \tag{5.6}$$

are satisfied for all $h \in \mathfrak{M}^+(0,\infty)$.

Let us recall the following Theorem. (see Theorem 4.1 and Theorem 4.4 in [3]

Theorem 5.1. *Let $0 < q < \infty$ and let u be a continuous weight. Let v and w be weights such that $0 < \int_0^x v(t)dt < \infty$ and $0 < \int_0^x w(t)dt < \infty$ for every $x \in (0,\infty)$.*

(i) Let $1 \le q$. Then the inequality

$$\left(\int_0^\infty \left[\sup_{t\le s<\infty}\frac{u(s)}{s}\int_0^s g(y)dy\right]^q w(t)dt\right)^{1/q} \lesssim \int_0^\infty g(t)v(t)dt \tag{5.7}$$

holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\sup_{x>0}\left(\left(\frac{\overline{u}(x)}{x}\right)^q\int_0^x w(t)dt + \int_x^\infty\left(\frac{\overline{u}(t)}{t}\right)^q w(t)dt\right)^{1/q}\operatorname*{ess\,sup}_{0<t<x}\frac{1}{v(t)} < \infty, \tag{5.8}$$

where

$$\overline{u}(t) = t\sup_{t\le\tau<\infty}\frac{u(\tau)}{\tau}, \; t \in (0,\infty).$$

(ii) Let $q < 1$. Then the inequality (5.7) holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\left(\int_0^\infty\left(\int_t^\infty\left(\frac{\overline{u}(s)}{s}\right)^q w(s)ds\right)^{\frac{q}{1-q}}\left(\frac{\overline{u}(t)}{t}\right)^q\left[\operatorname*{ess\,sup}_{0<\tau<t}\frac{1}{v(\tau)}\right]^{\frac{q}{1-q}} w(t)dt\right)^{\frac{1-q}{q}} < \infty, \tag{5.9}$$

and

$$\left(\int_0^\infty \left(\int_0^t w(s)ds\right)^{\frac{q}{1-q}} \left[\sup_{t\le\tau<\infty} \frac{\overline{u}(\tau)}{\tau} \operatorname*{ess\,sup}_{0<s<\tau} \frac{1}{\upsilon(s)}\right]^{\frac{q}{1-q}} w(t)dt\right)^{\frac{1-q}{q}} < \infty, \qquad (5.10)$$

The following Theorem is true.

Theorem 5.2. *Let $q = \infty$ and u be a continuous weight on $(0,\infty)$. Let υ and w be weights such that $0 < \operatorname{ess\,sup}_{0<t<x}(\upsilon(t))^{-1} < \infty$ and $0 < \operatorname{ess\,sup}_{0<t<x} w(t) < \infty$ for every $x \in (0,\infty)$. Then the inequality (5.7) holds on $\mathfrak{M}^+(0,\infty)$ if and only if*

$$\sup_{t>0} \left(\int_t^\infty \frac{u(s)}{s} \left(\operatorname*{ess\,sup}_{0<r<s} w(r)\right) ds\right) \operatorname*{ess\,sup}_{0<r<t} \frac{1}{\upsilon(r)} < \infty \qquad (5.11)$$

Proof. When $q = \infty$, the inequality (5.7) takes the form

$$\operatorname*{ess\,sup}_{t>0} \sup_{t\le s<\infty} w(t) \left(\frac{u(s)}{s} \int_0^s g(y)dy\right) \lesssim \int_0^\infty g(t)\upsilon(t)dt. \qquad (5.12)$$

Applying the Fubini Theorem to the left hand side of (5.12), we get the inequality

$$\sup_{s>0} \left(\frac{u(s)}{s} \operatorname*{ess\,sup}_{0<t\le s} w(t)\right) \int_0^s g(y)dy \lesssim \int_0^\infty g(t)\upsilon(t)dt. \qquad (5.13)$$

Since

$$\int_0^\infty w(x) \int_0^x f(t)dt \lesssim \int_0^\infty f(x)\upsilon(x)dx$$
$$\Leftrightarrow \sup_{r>0} \left(\int_r^\infty w(x)dx\right) \operatorname*{ess\,sup}_{0<x<r} \frac{1}{\upsilon(x)} < \infty$$

(see Theorem 2 on p.42 in [18]), the inequality (5.13) holds on $\mathfrak{M}^+(0,\infty)$ if and only if the condition (5.11) holds. □

The following Theorem is true.

Theorem 5.3. *Let $0 < q < \infty$ and let u be a continuous weight. Let υ and w be weights such that $0 < \operatorname{ess\,sup}_{t\le y<\infty} \upsilon(y) < \infty$ for any $t > 0$, $\operatorname{ess\,sup}_{t>0} \upsilon(t) = \infty$.*
Then the inequality

$$\left(\int_0^\infty \left[\sup_{t\le s<\infty} \frac{u(s)}{s} \int_0^s g(y)dy\right]^q w(t)dt\right)^{1/q} \lesssim \operatorname*{ess\,sup}_{t>0} \upsilon(t) \int_0^t g(s)ds \qquad (5.14)$$

holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\left(\int_0^\infty \left(\operatorname*{ess\,sup}_{t<s<\infty} \frac{u(s)}{s} \frac{1}{\operatorname{ess\,sup}_{s<y<\infty} \upsilon(y)}\right)^q w(t)dt\right)^{\frac{1}{q}} < \infty. \qquad (5.15)$$

Proof. Whenever F, G are non-negative functions on $(0,\infty)$ and F is non-decreasing, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{s\in(t,\infty)} G(s),\ t \in (0,\infty). \tag{5.16}$$

Therefore

$$\operatorname*{ess\,sup}_{t>0} v(t) \int_0^t g(s)ds = \operatorname*{ess\,sup}_{t>0} \left(\int_0^t g(s)ds \right) \operatorname*{ess\,sup}_{t<y<\infty} v(y). \tag{5.17}$$

At first let us to prove sufficiency. Assume that the condition (5.15) holds. Then

$$\begin{aligned}
&\left(\int_0^\infty \left[\sup_{t\le s<\infty} \frac{u(s)}{s} \int_0^s g(y)dy \right]^q w(t)dt \right)^{1/q} \\
&= \left(\int_0^\infty \left[\operatorname*{ess\,sup}_{t\le s<\infty} \frac{u(s)}{s} \frac{\operatorname{ess\,sup}_{s<y<\infty} v(y)}{\operatorname{ess\,sup}_{s<y<\infty} v(y)} \int_0^s g(y)dy \right]^q w(t)dt \right)^{1/q} \\
&\le \sup_{t>0} \operatorname*{ess\,sup}_{t<y<\infty} v(y) \int_0^t g(s)ds \times \\
&\times \left(\int_0^\infty \left(\operatorname*{ess\,sup}_{t<s<\infty} \frac{u(s)}{s} \frac{1}{\operatorname{ess\,sup}_{s<y<\infty} v(y)} \right)^q w(t)dt \right)^{\frac{1}{q}} \\
&\le c \sup_{t>0} \operatorname*{ess\,sup}_{t<y<\infty} v(y) \int_0^t g(s)ds.
\end{aligned} \tag{5.18}$$

To prove necessity note that, for every non-decreasing function Φ on $(0,\infty)$, there is a sequence $\{H_n\}_{n=1}^\infty$ of smooth increasing functions such that $H_n \nearrow \Phi$ as $n \to \infty$. The functions H_n, being smooth, can be represented as $H_n(t) = \int_0^t h_n(s)ds + H_n(0)$ for some positive measurable functions h_n on $(0,\infty)$. Applying this to the non-decreasing function $\Phi(t) = (\operatorname{ess\,sup}_{t<y<\infty} v(y))^{-1}$, let $\{h_n\}$ be a sequence of positive measurable functions on $(0,\infty)$, such that

$$\int_0^t h_n(s)ds \nearrow \Phi(t),\ n \to \infty \text{ a.e. on } (0,\infty). \tag{5.19}$$

For the right hand side of the inequality

$$\left(\int_0^\infty \left[\sup_{t\le s<\infty} \frac{u(s)}{s} \int_0^s h_n(y)dy \right]^q w(t)dt \right)^{1/q} \lesssim \operatorname*{ess\,sup}_{t>0} v(t) \int_0^t h_n(s)ds \tag{5.20}$$

we have

$$\operatorname*{ess\,sup}_{t>0} v(t) \int_0^t h_n(s)ds \le \operatorname*{ess\,sup}_{t>0} v(t) (\operatorname*{ess\,sup}_{t<y<\infty} v(y))^{-1} \le 1. \tag{5.21}$$

In view of the fact, that from (5.19) follows that

$$\sup_{t\le s<\infty} \frac{u(s)}{s} \int_0^s g(y)dy \nearrow \operatorname*{ess\,sup}_{t\le s<\infty} \frac{u(s)}{s} \Phi(s),\ n \to \infty \text{ a.e. on } (0,\infty),$$

(5.20) and (5.21), by Fatou's lemma, imply (5.15). □

Theorem 5.4. *Let $q = \infty$ and Let u be a continuous weight. Let υ and w be weights such that $0 < \operatorname{ess\,sup}_{t\le y<\infty} \upsilon(y) < \infty$ for any $t > 0$, $\operatorname{ess\,sup}_{t>0} \upsilon(t) = \infty$.*

Then the inequality (5.14) holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\operatorname*{ess\,sup}_{t>0} \frac{u(r)}{r} \frac{\operatorname{ess\,sup}_{0<t\le r} w_2(t)}{\operatorname{ess\,sup}_{r<y<\infty} \upsilon(y)} < \infty. \tag{5.22}$$

Proof. If $q = \infty$, then the inequality (5.14) takes the form

$$\operatorname*{ess\,sup}_{t>0} w_2(t) \sup_{t\le r<\infty} \frac{u(r)}{r} \int_0^r g(y)dy \lesssim \operatorname*{ess\,sup}_{t>0} w_1(t) \int_0^t g(y)dy. \tag{5.23}$$

Applying the Fubini theorem to the left hand side and in view of (5.17) for right hand side, we can write (5.23) in the following form

$$\operatorname*{ess\,sup}_{t>0} \frac{u(r)}{r} \left(\operatorname*{ess\,sup}_{0<t\le r} w_2(t)\right) \int_0^r g(y)dy \lesssim \operatorname*{ess\,sup}_{t>0} \left(\int_0^t g(s)ds\right) \operatorname*{ess\,sup}_{t<y<\infty} \upsilon(y). \tag{5.24}$$

Since

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t>0} \frac{u(r)}{r} \left(\operatorname*{ess\,sup}_{0<t\le r} w_2(t)\right) \int_0^r g(y)dy \\
&= \operatorname*{ess\,sup}_{t>0} \frac{u(r)}{r} \left(\operatorname*{ess\,sup}_{0<t\le r} w_2(t)\right) \frac{\operatorname{ess\,sup}_{r<y<\infty} \upsilon(y)}{\operatorname{ess\,sup}_{r<y<\infty} \upsilon(y)} \int_0^r g(y)dy \\
&\le \operatorname*{ess\,sup}_{t>0} \frac{u(r)}{r} \frac{\operatorname{ess\,sup}_{0<t\le r} w_2(t)}{\operatorname{ess\,sup}_{r<y<\infty} \upsilon(y)} \operatorname*{ess\,sup}_{t>0} \left(\int_0^t g(s)ds\right) \operatorname*{ess\,sup}_{t<y<\infty} \upsilon(y),
\end{aligned}$$

we get, that the condition (5.22) is sufficient for the inequality (5.23) to be hold.

The necessity part can be proved in similar way, as it was done in the proof of Theorem 5.3. □

From Theorem 5.3 immediately follows next Corollary.

Corollary 5.5. *Let $0 < \theta_1, \theta_2 < \infty$ and u be a continuous weight. Let w_1 and w_2 be weights such that $0 < \int_0^x \int_t^\infty w_1(s)dsdt < \infty$ and $0 < \int_0^x w_2(t)dt < \infty$ for every $x \in (0,\infty)$.*

(i) Let $\theta_1 \le \theta_2$. Then the inequality (5.6) holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\begin{aligned}
\sup_{x>0} \left(\left(\sup_{x\le\tau<\infty} (u(\tau))^{\theta_2}\right) \int_0^x \omega_2(t)dt + \int_x^\infty \left(\sup_{t\le\tau<\infty} (u(\tau))^{\theta_2}\right) \omega_2(t)dt\right)^{\frac{\theta_1}{\theta_2}} \times \\
\times \left(\int_x^\infty \omega_1(\tau)d\tau\right)^{-1} < \infty
\end{aligned} \tag{5.25}$$

(ii) Let $\theta_2 < \theta_1$. Then the inequality (5.6) holds on $\mathfrak{M}^+(0,\infty)$ if and only if

$$\left(\int_0^\infty \left(\int_t^\infty \left(\sup_{s\le\tau<\infty}(u(\tau))^{\theta_2}\right) w_2(s)ds\right)^{\frac{\theta_2}{\theta_1-\theta_2}} \left(\sup_{t\le\tau<\infty}(u(\tau))^{\theta_2}\right) \times \right.$$
$$\left. \times \left(\int_t^\infty \omega_1(s)ds\right)^{\frac{\theta_2}{\theta_2-\theta_1}} w_2(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_2}} < \infty, \tag{5.26}$$

and

$$\left(\int_0^\infty \left(\int_0^t w_2(s)ds\right)^{\frac{\theta_2}{\theta_1-\theta_2}} \left[\sup_{t\le\tau<\infty}\left(\sup_{\tau\le y<\infty}(u(y))^{\theta_1}\right) \times \right.\right.$$
$$\left.\left. \times \left(\int_\tau^\infty \omega_1(y)dy\right)^{-1}\right]^{\frac{\theta_2}{\theta_1-\theta_2}} w_2(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_2}} < \infty, \tag{5.27}$$

6. Sufficient conditions

Theorem 6.1. *Let $0 \le \alpha < n$, $0 < p_2 < \infty$, $0 < \theta_1, \theta_2 \le \infty$, $\omega_1 \in \Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$. Moreover, let $1 < \frac{p_2 n}{n+\alpha p_2} \le p_1$, or $\frac{p_2 n}{n+\alpha p_2} < 1 \le p_1$, or $\frac{p_2 n}{n+\alpha p_2} = 1 < p_1$.*

(i) Let $\theta_1 \le \theta_2$. If

$$\sup_{x>0} \frac{\left\|\left(\min\{\frac{t}{x},1\}\right)^{\frac{n}{p_1}-\alpha} \omega_2(t) t^{\alpha-n\left(\frac{1}{p_1}-\frac{1}{p_2}\right)}\right\|_{L_{\theta_2}(0,\infty)}}{\|\omega_1\|_{L_{\theta_1}(x,\infty)}} < \infty, \tag{6.1}$$

then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$ and from $GM_{p_1\theta_1,\omega_1}$ to $GM_{p_1\theta_2,\omega_2}$. (In the latter case we assume that $\omega_1 \in \Omega_{p_1,\theta_1}$, $\omega_2 \in \Omega_{p_2,\theta_2}$)

(ii) Let $\theta_2 < \theta_1$. If

$$\left(\int_0^\infty \left(\int_t^\infty s^{\theta_2\left(\alpha-\frac{n}{p_1}+\frac{n}{p_2}\right)} w_2^{\theta_2}(s)ds\right)^{\frac{\theta_2}{\theta_1-\theta_2}} t^{\theta_2\left(\alpha-\frac{n}{p_1}+\frac{n}{p_2}\right)} \times \right.$$
$$\left. \times \left(\int_t^\infty \omega_1^{\theta_1}(s)ds\right)^{\frac{\theta_2}{\theta_2-\theta_1}} w_2^{\theta_2}(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_2}} < \infty, \tag{6.2}$$

and

$$\left(\int_0^\infty \left(\int_0^t s^{\theta_2 \frac{n}{p_2}} w_2^{\theta_2}(s)ds\right)^{\frac{\theta_2}{\theta_1-\theta_2}} \left[\sup_{t\le\tau<\infty} \tau^{\theta_1\left(\alpha-\frac{n}{p_1}\right)} \times \right.\right.$$
$$\left.\left. \times \left(\int_\tau^\infty \omega_1^{\theta_1}(y)dy\right)^{-1}\right]^{\frac{\theta_2}{\theta_1-\theta_2}} t^{\theta_2\frac{n}{p_2}} w_2(t)dt\right)^{\frac{\theta_1-\theta_2}{\theta_2}} < \infty, \tag{6.3}$$

then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$ and from $GM_{p_1\theta_1,\omega_1}$ to $GM_{p_1\theta_2,\omega_2}$. (In the latter case we assume that $\omega_1 \in \Omega_{p_1,\theta_1}$, $\omega_2 \in \Omega_{p_2,\theta_2}$)

(iii) Let $0<\theta_1<\infty$, $\theta_2=\infty$. Moreover, assume that

$$0 < \|\omega_1\|^{-1}_{L_{\theta_1}(x,\infty)} < \infty$$

and $0 < \operatorname{ess\,sup}_{0<t<x} \omega_2(t) t^{\frac{n}{p_2}} < \infty$ for every $x \in (0,\infty)$. If

$$\sup_{t>0} \left\| s^{\alpha-\frac{n}{p_1}} \operatorname*{ess\,sup}_{0<r<s} \omega_2(r) r^{\frac{n}{p_2}} \right\|_{L_{\theta_1}(t,\infty)} \|\omega_1\|^{-1}_{L_{\theta_1}(t,\infty)} < \infty, \tag{6.4}$$

then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to LM_{p_1,∞,θ_2} and from $GM_{p_1,\theta_1,\omega_1}$ to GM_{p_1,∞,ω_2}. (In the latter case we assume that $\omega_1 \in \Omega_{p_1,\theta_1}$, $\omega_2 \in \Omega_{p_2,\infty}$)

(iv) Let $\theta_1=\infty$, $0<\theta_2<\infty$. Moreover, assume that $0 < \operatorname{ess\,sup}_{t\le y<\infty} \omega_1(y) < \infty$ for any $t>0$, $\operatorname{ess\,sup}_{t>0} \omega(t) = \infty$. If

$$\left\| \omega_2(t) t^{\frac{n}{p_2}} \operatorname*{ess\,sup}_{t<s<\infty} \frac{s^{\alpha-\frac{n}{p_1}}}{\operatorname{ess\,sup}_{s<y<\infty} \omega_1(y)} \right\|_{L_{\theta_2}(0,\infty)} < \infty, \tag{6.5}$$

then M_α is bounded from LM_{p_1,∞,ω_1} to $LM_{p_2,\theta_2,\omega_2}$ and from GM_{p_1,∞,ω_1} to $GM_{p_1,\theta_2,\omega_2}$. (In the latter case we assume that $\omega_1 \in \Omega_{p_1,\infty}$, $\omega_2 \in \Omega_{p_2,\theta_2}$)

(v) Let $\theta_1=\theta_2=\infty$. Moreover, assume that $0 < \operatorname{ess\,sup}_{t\le y<\infty} \omega_1(y) < \infty$ for any $t>0$, $\operatorname{ess\,sup}_{t>0} \omega(t) = \infty$. If

$$\operatorname*{ess\,sup}_{r>0} r^{\alpha-\frac{n}{p_1}} \frac{\operatorname{ess\,sup}_{0<t\le r} \omega_2(t) t^{\frac{n}{p_2}}}{\operatorname{ess\,sup}_{r<t<\infty} \omega_1(t)} < \infty, \tag{6.6}$$

then M_α is bounded from LM_{p_1,∞,ω_1} to LM_{p_2,∞,ω_2} and from GM_{p_1,∞,ω_1} to GM_{p_1,∞,ω_2}. (In the latter case we assume that $\omega_1 \in \Omega_{p_1,\infty}$, $\omega_2 \in \Omega_{p_2,\infty}$)

Proof. (iii) Let $0<\theta_1<\infty$, $\theta_2=\infty$. Theorem 4.2 states, that if

$$\operatorname*{ess\,sup}_{t>0} \omega_2(t) t^{\frac{n}{p_2}} \sup_{t\le r<\infty} r^{\alpha-\frac{n}{p_1}} \varphi(r) \lesssim \left(\int_0^\infty \omega_1^{\theta_1}(t) \left(\varphi(t)\right)^{\theta_1} dt\right)^{\frac{1}{\theta_1}} \text{ on } \mathbb{A}, \tag{6.7}$$

then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to LM_{p_1,∞,θ_2}. But the inequality is equivalent to the following inequality

$$\operatorname*{ess\,sup}_{t>0} \omega_2^{\theta_1}(t) t^{\theta_1\frac{n}{p_2}} \operatorname*{ess\,sup}_{t\le r<\infty} r^{\theta_1\left(\alpha-\frac{n}{p_1}\right)} g(r) \lesssim \int_0^\infty g(t) \left(\int_t^\infty \omega_1^{\theta_1}(s)ds\right) \tag{6.8}$$

on $\mathfrak{M}^+(0,\infty)$ (see Section 6). By Theorem 5.2, we get, that the inequality (6.8) holds if and only if the condition

$$\sup_{t>0}\left(\int_t^\infty s^{\theta_1\left(\alpha-\frac{n}{p_1}\right)}\left(\operatorname*{ess\,sup}_{0<r<s}\omega_2^{\theta_1}(r)r^{\theta_1\frac{n}{p_2}}\right)ds\right)\left(\int_t^\infty \omega_1^{\theta_1}(\tau)d\tau\right)^{-1}<\infty, \quad (6.9)$$

that is, the condition (6.4) holds.

(iv) Let $\theta_1=\infty$, $0<\theta_2<\infty$. Theorem 4.2 and argumentations at the beginning of Section show, that if

$$\left(\int_0^\infty \omega_2^{\theta_2}(t)t^{\theta_2\frac{n}{p_2}}\left[\sup_{t\le r<\infty} r^{\alpha-\frac{n}{p_1}}\int_0^r g(s)ds\right]^{\theta_2}dt\right)^{\frac{1}{\theta_2}} \lesssim \operatorname*{ess\,sup}_{t>0}\omega_1(r)\int_0^r g(s)ds \quad (6.10)$$

on $\mathfrak{M}^+(0,\infty)$, then M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to LM_{p_1,∞,θ_2}. By Theorem 5.3, the inequality (6.10) holds if and only if the condition

$$\left(\int_0^\infty\left(\operatorname*{ess\,sup}_{t<s<\infty}\frac{s^{\alpha-\frac{n}{p_1}}}{\operatorname{ess\,sup}_{s<y<\infty}\omega_1(y)}\right)^{\theta_2}\omega_2^{\theta_2}(t)t^{\theta_2\frac{n}{p_2}}dt\right)^{\frac{1}{\theta_2}}<\infty,$$

that is, the condition (6.5) holds.

(v) Let $\theta_1=\theta_2=\infty$. Theorem 4.2 and argumentations at the beginning of Section show, that if

$$\operatorname*{ess\,sup}_{t>0}\omega_2(t)t^{\frac{n}{p_2}}\sup_{t\le r<\infty} r^{\alpha-\frac{n}{p_1}}\int_0^r g(s)ds \lesssim \operatorname*{ess\,sup}_{t>0}\omega_1(r)\int_0^r g(s)ds \quad (6.11)$$

on $\mathfrak{M}^+(0,\infty)$, then M_α is bounded from LM_{p_1,∞,ω_1} to LM_{p_1,∞,θ_2}. By Theorem 5.3, the inequality (6.10) holds if and only if the condition

$$\operatorname*{ess\,sup}_{r>0} r^{\alpha-\frac{n}{p_1}}\frac{\operatorname{ess\,sup}_{0<t\le r}\omega_2(t)t^{\frac{n}{p_2}}}{\operatorname{ess\,sup}_{r<t<\infty}\omega_1(t)}<\infty$$

holds. □

Remark 6.2. Let $0\le\alpha<n$, $0<p_2<\infty$, $1<\frac{p_2n}{n+\alpha p_2}\le p_1$, or $\frac{p_2n}{n+\alpha p_2}<1\le p_1$, or $\frac{p_2n}{n+\alpha p_2}=1<p_1$. Moreover, let $1<\theta_1\le\theta_2<\infty$, and β such that

$$\beta+\frac{1}{\theta_2}<0,\ \beta+\frac{n}{p_2}+\frac{1}{\theta_2}>0,\ \beta+\frac{n}{p_2}+\frac{1}{\theta_2}+\alpha-\frac{n}{p_1}<0,$$

then it is easy calculate that the weight functions $\omega_1(t)=t^{\beta+\frac{n}{p_2}+\frac{1}{\theta_2}+\alpha-\frac{n}{p_1}-\frac{1}{\theta_1}}$, $\omega_2(t)=t^\beta$ satisfy the condition (i) of the Theorem 6.1. Thus M_α is bounded from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$. But no power function can satisfy the condition (1.3).

Remark 6.3. For $1 < p_1 < p_2 < \infty$, $\alpha = n(1/p_1 - p_2)$, $\omega_1 \in \Omega_{p_1,\infty}$, $\omega_2 \in \Omega_{p_2,\infty}$ Theorem 6.1 states that if

$$\operatorname*{ess\,sup}_{r>0} r^{\alpha-\frac{n}{p_1}} \frac{\operatorname{ess\,sup}_{0<t\le r} \omega_2(t) t^{\frac{n}{p_2}}}{\operatorname{ess\,sup}_{r<t<\infty} \omega_1(t)} < \infty, \tag{6.12}$$

then M_α is bounded from $\mathcal{M}_{p_1,\omega_1}$ to $\mathcal{M}_{p_2,\omega_2}$. Let $\omega = \omega_1 = \omega_2$ and (1.1) holds. Then (6.13) takes the form

$$\operatorname*{ess\,sup}_{0<t\le r} \omega(t) t^{\frac{n}{p_2}} \lesssim \omega(r) r^{\frac{n}{p_1}-\alpha}. \tag{6.13}$$

Note that the condition (6.13) follows from the condition (1.2). Indeed, if the condition (1.2) holds, then for any $t \le r$ we get

$$\int_r^{2r} \frac{ds}{w^{p_1}(s) s^{n+1-\alpha p_1}} \lesssim \frac{1}{w^{p_1}(t) t^{np_1/p_2}}.$$

By (1.1), we have

$$\frac{1}{w^{p_1}(r) r^{n-\alpha p_1}} \lesssim \frac{1}{w^{p_1}(t) t^{np_1/p_2}}.$$

Hence

$$\omega(t) t^{\frac{n}{p_2}} \lesssim \omega(r) r^{\frac{n}{p_1}-\alpha}.$$

Therefore

$$\operatorname*{ess\,sup}_{0<t\le r} \omega(t) t^{\frac{n}{p_2}} \lesssim \omega(r) r^{\frac{n}{p_1}-\alpha}.$$

The weight function $\omega(t) = t^{\alpha-\frac{n}{p_1}}$ satisfies our condition and it is easy to see that the condition (1.2) does not hold for ω, since

$$\int_t^{\infty} \frac{ds}{\omega(s)^{p_1} s^{n+1-\alpha p_1}} = \infty.$$

Theorem 6.4. *Let $1 < p_1 \le p_2 < \infty$, $0 < \theta_1 \le \theta_2 \le \infty$, $\alpha = n(1/p_1 - 1/p_2)$, $\omega_1 \in \Omega_{\theta_1}$ and $\omega_2 \in \Omega_{\theta_2}$, then the condition*

$$\left\| \omega_2(r) \left(\frac{r}{t+r} \right)^{n/p_2} \right\|_{L_{\theta_2}(0,\infty)} \le c \|\omega_1\|_{L_{\theta_1}(t,\infty)} \tag{6.14}$$

for all $t > 0$, where $c > 0$ is independent of t, is necessary and sufficient for the boundedness of M_α from $LM_{p_1,\theta_1,\omega_1}$ to $LM_{p_2,\theta_2,\omega_2}$.

Proof. The necessity foolows from Theorem 1.5 item (1). The sufficiency follows from Theorem 6.1 item (i). ☐

Remark 6.5. Note that, in the case $\theta_1 \le p_1$ Theorem 6.4 was proved in [8].

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