



On a singular Cauchy problem for functional differential equations with non-increasing non-linearities*

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Abstract

We obtain general conditions sufficient for the solvability of a singular Cauchy problem for functional differential equations with non-increasing non-linearities.

1 Problem setting and introduction

The aim of this note is to establish some general conditions sufficient for the existence of a solution of a singular Cauchy problem for a class of non-linear functional differential equations. More precisely, we consider the equation

$$(1) \quad u'(t) = (gu)(t), \quad t \in (a, b],$$

where $-\infty < a < b < \infty$ and $g : C((a, b], \mathbb{R}) \rightarrow L_{1;\text{loc}}((a, b], \mathbb{R})$ is a certain (generally speaking, non-linear) mapping which is assumed to be non-increasing with respect to the natural pointwise ordering (see Definition 4). Solutions of equation (1) are sought for in the class of locally absolutely continuous functions and, in particular, may be unbounded in a neighbourhood of the point a (the precise notion of a solution is given by Definition 2 below).

Definition 1. One says that a function $u : (a, b] \rightarrow \mathbb{R}$ is *locally absolutely continuous* if its restriction $u|_{[a+\varepsilon, b]}$ to the interval $[a + \varepsilon, b]$ is absolutely continuous for any $\varepsilon \in (0, b - a)$.

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We are interested in conditions under which equation (1) has solutions satisfying the additional condition

$$(2) \quad \lim_{t \rightarrow a^+} h(t)u(t) = 0,$$

where $h : (a, b] \rightarrow \mathbb{R}$ is a given continuous function such that

$$(3) \quad \lim_{t \rightarrow a^+} h(t) = 0$$

and

$$(4) \quad h(t) > 0 \quad \text{for all } t \in (a, b].$$

Definition 2. By a *solution* of equation (1), we mean a locally absolutely continuous function $u : (a, b] \rightarrow \mathbb{R}$ which possesses the property $hu' \in L_1((a, b], \mathbb{R})$ and satisfies equality (1) almost everywhere on the interval $(a, b]$.

It is important to point out that the derivative of a locally absolutely continuous function satisfying equality (1) may have a non-integrable singularity in a neighbourhood of the point a . For example, the function

$$(5) \quad u(t) = \frac{\lambda}{t^4}, \quad t \in (0, 1],$$

for any real λ satisfies the equality

$$(6) \quad u'(t) = -\frac{4}{t^3}u(\sqrt{t}), \quad t \in (0, 1],$$

where the coefficient function $(0, 1] \ni t \mapsto -4t^{-3}$ is non-integrable. The presence of a certain weight function is thus expected when dealing with the problems of the kind specified. As such a weight function, our definition of a solution of equation (1) uses the same function h with properties (3) and (4) that appears in condition (3). For example, function (5) turns out to be a solution of equation (6) if $h(t) := t^\gamma$, $t \in (0, 1]$, with $\gamma > 4$.

For the sake of simplicity, we assume throughout the paper that the above mentioned function h has the following properties:

$$(7) \quad \text{The function } h : (a, b] \rightarrow (0, +\infty) \text{ is absolutely continuous and non-decreasing, and possesses property (3).}$$

These technical assumptions are not very restrictive because only the qualitative behaviour of h in a neighbourhood of the point a has influence on the formulation of the problems below and, thus, h can be redefined in an arbitrary suitable manner in a neighbourhood of b . The function

$$h(t) = (t - a)^\gamma, \quad t \in (a, b],$$

where $\gamma \in (0, +\infty)$, is a typical example of a function satisfying conditions (7).

Definition 3. By a *solution of problem* (1), (2), a solution of equation (1) possessing property (2) is meant.

We emphasize that the definition of a solution of the functional differential equation (1) is constructed here so that it involves the same function h that determines the additional condition (2), i. e., Definition 2 in each case is adjusted to condition (2). For example, one can check directly that function (5) for arbitrary $\lambda \in \mathbb{R}$ and $\varepsilon \in (0, +\infty)$ is a solution of the problem

$$(8) \quad \lim_{t \in (0,1]} t^{4+\varepsilon} u(t) = 0$$

for equation (6) in the sense of the definition above.

It should be noted that the property of the uniqueness of a solution is not typical for the problems of the class indicated. For example, the homogeneous problem (6), (8), as we have already seen above, has infinitely many solutions if ε is positive.

Problem (1), (2) with h satisfying (3) is usually referred to as the *singular Cauchy problem* (see, e. g., [1]). It reduces, in a natural way, to the classical regular Cauchy problem if h is equal identically to a non-zero constant. Regular and singular Cauchy problems for various classes of functional differential equations are treated, in particular, in [1–9]. A problem on regular solutions possessing properties of type (2) is studied in [10].

2 Notation

The following notation is used throughout the paper.

1. $\mathbb{R} := (-\infty, \infty)$, $\mathbb{N} := \{1, 2, 3, \dots\}$.
2. If $-\infty < a < b < \infty$ and $A \subseteq [a, b]$ is a measurable set, then $L_1(A, \mathbb{R})$ is the Banach space of all the Lebesgue integrable functions $u : A \rightarrow \mathbb{R}$ with the standard norm

$$L_1(A, \mathbb{R}) \ni u \longmapsto \int_A |u(t)| dt.$$
3. $L_{1;\text{loc}}((a, b], \mathbb{R})$ is the set of functions $u : (a, b] \rightarrow \mathbb{R}$ such that $u|_{[a+\varepsilon, b]} \in L_1([a+\varepsilon, b], \mathbb{R})$ for any $\varepsilon \in (0, b-a)$.
4. $C((a, b], \mathbb{R})$ is the linear manifold of all the continuous functions $u : (a, b] \rightarrow \mathbb{R}$.
5. $AC_{\text{loc}}((a, b], \mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a, b] \rightarrow \mathbb{R}$.

6. $AC_{\text{loc};h}((a,b],\mathbb{R})$ is the set of all the locally absolutely continuous functions $u : (a,b] \rightarrow \mathbb{R}$ such that $hu' \in L_1([a,b],\mathbb{R})$ and

$$(9) \quad \sup_{t \in (a,b]} h(t) |u(t)| < +\infty.$$

3 General theorem

Let us consider the functional differential equation

$$(10) \quad u'(t) = (gu)(t), \quad t \in [a,b],$$

where $g : C((a,b],\mathbb{R}) \rightarrow L_{1;\text{loc}}((a,b],\mathbb{R})$ is a certain mapping. Our main object here is the problem on finding solutions of equation (10) possessing property (2).

Definition 4. An operator $g : C((a,b],\mathbb{R}) \rightarrow L_{1;\text{loc}}((a,b],\mathbb{R})$ is said to be *non-increasing* if the condition

$$(gu_1)(t) \leq (gu_0)(t), \quad t \in (a,b],$$

is satisfied for arbitrary pairs of functions $\{u_0, u_1\} \subset C((a,b],\mathbb{R})$ such that

$$u_1(t) \geq u_0(t), \quad t \in (a,b].$$

The standard Definition 4 here restricts the class of non-linearities that the right-hand side of equation (10) may contain under conditions of the theorems formulated below.

Definition 5. We say that a locally absolutely continuous function $u : (a,b] \rightarrow \mathbb{R}$ belongs to the set $AC_{\text{loc};h}((a,b],\mathbb{R})$ if $hu' \in L_1([a,b],\mathbb{R})$ and relation (9) is satisfied.

The theorem given below provides general conditions guaranteeing the existence of the solutions satisfying the condition

$$(11) \quad \lim_{t \rightarrow a+} h(t)u(t) = 0,$$

that is, the singular Cauchy problem (10), (11) is considered.

Theorem 1. Let the mapping $g : C((a,b],\mathbb{R}) \rightarrow L_{1;\text{loc}}((a,b],\mathbb{R})$ in equation (10) be non-increasing and, moreover, the conditions

$$(12) \quad hg \left(\frac{\lambda}{h} \right) \in L_1((a,b],\mathbb{R})$$

and

$$(13) \quad \lim_{t \rightarrow a^+} h(t) \int_t^b g\left(\frac{\lambda}{h}\right)(s) ds = 0$$

be satisfied for any $\lambda \in \mathbb{R}$.

Furthermore, let there exist certain functions ψ_0 and ψ_1 from $AC_{\text{loc};h}((a, b], \mathbb{R})$ such that

$$(14) \quad \lim_{t \rightarrow a^+} h(t)\psi_i(t) = 0, \quad i = 0, 1,$$

and the inequalities

$$(15) \quad (-1)^i(\psi_1^{(i)}(t) - \psi_0^{(i)}(t)) \geq 0, \quad t \in (a, b], i = 0, 1,$$

and

$$(16) \quad (-1)^i(\psi_i'(t) - (g\psi_i)(t)) \geq 0, \quad t \in (a, b], i = 0, 1,$$

are satisfied.

Then equation (10) has a solution $u : (a, b] \rightarrow \mathbb{R}$ possessing property (11) and, moreover, the estimates

$$(17) \quad \psi_0(t) \leq u(t) \leq \psi_1(t), \quad t \in (a, b],$$

and

$$(18) \quad \psi_0'(t) \geq u'(t) \geq \psi_1'(t), \quad t \in (a, b],$$

are true.

Note that, due to (7), the function λ/h , in particular, is continuous at every point from $(a, b]$ and, therefore, the left-hand side terms of relations (12) and (13) are well defined.

Remark 1. For g homogeneous, it is sufficient to suppose that

$$(19) \quad hg\left(\frac{1}{h}\right) \in L_1((a, b], \mathbb{R})$$

because (19) guarantees the fulfilment of (12) for any λ in that case. In the general case, however, condition (19) may not imply the property indicated. Indeed, let us take $a = 0$, $b = 1$, fix a certain $\varepsilon \in (0, 1)$, and, for any non-negative u from $C((0, 1], \mathbb{R})$, put

$$(20) \quad (gu)(t) := -\frac{1}{t^{\varepsilon} h(t)}, \quad t \in (0, 1],$$

where

$$m_u := \begin{cases} 1 - \varepsilon & \text{if } \mu_u \leq 1, \\ 1 & \text{if } \mu_u > 1, \end{cases}$$

and $\mu_u := \sup_{t \in (a, b]} h(t)|u(t)|$. For all other functions u from $C((0, 1], \mathbb{R})$, we put $gu := 0$. Then, as can be checked, the operator $g : C((a, b], \mathbb{R}) \rightarrow L_{1; \text{loc}}((a, b], \mathbb{R})$ given by equality (20) is non-increasing in the sense of Definition 4.

Since $m_{\frac{1}{h}} = 1 - \varepsilon$, relation (12) is satisfied for $\lambda \leq 1$ (and, in particular, (19) is true). However, relation (12) does not hold if $\lambda > 1$.

Remark 2. The solution of problem (10), (11), the existence of which is stated in Theorem 1, can be found approximately by using a convergent monotone two-sided iteration procedure.

4 Corollaries for linear equations

Let us formulate corollaries for the case where the functional differential equation (10) has the form

$$(21) \quad u'(t) = (lu)(t) + q(t), \quad t \in [a, b],$$

where $l : C((a, b], \mathbb{R}) \rightarrow L_{1; \text{loc}}((a, b], \mathbb{R})$ is a certain linear mapping and q is a given function from $L_{1; \text{loc}}((a, b], \mathbb{R})$.

Definition 6. An operator $l : C((a, b], \mathbb{R}) \rightarrow L_{1; \text{loc}}((a, b], \mathbb{R})$ is said to be *negative* if the inequality $(lu)(t) \leq 0$ is satisfied for a. e. $t \in (a, b]$ whenever $u \in C((a, b], \mathbb{R})$ is such that

$$\inf_{t \in (a, b]} u(t) \geq 0.$$

Definition 7. Let u and v be some functions from $AC_{\text{loc}}((a, b], \mathbb{R})$. We say that a function $q : (a, b] \rightarrow \mathbb{R}$ belongs to the set $M_l(u, v)$ if it almost everywhere on $(a, b]$ satisfies the estimate

$$v'(t) - (lv)(t) \leq q(t) \leq u'(t) - (lu)(t), \quad t \in (a, b].$$

The following statement on problem (21), (11) is true.

Corollary 1. *Let the mapping l in (21) be negative, the condition*

$$(22) \quad hl \left(\frac{1}{h} \right) \in L_1((a, b], \mathbb{R})$$

be satisfied, and, moreover,

$$(23) \quad \lim_{t \rightarrow a^+} h(t) \int_t^b l\left(\frac{1}{h}\right)(s) ds = 0.$$

Furthermore, let the function $q \in L_{1;\text{loc}}((a, b], \mathbb{R})$ be such that

$$(24) \quad hq \in L_1((a, b], \mathbb{R}), \quad \lim_{t \rightarrow a^+} h(t) \int_t^b q(s) ds = 0,$$

and $q \in M_l(\psi_0, \psi_1)$, where ψ_0 and ψ_1 are certain functions from $AC_{\text{loc};h}((a, b], \mathbb{R})$ satisfying inequalities (14), (15), (16).

Then equation (21) has a solution $u : (a, b] \rightarrow \mathbb{R}$ possessing property (11) and satisfying estimates (17) and (18).

Corollary 2. Let the mapping l in (21) be negative and possess properties (22) and (23). Furthermore, assume that the function $q \in L_{1;\text{loc}}((a, b], \mathbb{R})$ satisfies conditions (24) and, for some $i \in \{0, 1\}$, the inequality

$$(25) \quad (-1)^i(\psi'(t) - (l\psi)(t)) \leq (-1)^i q(t) \leq 0, \quad t \in (a, b],$$

holds with a certain function $\psi \in AC_{\text{loc};h}((a, b], \mathbb{R})$ such that

$$\lim_{t \rightarrow a^+} h(t)\psi(t) = 0$$

and

$$(-1)^i \psi(t) \geq 0, \quad t \in (a, b].$$

Then equation (21) has a solution $u : (a, b] \rightarrow \mathbb{R}$ possessing property (11). Moreover, this solution and its derivative admit the estimates

$$(26) \quad 0 \leq (-1)^{i+k} u^{(k)}(t) \leq (-1)^{i+k} \psi^{(k)}(t), \quad t \in (a, b], \quad i, k = 0, 1.$$

Note that the lack of extra assumptions on the functions ψ_0 and ψ_1 is explained by the unpleasant fact that the set $M_l(\psi_0, \psi_1)$ may turn out to be empty. This never happens, for example, if the mapping $-l$ is negative, because, as can be verified, then

$$M_l(\psi_0, \psi_1) \neq \emptyset$$

for arbitrary functions ψ_0 and ψ_1 satisfying estimates (15). In our case, however, the operator l is negative and, therefore, the property mentioned essentially depends on the choice of ψ_0 and ψ_1 .

5 Auxiliary statements and proofs

In the sequel we need an abstract theorem on operators in partially ordered normed spaces [11, Theorem 4.1]. In order to state it, we first formulate definitions. We use [11] as the main reference (see also [12, 13]).

5.1 General notions

Let X be a normed space over \mathbb{R} and P be a cone [11] in X , i. e., a non-empty closed subset of X possessing the properties $P \cap (-P) = \{0\}$ and $\alpha_1 P + \alpha_2 P \subseteq P$ for all $\{\alpha_1, \alpha_2\} \subset [0, +\infty)$. A cone P generates a natural partial ordering of X . As usual, we shall write $u \leq_P v$ and $v \geq_P u$ if and only if $v - u \in P$.

Definition 8 ([11]). One says that an operator $T : X \rightarrow X$ is *non-decreasing* (with respect to P) if $Tu \geq_P Tv$ for any u and v from X such that $u \geq_P v$.

Theorem 2 ([11]). *Let $T : X \rightarrow X$ be a non-decreasing and continuous operator. Let there exist some elements $\{\psi_0, \psi_1\} \subset X$ such that $\psi_0 \leq_P \psi_1$ and, moreover,*

$$(27) \quad T\psi_0 \geq_P \psi_0,$$

$$(28) \quad T\psi_1 \leq_P \psi_1.$$

Moreover, assume that the cone P is regular. Then the operator T has at least one fixed point $u \in X$ such that

$$(29) \quad \psi_0 \leq_P u \leq_P \psi_1.$$

We assume that the function h involved in the non-local condition (11) possesses properties (7).

Lemma 1. *The set $AC_{\text{loc};h}((a, b], \mathbb{R})$ is a Banach space with respect to the norm*

$$(30) \quad AC_{\text{loc};h}((a, b], \mathbb{R}) \ni u \longmapsto \|u\| := \int_a^b h(s) |u'(s)| ds + \sup_{\xi \in (a, b]} h(\xi) |u(\xi)|.$$

Recall that the set $AC_{\text{loc};h}((a, b], \mathbb{R})$ is introduced by Definition 5.

Lemma 2. *The set*

$$(31) \quad K := \left\{ u \in AC_{\text{loc};h}((a, b], \mathbb{R}) : \inf_{t \in (a, b]} u(t) \geq 0 \text{ and } \text{ess sup}_{t \in (a, b]} u'(t) \leq 0 \right\}$$

is a regular cone in the space $AC_{\text{loc};h}((a, b], \mathbb{R})$.

Lemmata 1 and 2 are proved in [14].

Lemma 3. *If the mapping $g : C((a, b], \mathbb{R}) \rightarrow L_{1;\text{loc}}((a, b], \mathbb{R})$ is non-increasing, then, for any u from $AC_{\text{loc};h}((a, b], \mathbb{R})$, there exists a certain $\mu_u \in [0, +\infty)$ such that*

$$(32) \quad g\left(\frac{\mu_u}{h}\right)(t) \leq (gu)(t) \leq g\left(\frac{-\mu_u}{h}\right)(t)$$

for a. e. $t \in (a, b]$.

Proof. Let $u \in AC_{\text{loc};h}((a, b], \mathbb{R})$ be an arbitrary function. Then

$$\mu_u := \sup_{t \in (a, b]} h(t)|u(t)| < +\infty,$$

and, hence, the estimate

$$|u(t)| \leq \frac{\mu_u}{h(t)}$$

holds for any $t \in (a, b]$. Therefore,

$$(33) \quad -\frac{\mu_u}{h(t)} \leq -|u(t)| \leq u(t) \leq |u(t)| \leq \frac{\mu_u}{h(t)}, \quad t \in (a, b].$$

By assumption, the mapping g is non-decreasing in the sense of Definition 4. Therefore, relation (33) yields

$$(34) \quad g\left(-\frac{\mu_u}{h}\right)(t) \geq g(-|u|)(t) \geq (gu)(t) \geq (g|u|)(t) \geq g\left(\frac{\mu_u}{h}\right)(t)$$

for a. e. $t \in (a, b]$, whence estimate (32) follows immediately. \square

Lemma 4. *The set*

$$(35) \quad \Pi_0(h) := \left\{ u \in AC_{\text{loc};h}((a, b], \mathbb{R}) \mid \lim_{t \rightarrow a^+} h(t)u(t) = 0 \right\}$$

is a closed linear subspace in $AC_{\text{loc};h}((a, b], \mathbb{R})$.

Proof. Let $\{u_m \mid m \geq 1\} \subset \Pi_0(h)$ be an arbitrary sequence convergent, in terms of norm (30), to a certain element $u \in AC_{\text{loc};h}((a, b], \mathbb{R})$. This means, in particular, that

$$(36) \quad \lim_{m \rightarrow +\infty} \sup_{t \in (a, b]} h(t)|u_m(t) - u(t)| = 0.$$

Let us fix an arbitrary positive ε . In view of (36), there exists a certain $m_\varepsilon \in \mathbb{N}$ such that

$$(37) \quad h(t) |u_m(t) - u(t)| \leq \frac{\varepsilon}{2}$$

for any $t \in (a, b]$ and $m \geq m_\varepsilon$. By assumption, $u_{m_\varepsilon} \in \Pi_0(h)$ and, therefore, there exists some $\delta_\varepsilon > 0$ such that

$$(38) \quad |u_{m_\varepsilon}(t)|h(t) < \frac{\varepsilon}{2}$$

whenever $t \in (a, b]$ is such that $|t - a| < \delta_\varepsilon$. Combining (37) and (38), we find that

$$(39) \quad |u(t)|h(t) \leq |u_{m_\varepsilon}(t)|h(t) + h(t) |u_m(t) - u(t)| < \varepsilon$$

for $|t - a| < \delta_\varepsilon$. The arbitrariness of ε now implies that $u \in \Pi_0(h)$, which, in view of the arbitrariness of the sequence $\{u_m \mid m \geq 1\}$, proves our lemma. \square

Let us put

$$(40) \quad (Tu)(t) := u(b) - \int_t^b (gu)(s) ds, \quad t \in (a, b],$$

for any u from $AC_{\text{loc};h}((a, b], \mathbb{R})$.

Lemma 5. *Assume that $g : C((a, b], \mathbb{R}) \rightarrow L_{1;\text{loc}}((a, b], \mathbb{R})$ is non-increasing, the function h possesses properties (7), and condition (12) holds. Then*

1. *T is a well-defined mapping from $AC_{\text{loc};h}((a, b], \mathbb{R})$ to $AC_{\text{loc};h}((a, b], \mathbb{R})$;*
2. *The mapping $T : AC_{\text{loc};h}((a, b], \mathbb{R}) \rightarrow AC_{\text{loc};h}((a, b], \mathbb{R})$ is continuous with respect to norm (30);*
3. *$T : AC_{\text{loc};h}((a, b], \mathbb{R}) \rightarrow AC_{\text{loc};h}((a, b], \mathbb{R})$ is a mapping non-decreasing with respect to cone (31);*
4. *If condition (13) holds, then the inclusion*

$$(41) \quad T(\Pi_0(h)) \subseteq \Pi_0(h)$$

is satisfied.

Proof. Assertions 1–3 are established similarly to the argument of [14], and we thus prove Assertion 4 only.

Let $u \in \Pi_0(h)$ be arbitrary. Then, according to (35), u possesses property (11) and, by virtue of Lemma 3, one can specify a certain constant $\mu_u \in (0, +\infty)$ such that estimate (32) is true. Then

$$(42) \quad h(t) \int_t^b g\left(\frac{\mu_u}{h}\right)(s) ds \leq h(t) \int_t^b (gu)(s) ds \leq h(t) \int_t^b g\left(\frac{-\mu_u}{h}\right)(s) ds$$

for a. e. t from $(a, b]$. Estimate (42) and assumption (13) yield

$$(43) \quad \lim_{t \rightarrow a^+} h(t) \int_t^b (gu)(s) ds = 0.$$

Taking relation (43) and definition (40) of the operator T into account, we obtain

$$\lim_{t \rightarrow a^+} h(t)(Tu)(t) = 0,$$

whence it follows that the function Tu belongs to the set $\Pi_0(h)$. The arbitrariness of u from $\Pi_0(h)$ thus leads us to inclusion (41). \square

Lemma 6. *A function $u : (a, b] \rightarrow \mathbb{R}$ from $AC_{\text{loc};h}((a, b], \mathbb{R})$ is a solution of problem (10), (11) if and only if it is a fixed point of mapping (40) lying in the set $\Pi_0(h)$.*

These last lemma is an immediate consequence of relations (35), (40) and Definition 5 of the set $AC_{\text{loc};h}((a, b], \mathbb{R})$.

5.2 Proof of Theorem 1

We are going to use Theorem 2. By Lemma 4,

$$(44) \quad X := \Pi_0(h)$$

is a subspace of $AC_{\text{loc};h}((a, b], \mathbb{R})$ and, hence, can be regarded as an independent Banach space with the same norm (30). Lemmata 1 and 2 guarantee that the set

$$K_0 := K \cap \Pi_0(h),$$

where K is given by (31), is a regular cone in the Banach space $\Pi_0(h)$. By Lemma 5, the operator $T : AC_{\text{loc};h}((a, b], \mathbb{R}) \rightarrow AC_{\text{loc};h}((a, b], \mathbb{R})$ defined by formula (40) has property (41) and, thus, we can pass to its restriction to $\Pi_0(h)$. The same Lemma 5 then ensures that $T : \Pi_0(h) \rightarrow \Pi_0(h)$ is continuous and non-decreasing with respect to the cone K_0 .

The functions ψ_0 and ψ_1 appearing in the formulation of the theorem satisfy relations (27) and (28) with respect to cone (31), i. e.,

$$(45) \quad (-1)^i((T\psi_0)^{(i)}(t) - \psi_0^{(i)}(t)) \geq 0, \quad t \in (a, b], i = 0, 1,$$

and

$$(46) \quad (-1)^i((T\psi_1)^{(i)}(t) - \psi_1^{(i)}(t)) \leq 0, \quad t \in (a, b], i = 0, 1,$$

and, hence, Theorem 2 can be applied. Indeed, assumption (14) ensures that $\{\psi_0, \psi_1\} \subset \Pi_0(h)$. It follows immediately from (40) that relations (45) and (46) with $i = 1$ are satisfied in view of (16). Relations (45) and (46) with $i = 0$ are obtained from the corresponding estimates with $i = 1$ by integrating them from $t \in (a, b]$ to b .

Applying Theorem 2 with X given by equality (44) and $P := K_0$ and using Lemma 6, we obtain the existence of at least one solution u of problem (10), (11). Finally, noticing that estimates (17) and (18) follow from (29), we complete the proof of Theorem 1.

5.3 Proofs of Corollaries 1 and 2

In order to obtain Corollary 1, it is sufficient to notice that, under assumptions (22) and (23), the mapping $g : C((a, b], \mathbb{R}) \rightarrow L_{1; \text{loc}}((a, b], \mathbb{R})$ given by the formula

$$(gu)(t) := (lu)(t) + q(t), \quad t \in (a, b],$$

satisfies conditions (12) and (13) for any λ , by virtue of the linearity of l . Therefore, Theorem 1 can be applied. The assertion of Corollary 2 is an immediate consequence of Corollary 1 in the case where

$$\psi_k := ((1 - k)i + k(1 - i))\psi$$

for $k = 0, 1$.

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