Preprint, Institute of Mathematics, AS CR, Prague. 2008-2-6

On compactness of the velocity field in the incompressible limit of the full Navier-Stokes-Fourier system on large domains

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1 Introduction

Many problems in continuum fluid mechanics are considered on *unbounded* spatial domains, in particular on the whole space R^3. Although it seems intuitively clear that any observable physical space is necessarily bounded, the concept of unbounded domain offers a useful approximation when the influence of the boundary on the motion is negligible. For instance, the presence of *acoustic waves* is usually neglected in meteorological models, where the underlying physical domain is large and the speed of sound dominates the characteristic speed of the fluid (see Klein [9]). Under these circumstances, a relevant mathematical description can be obtained through a suitable scaling of the primitive equations typically represented by the complete Navier-Stokes-Fourier system.

We examine the situation when the characteristic velocity of the fluid $\mathbf{u}_{\rm char} = \varepsilon$, the characteristic time $t_{\rm char} = 1/\varepsilon$ as well as the characteristic viscosity $\mu_{\rm char} = \varepsilon$ are given in terms of a small parameter $\varepsilon > 0$. The motion of the fluid is governed by the standard Navier-Stokes-Fourier system in the dimensionless form:

$$\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p = {\rm div}_x \mathbb{S}, \tag{1.2}$$

$$\partial_t(\varrho s) + {\rm div}_x(\varrho s \mathbf{u}) + {\rm div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma, \tag{1.3}$$

where $\varrho = \varrho(t,x)$ denotes the density, $\mathbf{u} = \mathbf{u}(t,x)$ is the velocity field, and $\vartheta = \vartheta(t,x)$ is the absolute temperature. The pressure $p = p(\varrho, \vartheta)$ and the

*The work of E.F. was supported by Grant IAA100190606 of GA ASCR in the framework of the general research programme of the Academy of Sciences of the Czech Republic, Institutional Research Plan AV0Z10190503

specific entropy $s = s(\varrho, \vartheta)$ are given functions of the state variables ϱ, ϑ. The symbol $\mathbb{S}$ denotes the viscous stress tensor assumed to satisfy the standard Newton's rheological law

$$\mathbb{S} = \mu\Big(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\Big), \tag{1.4}$$

while $\mathbf{q}$ denotes the heat flux obeying Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.5}$$

Finally, the entropy production σ satisfies

$$\sigma = \frac{1}{\vartheta}\Big(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\Big). \tag{1.6}$$

The singular coefficient in the pressure term in (1.3) corresponds to the Mach number proportional to ε (Klein et al. [10]).

System (1.1 - 1.6) is considered on a spatial domain Ω_ε large enough in order to eliminate the effect of the boundary on propagation of the acoustic waves. Seeing that the speed of sound in (1.1 - 1.6) is proportional to $1/\varepsilon$ we shall assume that the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ enjoys the following property:

Property (L)

For any $x \in R^3$, there is $\varepsilon_0 = \varepsilon_0(x)$ such that $x \in \Omega_\varepsilon$ for all $0 < \varepsilon < \varepsilon_0$. Moreover, there exists a function h, $\lim_{z\to\infty} h(z)/z = \infty$ such that

$$\mathrm{dist}[x, \partial\Omega_\varepsilon] > h(1/\varepsilon) \ \textit{for all}\ 0 < \varepsilon < \varepsilon_0. \tag{1.7}$$

In addition to (1.7), we suppose that the initial distribution of the density and the temperature are close to a spatially homogeneous state. More specifically,

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \tag{1.8}$$

$$\vartheta(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon}, \tag{1.9}$$

where $\overline{\varrho}$, $\overline{\vartheta}$ are positive constants and

$$\int_{\Omega_\varepsilon} \Big(|\varrho^{(1)}_{0,\varepsilon}|^2 + |\vartheta^{(1)}_{0,\varepsilon}|^2 \Big)\ \mathrm{d}x \le c \tag{1.10}$$

uniformly for $\varepsilon \to 0$.

We consider a family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of (weak) solutions to problem (1.1 - 1.6) on a compact time interval $(0,T)$ emanating from the initial state satisfying (1.8 - 1.10). The main goal of the present paper is to show that

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T; L^2(B;R^3)) \text{ for any bounded ball } B \subset R^3, \tag{1.11}$$

at least for a suitable subsequence $\varepsilon \to 0$, where the limit velocity field complies with the standard incompressibility constraint

$$\mathrm{div}_x \mathbf{u} = 0. \tag{1.12}$$

As already pointed out, the result should be independent of the behavior of $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ "far away" from the set B, in particular we do not impose any specific boundary conditions. On the other hand, certain restrictions have to be made in order to prevent the energy to be "pumped" into the system at infinity. Specifically, the following hypotheses are required:

- The total mass of the fluid contained in Ω_ε is a constant of motion.
- The system dissipates energy, specifically, the total energy of the fluid contained in Ω_ε is non-increasing in time.
- The system produces entropy, in particular, the total entropy of the system is non-decreasing in time.

Apart from the general stipulations stated above, we assume that the quantities $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ solve (1.1 - 1.5) in the sense of distributions while (1.6) is replaced by an inequality

$$\sigma \geq \frac{1}{\vartheta}\Big(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\Big) \tag{1.13}$$

in the spirit of the existence theory developed in [4].

Our technique is based on uniform estimates of the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ resulting from the dissipation inequality deduced in a similar way as in [6] (see Section 2). The time evolution of the acoustic waves is governed by a wave equation (acoustic equation) derived in Section 3. At this stage, the finite speed of propagation of the waves is used in order to reduce the problem to a bounded spatial domain (Section 4). Finally, we use the dispersive estimates for the acoustic equation in order to obtain the desired conclusion stated in (1.11) (see Section 5). The paper is concluded by a rigorous formulation of the main result in Section 6.

A similar problem for the Navier-Stokes system in the isentropic regime posed on the whole space R^3 was addressed by Desjardins and Grenier [3]. Related results for the isentropic system were also obtained by Lions and Masmoudi [11] (see also the survey paper by Masmoudi [12]), singular problems on unbounded domains are investigated in the monograph by Chemin et al. [2]. In contrast to this reference material, the acoustic equation for the complete system contains the contribution of "thermal" waves including the entropy production rate σ being merely a positive measure. In order to handle this additional difficulty, a regularization and "time lifting" technique is used in combination the local decay estimates for the wave equation obtained recently by Burq [1] and Metcalfe [13] (see Section 5).

2 Uniform estimates

2.1 Estimates based on the hypothesis of thermodynamics stability

In accordance with the principle of thermodynamics stability, we shall assume that

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial \vartheta} > 0, \tag{2.1}$$

where $e = e(\varrho,\vartheta)$ is the specific internal energy interrelated to p and s through Gibbs' equation

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + p(\varrho,\vartheta) D\Big(\frac{1}{\varrho}\Big). \tag{2.2}$$

The former condition in (2.1) asserts that the compressibility of the fluid is always positive while the latter says that the specific heat at constant volume is positive (see Gallavotti [8]).

In accordance with the general principles delineated in the previous section, we shall assume that the total mass is a conserved quantity, specifically,

$$\int_{\Omega_\varepsilon} \Big(\varrho_\varepsilon(t,\cdot) - \overline{\varrho}\Big) \ \mathrm{d}x = 0 \text{ for a.a. } t \in (0,T), \tag{2.3}$$

in particular, we take

$$\int_{\Omega_\varepsilon} \varrho^{(1)}_{0,\varepsilon} \ \mathrm{d}x = 0 \tag{2.4}$$

in (1.8).

Similarly, we assume that the total energy is a non-decreasing function of time, meaning

$$\int_{\Omega_\varepsilon} \Big[\frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2(t) + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)(t) - \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2(0) - \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)(0)\Big] \ \mathrm{d}x \le 0 \tag{2.5}$$

while the entropy is being produced:

$$\int_{\Omega_\varepsilon} \Big[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)(t) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)(0)\Big] \ \mathrm{d}x = \sigma_\varepsilon[[0,t] \times \overline{\Omega}_\varepsilon] \tag{2.6}$$

for a.a. $t \in (0,T)$, where the entropy production rate σ_ε is a non-negative measure satisfying

$$\sigma_\varepsilon \ge \frac{1}{\vartheta_\varepsilon}\Big(\varepsilon^2 \frac{\mu}{2}\Big|\nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I}\Big|^2 + \frac{\kappa|\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon}\Big). \tag{2.7}$$

Combining (2.3) with (2.5), (2.6) we get the so-called dissipation inequality

$$\int_{\Omega_\varepsilon} \Big[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon) - \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho_\varepsilon - \overline{\varrho}) - H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big)\Big](t) \ \mathrm{d}x \tag{2.8}$$

$$+\frac{\overline{\vartheta}}{\varepsilon^2}\sigma_\varepsilon[[0,t]\times\overline{\Omega}_\varepsilon]$$

$$\leq \int_{\Omega_\varepsilon}\left[\frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2+\frac{1}{\varepsilon^2}\Big(H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon)-\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})(\varrho_\varepsilon-\overline{\varrho})-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big)\right](0)\ \mathrm{d}x$$

for a.a. $t\in[0,T]$, where we have introduced

$$H_{\overline{\vartheta}}(\varrho,\vartheta)=\varrho e(\varrho,\vartheta)-\overline{\vartheta}\varrho s(\varrho,\vartheta). \tag{2.9}$$

Since, by virtue of Gibbs' relation (2.2),

$$\frac{\partial^2 H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})}{\partial\varrho^2}=\frac{1}{\varrho}\frac{\partial p(\varrho,\overline{\vartheta})}{\partial\varrho},\ \ \frac{\partial H_{\overline{\vartheta}}(\varrho,\vartheta)}{\partial\vartheta}=\frac{\varrho}{\vartheta}(\vartheta-\overline{\vartheta})\frac{\partial e(\varrho,\vartheta)}{\partial\vartheta},$$

the thermodynamics stability hypothesis (2.1) implies that

$$\varrho\mapsto H_{\overline{\vartheta}}(\varrho,\overline{\vartheta})\text{ is strictly convex on }(0,\infty),$$

and

$$\vartheta\mapsto H_{\overline{\vartheta}}(\varrho,\vartheta)\text{ is decreasing for }\vartheta<\overline{\vartheta}\text{ and increasing for }\vartheta>\overline{\vartheta}.$$

Introducing the essential and residual set of values as follows

$$\mathcal{M}_{\rm ess}=\{(\varrho,\vartheta)\mid\overline{\varrho}/2<\varrho<2\overline{\varrho},\ \overline{\vartheta}/2<\vartheta<2\overline{\vartheta}\},\ \mathcal{M}_{\rm res}=[0,\infty)^2\setminus\mathcal{M}_{\rm ess}$$

we report the following structural properties of the function $H_{\overline{\vartheta}}$ (see Lemma 2.1 in [5]):

$$c_1\Big(|\varrho-\overline{\varrho}|^2+|\vartheta-\overline{\vartheta}|^2\Big) \tag{2.10}$$
$$\leq H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})$$
$$\leq c_2\Big(|\varrho-\overline{\varrho}|^2+|\vartheta-\overline{\vartheta}|^2\Big)\text{ for all }(\varrho,\vartheta)\in\mathcal{M}_{\rm ess},$$

$$H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \tag{2.11}$$
$$\geq\inf_{(r,\Theta)\in\partial\mathcal{M}_{\rm ess}}\Big\{H_{\overline{\vartheta}}(r,\Theta)-(r-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\Big\}>0\text{ for all }(\varrho,\vartheta)\in\mathcal{M}_{\rm res},$$

and

$$H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\frac{\partial H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta}) \tag{2.12}$$
$$\geq c\Big(\varrho e(\varrho,\vartheta)+\varrho|s(\varrho,\vartheta)|\Big)\text{ for all }(\varrho,\vartheta)\in\mathcal{M}_{\rm res}.$$

It follows from (2.10) that the integral on the right-hand side of the dissipation inequality (2.8) is bounded uniformly with respect to $\varepsilon\to 0$ as soon as the initial data satisfy (1.8), (1.9), and, in addition,

$$\mathbf{u}_\varepsilon(0,\cdot)=\mathbf{u}_{0,\varepsilon}, \tag{2.13}$$

where

$$\|\varrho^{(1)}_{0,\varepsilon}\|_{L^1\cap L^\infty(\Omega_\varepsilon)} + \|\vartheta^{(1)}_{0,\varepsilon}\|_{L^1\cap L^\infty(\Omega_\varepsilon)} + \|\mathbf{u}_{0,\varepsilon}\|_{L^2\cap L^\infty(\Omega_\varepsilon;R^3)} \leq c, \tag{2.14}$$

with c independent of ε.

Thus relation (2.8), together with the structural properties of the function $H_{\overline{\vartheta}}$ listed in (2.10 - 2.12), can be used to deduce uniform bounds on the family $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$. To this end, it is convenient to associate to a family $\{h_\varepsilon\}_{\varepsilon>0}$ its *essential* and *residual* part as follows:

$$[h_\varepsilon]_{\rm ess} = h_\varepsilon 1_{\{(t,x)\ |\ (\varrho_\varepsilon,\vartheta_\varepsilon)(t,x)\in\mathcal{M}_{\rm ess}\}},\ [h_\varepsilon]_{\rm res} = h_\varepsilon 1_{\{(t,x)\ |\ (\varrho_\varepsilon,\vartheta_\varepsilon)(t,x)\in\mathcal{M}_{\rm res}\}}.$$

Thus the dissipation inequality (2.8) gives rise to the following uniform estimates:

$$\text{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\|_{L^2(\Omega_\varepsilon;R^3)} \leq c, \tag{2.15}$$

$$\text{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\rm ess}\right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{2.16}$$

$$\text{ess}\sup_{t\in(0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}\right]_{\rm ess}\right\|_{L^2(\Omega_\varepsilon)} \leq c, \tag{2.17}$$

$$\text{ess}\sup_{t\in(0,T)} \|\ [\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{2.18}$$

$$\text{ess}\sup_{t\in(0,T)} \|\ [\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c, \tag{2.19}$$

and

$$\|\sigma_\varepsilon\|_{\mathcal{M}^+([0,T]\times\overline{\Omega}_\varepsilon} \leq \varepsilon^2 c. \tag{2.20}$$

Moreover, the measure of the "residual" set is small, specifically,

$$\text{ess}\sup_{t\in(0,T)} \|[1]_{\rm res}\|_{L^1(\Omega_\varepsilon)} \leq \varepsilon^2 c. \tag{2.21}$$

Finally, combining (2.7), (2.20) we conclude that

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\mu}{\vartheta}\Big|\nabla_x\mathbf{u}_\varepsilon + \nabla_x^t\mathbf{u}_\varepsilon - \frac{2}{3}{\rm div}_x\mathbf{u}_\varepsilon\mathbb{I}\Big|^2\ {\rm d}x\ {\rm d}t \leq c, \tag{2.22}$$

and

$$\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa}{\vartheta^2}|\nabla_x\vartheta_\varepsilon|^2\ {\rm d}x\ {\rm d}t \leq \varepsilon^2 c. \tag{2.23}$$

Note that all bounds established in (2.15 - 2.23) have been obtained assuming only the thermodynamics stability hypothesis (2.1), the uniform bound on the data (2.14), and the general physical principles (2.2), (2.3), (2.5), and (2.6). In particular, these bounds are independent of the specific form of the constitutive relations.

2.2 Estimates based on constitutive relations

Unlike the uniform bounds established in the previous part, the following estimates are derived under certain restrictions imposed on the material properties of the fluid. The purpose of these estimates is to control the residual part of the quantities appearing in the acoustic equation introduced in Section 3 below. Note that all restrictions here are technical and by no means optimal.

Motivated by the existence theory developed in [4], we consider the state equation for the pressure in the form

$$p(\varrho,\vartheta) = \underbrace{p_M(\varrho,\vartheta)}_{\text{molecular pressure}} + \underbrace{p_R(\vartheta)}_{\text{radiation pressure}}, \ p_M = \vartheta^{\frac{5}{2}} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ p_R = \frac{a}{3}\vartheta^4,\ a > 0, \tag{2.24}$$

while the integral energy reads

$$e(\varrho,\vartheta) = e_M(\varrho,\vartheta) + e_R(\varrho,\vartheta),\ e_M = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ e_R = a\frac{\vartheta^4}{\varrho}, \tag{2.25}$$

and, in accordance with Gibbs' relation (2.2),

$$s(\varrho,\vartheta) = s_M(\varrho,\vartheta) + s_R(\varrho,\vartheta),\ s_M(\varrho,\vartheta) = S\Big(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\Big),\ s_R = \frac{4}{3}a\frac{\vartheta^3}{\varrho}, \tag{2.26}$$

where

$$S'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z^2} \text{ for all } Z > 0. \tag{2.27}$$

The thermodynamics stability hypothesis (2.1) reformulated in terms of the structural properties of P gives rise to

$$P \in C^1[0,\infty) \cap C^2(0,\infty),\ P(0) = 0,\ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.28}$$

$$0 < \frac{\frac{5}{3}P(Z) - ZP'(Z)}{Z} \leq \sup_{z>0} \frac{\frac{5}{3}P(z) - zP'(z)}{z} < \infty. \tag{2.29}$$

Furthermore, it follows from (2.29) that $P(Z)/Z^{5/3}$ is a decreasing function of Z, and we assume that

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{\frac{5}{3}}} = p_\infty > 0. \tag{2.30}$$

The transport coefficients μ and κ are continuously differentiable functions of the temperature ϑ satisfying the growth restrictions

$$\left\{ \begin{array}{r} 0 < \underline{\mu}(1+\vartheta) \leq \mu(\vartheta) \leq \overline{\mu}(1+\vartheta), \\ \\ 0 < \underline{\kappa}(1+\vartheta^3) \leq \kappa(\vartheta) \leq \overline{\kappa}(1+\vartheta^3) \text{ for all } \vartheta \geq 0, \end{array} \right\} \tag{2.31}$$

where $\underline{\mu}$, $\overline{\mu}$, $\underline{\kappa}$, and $\overline{\kappa}$ are positive constants.

By virtue of (2.31), the uniform estimate (2.22) yields

$$\int_0^T \| \nabla_x \mathbf{u}_\varepsilon + \nabla_x^t \mathbf{u}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{u}_\varepsilon \mathbb{I} \|^2_{L^2(\Omega_\varepsilon; R^{3\times 3})} \ \mathrm{d}t \leq c, \tag{2.32}$$

with c independent of $\varepsilon \to 0$.

In order to get more information, we need the following version of Korn's inequality proved in [7, Proposition 6.1].

Proposition 2.1 *Let $\Omega \subset R^3$ be a bounded Lipschitz domain. Let $r \geq 0$ be a function such that*

$$0 < m = \int_\Omega r \ \mathrm{d}x, \ \int_\Omega r^\gamma \ \mathrm{d}x < K \text{ for a certain } \gamma > 6/5.$$

Then

$$\|\mathbf{v}\|^2_{W^{1,2}(\Omega;R^3)} \leq c(m,k,\Omega) \Big(\| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3} \mathrm{div}_x \mathbf{v} \mathbb{I} \|^2_{L^2(\Omega;R^{3\times 3})} + \int_\Omega r |\mathbf{v}|^2 \ \mathrm{d}x \Big)$$

for any $\mathbf{v} \in W^{1,2}(\Omega; R^3)$.

Taking $r = [\varrho_\varepsilon]_{\mathrm{ess}}$, $\mathbf{v} = \mathbf{u}_\varepsilon$ we can cover the domains Ω_ε by a finite number of cubes and apply Proposition 2.1 in order to conclude that

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega_\varepsilon;R^3)} \ \mathrm{d}t \leq c \text{ uniformly for } \varepsilon \to 0, \tag{2.33}$$

where we have used estimates (2.15), (2.32), together with "smallness" of the residual set stated in (2.21).

Similarly, we can use estimates (2.17), (2.23) in order to obtain

$$\int_0^T \|\vartheta_\varepsilon - \overline{\vartheta}\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t + \int_0^T \| \log(\vartheta_\varepsilon) - \log(\overline{\vartheta})\|^2_{W^{1,2}(\Omega_\varepsilon)} \ \mathrm{d}t \leq \varepsilon^2 c. \tag{2.34}$$

Finally, a combination of (2.18), (2.30) yields

$$\mathrm{ess} \sup_{t\in(0,T)} \int_{\Omega_\varepsilon} [\varrho_\varepsilon]^{5/3}_{\mathrm{res}} \ \mathrm{d}x \leq \varepsilon^2 c. \tag{2.35}$$

3 Acoustic equation

Acoustic equation is a wave equation governing the time evolution of the acoustic waves. It can be viewed as a linearization of system (1.1 - 1.3) around the static state $\{\overline{\varrho}, 0, \overline{\vartheta}\}$. If $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ satisfy (1.1 - 1.3) in the sense of distributions, we get

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big)\partial_t\varphi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi\Big] \ \mathrm{d}x \ \mathrm{d}t = 0 \tag{3.1}$$

for any test function $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon)$;

$$\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\partial_t\varphi \ \mathrm{d}x \ \mathrm{d}t \tag{3.2}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \frac{\kappa\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - < \sigma_\varepsilon, \varphi >$$

for any test function $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon)$; and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi + \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\mathrm{div}_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{3.3}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \varepsilon\Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\Big) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon; R^3)$.

Thus, after a simple manipulation, we obtain

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon r_\varepsilon \partial_t\varphi + A(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\nabla_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{3.4}$$

$$= B\int_0^T \int_{\Omega_\varepsilon} \varepsilon\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+B\int_0^T \int_{\Omega_\varepsilon} \frac{\kappa\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} \cdot \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t - B < \sigma_\varepsilon, \varphi >$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon)$, and

$$\int_0^T \int_{\Omega_\varepsilon} \Big[\varepsilon(\varrho_\varepsilon\mathbf{u}_\varepsilon)\cdot\partial_t\varphi + r_\varepsilon\mathrm{div}_x\varphi\Big] \ \mathrm{d}x \ \mathrm{d}t \tag{3.5}$$

$$= \int_0^T \int_{\Omega_\varepsilon} \Big[r_\varepsilon - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big)\Big]\mathrm{div}_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega_\varepsilon} \varepsilon\Big(\mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon\Big) : \nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$, where we have set

$$r_\varepsilon = A\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big) + B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big), \tag{3.6}$$

with A, B determined through

$$B\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta},\ A + B\overline{\varrho}\frac{\partial s(\overline{\varrho},\overline{\vartheta})}{\partial\varrho} = \frac{\partial p(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}. \tag{3.7}$$

As a direct consequence of Gibbs' relation (2.2), we have

$$\frac{\partial s}{\partial\varrho} = -\frac{1}{\varrho^2}\frac{\partial p}{\partial\vartheta},$$

in particular, $A > 0$ as soon as e, p comply with the thermodynamics stability hypotheses (2.1).

Finally, introducing the "time lifting" Σ_ε of the measure σ_ε as

$$\Sigma_\varepsilon \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega}_\varepsilon)),\ <\Sigma_\varepsilon,\psi> = <\sigma_\varepsilon,\Psi>,\ \Psi(t,x) = \int_0^t \psi(s,x)\,\mathrm{d}s \tag{3.8}$$

we can rewrite system (3.4), (3.5) in a concise form

$$\int_0^T\int_{\Omega_\varepsilon}\Big[\varepsilon Z_\varepsilon\partial_t\varphi + A\mathbf{V}_\varepsilon\cdot\nabla_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon}\varepsilon\mathbf{F}^1_\varepsilon\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t \tag{3.9}$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon)$,

$$\int_0^T\int_{\Omega_\varepsilon}\Big[\varepsilon\mathbf{V}_\varepsilon\cdot\partial_t\varphi + Z_\varepsilon\mathrm{div}_x\varphi\Big]\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\Omega_\varepsilon}\Big(\varepsilon\mathbb{F}^2_\varepsilon : \nabla_x\varphi + \varepsilon F^3_\varepsilon\mathrm{div}_x\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.10}$$

for all $\varphi \in \mathcal{D}((0,T)\times\Omega_\varepsilon;R^3)$,

where we have set

$$Z_\varepsilon = A\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\Big) + B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}\Big) + \frac{B}{\varepsilon}\Sigma_\varepsilon,\ \mathbf{V}_\varepsilon = \varrho_\varepsilon\mathbf{u}_\varepsilon, \tag{3.11}$$

$$\mathbf{F}^1_\varepsilon = B\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon + B\frac{\kappa\nabla_x\vartheta_\varepsilon}{\varepsilon\vartheta_\varepsilon}, \tag{3.12}$$

$$\mathbb{F}^2_\varepsilon = \mathbb{S}_\varepsilon - \varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon, \tag{3.13}$$

and

$$F^3_\varepsilon = \frac{B}{\varepsilon^2}\Sigma_\varepsilon + A\Big(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2}\Big) + B\varrho_\varepsilon\Big(\frac{s(\varrho_\varepsilon,\vartheta_\varepsilon) - s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big) - \Big(\frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\Big). \tag{3.14}$$

4 Regularization and extension to R^3

4.1 Uniform estimates

To begin, we establish uniform estimates for all terms appearing on the right-hand side of acoustic equation (3.9), (3.10).

Writing

$$\varrho_\varepsilon\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)$$

$$=[\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)+[\varrho_\varepsilon]_{\rm res}\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big),$$

we can use the uniform bounds (2.16), (2.17) in order to obtain

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|[\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\Big\|_{L^2(\Omega_\varepsilon)}\leq c. \tag{4.1}$$

Furthermore, estimate (4.1) combined with (2.33) yields

$$\int_0^T\Big\|[\varrho_\varepsilon]_{\rm ess}\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\mathbf{u}_\varepsilon\Big\|^2_{L^1(\Omega_\varepsilon;R^3)}\leq c, \tag{4.2}$$

where both (4.1) and (4.2) are uniform for $\varepsilon\to 0$.

On the other hand, in accordance with (2.18), (2.21), and (2.35),

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|[\varrho_\varepsilon]_{\rm res}\Big(\frac{s(\overline{\varrho},\overline{\vartheta})-s(\varrho_\varepsilon,\vartheta_\varepsilon)}{\varepsilon}\Big)\Big\|_{L^1(\Omega_\varepsilon)}\leq \varepsilon c, \tag{4.3}$$

Next it follows from the structural hypotheses (2.27 - 2.29) that

$$|\varrho s_M(\varrho,\vartheta)|\leq c(1+\varrho|\log(\varrho)|+\varrho|\log(\vartheta)|)\ \text{for all positive}\ \varrho,\ \vartheta.$$

In particular, we deduce from (2.21), (2.35) that

$$\operatorname*{ess\ sup}_{t\in(0,T)}\Big\|\frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon}\Big\|_{L^{6/5}(\Omega_\varepsilon)}\leq c, \tag{4.4}$$

which, together with (2.33) and the Sobolev embedding relation $W^{1,2}(R^3)\hookrightarrow L^2\cap L^6(R^3)$, gives rise to the uniform bound

$$\int_0^T\Big\|\frac{[\varrho_\varepsilon]_{\rm res}|\log(\varrho_\varepsilon)|}{\varepsilon}\mathbf{u}_\varepsilon\Big\|^2_{L^1(\Omega_\varepsilon)}\,{\rm d}t\leq c. \tag{4.5}$$

Similarly, we can write

$$\Big|\frac{[\varrho_\varepsilon]_{\rm res}|\log(\vartheta_\varepsilon)|\mathbf{u}_\varepsilon}{\varepsilon}\Big|$$

$$\leq\sqrt{[\varrho_\varepsilon]_{\rm res}}\frac{|\log(\vartheta_\varepsilon)-\log(\overline{\vartheta})|}{\varepsilon}\sqrt{[\varrho_\varepsilon]_{\rm res}}\ |\mathbf{u}_\varepsilon|+\frac{[\varrho_\varepsilon]_{\rm res}}{\varepsilon}|\mathbf{u}_\varepsilon|\ |\log(\overline{\vartheta})|$$

and use the uniform estimates (2.15), (2.21), (2.34), and (2.35) in order to conclude that

$$\int_0^T \left\| \frac{[\varrho_\varepsilon]_{\rm res} |\log(\vartheta_\varepsilon)|}{\varepsilon} \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon)} \ {\rm d}t \leq c. \tag{4.6}$$

Since exactly the same estimates can be deduced also for the radiation component $\varrho_\varepsilon s_R(\varrho_\varepsilon, \vartheta_\varepsilon) \approx \vartheta_\varepsilon^3$, we infer that

$$\int_0^T \left\| [\varrho_\varepsilon]_{\rm res} \Big(\frac{s(\overline{\varrho}, \overline{\vartheta}) - s(\varrho_\varepsilon, \vartheta_\varepsilon)}{\varepsilon} \Big) \mathbf{u}_\varepsilon \right\|^2_{L^1(\Omega_\varepsilon; R^3)} \leq c, \tag{4.7}$$

Using estimates (2.22), (2.23), we get

$$\int_0^T \Big(\| \, [\, \mathbb{S}_\varepsilon]_{\rm ess} \|^2_{L^2(\Omega_\varepsilon; R^{3\times 3})} + \left\| \, [\kappa]_{\rm ess} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^2(\Omega_\varepsilon; R^3)} \Big) \ {\rm d}t \leq c. \tag{4.8}$$

Finally, the contribution of the radiation energy in (2.18) gives rise to a bound

$$\mathrm{ess} \sup_{t \in (0,T)} \int_{\Omega_\varepsilon} [\vartheta_\varepsilon]^4_{\rm res} \ {\rm d}x \leq \varepsilon^2 c \tag{4.9}$$

which can be used in combination with (2.22), (2.23) in order to infer that

$$\int_0^T \Big(\| \, [\, \mathbb{S}_\varepsilon]_{\rm res} \|^2_{L^2(\Omega_\varepsilon; R^{3\times 3})} + \left\| \, [\kappa]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|^2_{L^1(\Omega_\varepsilon; R^3)} \Big) \ {\rm d}t \leq c. \tag{4.10}$$

As a matter of fact, it can be shown that the presence of the radiation terms is not necessary, however we would have to content ourselves with a weaker bound

$$\int_0^T \Big(\| \, [\, \mathbb{S}_\varepsilon]_{\rm res} \|_{L^2(\Omega_\varepsilon; R^{3\times 3})} + \left\| \, [\kappa]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon \vartheta_\varepsilon} \right\|_{L^1(\Omega_\varepsilon; R^3)} \Big) \ {\rm d}t \leq c.$$

Having established all the preliminary estimates we are ready to deduce uniform bounds on all quantities appearing in the acoustic equation (3.9), (3.10).

To begin, it follows from (2.16), (2.20), (2.21), (4.1), and (4.3) that

$$Z_\varepsilon = Z^1_\varepsilon + Z^2_\varepsilon + Z^3_\varepsilon, \tag{4.11}$$

with

$$\left\{ \begin{array}{c} \{Z^1_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^2(\Omega_\varepsilon)), \\ \\ \{Z^2_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; L^1(\Omega_\varepsilon)), \\ \\ \{Z^3_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T; \mathcal{M}^+(\Omega_\varepsilon)). \end{array} \right\} \tag{4.12}$$

Similarly, using (2.15), (2.21) together with (2.35), we obtain

$$\mathbf{V}_\varepsilon = \mathbf{V}^1_\varepsilon + \mathbf{V}^2_\varepsilon, \tag{4.13}$$

where

$$\left\{ \begin{array}{c} \{\mathbf{V}_\varepsilon^1\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^2(\Omega_\varepsilon;R^3)), \\ \\ \{\mathbf{V}_\varepsilon^2\}_{\varepsilon>0} \text{ is bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon;R^3)). \end{array} \right\} \tag{4.14}$$

Furthermore, in accordance with (4.2), (4.7 - 4.10),

$$\mathbf{F}_\varepsilon^1 = \mathbf{F}_\varepsilon^{1,1} + \mathbf{F}_\varepsilon^{1,2}, \tag{4.15}$$

with

$$\left\{ \begin{array}{c} \{\mathbf{F}_\varepsilon^{1,1}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^2(\Omega_\varepsilon;R^3)), \\ \\ \{\mathbf{F}_\varepsilon^{1,2}\}_{\varepsilon>0} \text{ bounded in } L^2(0,T;L^1(\Omega_\varepsilon;R^3)). \end{array} \right\} \tag{4.16}$$

By the same token, estimate (4.10) yields

$$\mathbb{F}_\varepsilon^2 = \mathbb{F}_\varepsilon^{2,1} + \mathbb{F}_\varepsilon^{2,2}, \tag{4.17}$$

where

$$\left\{ \begin{array}{c} \{\mathbb{F}_\varepsilon^{2,1}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^2(\Omega_\varepsilon;R^{3\times 3})), \\ \\ \{\mathbb{F}_\varepsilon^{2,2}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;L^1(\Omega_\varepsilon;R^{3\times 3})). \end{array} \right\} \tag{4.18}$$

Finally, by virtue of our choice of the parameters A, B in (3.7), we conclude, by help of (2.16 - 2.21), that

$$F_\varepsilon^3 = F_\varepsilon^{3,1} + F_\varepsilon^{3,2}, \tag{4.19}$$

with

$$\left\{ \begin{array}{c} \{F_\varepsilon^{3,1}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;L^1(\Omega_\varepsilon)), \\ \\ \{F_\varepsilon^{3,2}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;\mathcal{M}^+(\Omega_\varepsilon)). \end{array} \right\} \tag{4.20}$$

4.2 Regularization

Our goal is to show strong convergence of the velocity fields claimed in (1.11). By virtue of the uniform estimates (2.33), we already have

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T;W^{1,2}(B;R^3)) \text{ for any bounded domain } B \subset R^3 \tag{4.21}$$

passing to a suitable subsequence (independent of B) as the case may be.

Let

$$[\mathbf{v}]_\delta(t,x) = \int_{R^3} \eta_\delta(x-y)\mathbf{v}(t,y) \ \mathrm{d}y$$

denote the smoothing operator associated to a family $\{\eta_\delta\}_{\delta>0}$ of smooth regularizing kernels $\mathrm{supp}[\eta_\delta] \subset \{|y| < \delta\}$. We claim that the desired relation (1.11) follows as soon as we are able to show

$$\begin{array}{c} [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta \to \overline{\varrho}[\mathbf{u}]_\delta \text{ in } L^2(0,T;L^2(B;R^3)) \ \text{ as } \varepsilon \to 0 \\ \\ \text{for any bounded domain } B \subset R^3, \text{ and any fixed } \delta > 0. \end{array} \tag{4.22}$$

Indeed relation (4.22) implies

$$[\overline{\varrho}\mathbf{u}_\varepsilon]_\delta = \varepsilon[\frac{\overline{\varrho}-\varrho_\varepsilon}{\varepsilon}\mathbf{u}_\varepsilon]_\delta + [\varrho_\varepsilon\mathbf{u}_\varepsilon]_\delta \to \overline{\varrho}[\mathbf{u}]_\delta,$$

meaning

$$[\mathbf{u}_\varepsilon]_\delta \to [\mathbf{u}]_\delta \text{ in } L^2(0,T;L^2(B;R^3)) \text{ for any bounded } B\subset R^3;$$

whence the desired conclusion follows from compactness of the Sobolev embedding $W^{1,2}(B;R^3)\hookrightarrow\hookrightarrow L^2(B;R^3))$.

In order to see (4.22), we regularize the acoustic equation, that means, we take $\varphi_x(t,y)=\psi(t)\eta_\delta(x-y)$, $\psi\in\mathcal{D}(0,T)$, as test function in (3.9), (3.10). The resulting equation reads

$$\left\{\begin{array}{c} \varepsilon\partial_t[Z_\varepsilon]_\delta + A\mathrm{div}_x[\mathbf{V}_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\Big(\mathbf{G}^1_{\varepsilon,\delta}+\mathbf{G}^2_{\varepsilon,\delta}\Big) \\ \\ \varepsilon\partial_t[\mathbf{V}_\varepsilon]_\delta + \nabla_x[Z_\varepsilon]_\delta = \varepsilon\mathrm{div}_x\Big(\mathbb{H}^1_{\varepsilon,\delta}+\mathbb{H}^2_{\varepsilon,\delta}\Big), \end{array}\right\} \text{a.a. in } (0,T)\times\Omega_\varepsilon, \tag{4.23}$$

where, by virtue of the uniform estimates (4.16), (4.18), and (4.20)

$$\begin{aligned} &\{\mathbf{G}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,1}(\Omega_\varepsilon;R^3)),\\ &\{\mathbf{G}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\Omega_\varepsilon;R^3)),\\ &\{\mathbb{H}^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,1}(\Omega_\varepsilon;R^{3\times3})),\\ &\{\mathbb{H}^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ is bounded in } L^2(0,T;W^{k,2}(\Omega_\varepsilon;R^{3\times3})). \end{aligned} \tag{4.24}$$

Moreover,

$$[Z_\varepsilon]_\delta = Z^1_{\varepsilon,\delta}+Z^2_{\varepsilon,\delta},\ [\mathbf{V}_\varepsilon]_\delta = [\varrho_\varepsilon\mathbf{u}_\varepsilon]_\delta,$$

with

$$\begin{aligned} &\{Z^1_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;W^{k,1}(\Omega_\varepsilon))\\ &\{Z^2_{\varepsilon,\delta}\}_{\varepsilon>0} \text{ bounded in } L^\infty(0,T;W^{k,2}(\Omega_\varepsilon)) \end{aligned} \tag{4.25}$$

for any $k=0,1,\ldots$, where all bounds depend on k and δ but they are uniform for $\varepsilon\to0$.

4.3 Extension to the whole space R^3

The acoustic equation (4.23) admits a finite speed of propagation proportional to ε^{-1}. Indeed multiplying the left-hand side of (4.23) on $[Z_\varepsilon, A\mathbf{V}_\varepsilon]$ we get the expression

$$\partial_t(|Z_\varepsilon|^2 + A|\mathbf{V}_\varepsilon|^2) + \frac{2A}{\varepsilon}\mathrm{div}_x(Z_\varepsilon \mathbf{V}_\varepsilon);$$

whence the desired result follows by integration over an appropriate space-time cone.

From now on we fix a bounded ball $B \subset R^3$. Since our goal is to show strong convergence of $\{[\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta\}_{\varepsilon>0}$ on B as claimed in (4.22), the family $\{\Omega_\varepsilon\}_{\varepsilon>0}$ enjoys Property L formulated in Section 1, meaning the boundaries $\partial\Omega_\varepsilon$ are "far away" from B, and equation (4.23) admits the finite speed of propagation, we can extend all quantities in (4.23) onto the whole space R^3 in such a way that

- the acoustic equation (4.23) is satisfied a.a. in the set $(0,T) \times R^3$;
- the uniform bounds established in (4.24 - 4.25) hold with Ω_ε replaced by R^3;
- $$\{[\mathbf{V}_\varepsilon]_\delta(0,\cdot)\}_{\varepsilon>0} \text{ is bounded in } W^{k,1}(R^3;R^3) \text{ for any } k = 0,1,\dots \tag{4.26}$$
  (see (2.14));
- $$\int_{R^3} [Z_\varepsilon]_\delta(0,x)\ \mathrm{d}\mathbf{x} = 0; \tag{4.27}$$
- all quantities appearing in (4.23) have compact support in R^3, the radius of which depends on ε.

5 Dispersion estimates and time-decay of the acoustic waves

The problem being reduced to the situation described in Section 4, the proof of the desired relation (4.22) follows from the dispersive estimates for the acoustic equation (4.23). Note that, integrating the first equation in (4.23) and using (4.27), we get

$$\int_{R^3} [Z_\varepsilon]_\delta(t,x)\ \mathrm{d}\mathbf{x} = 0 \text{ for all } t \in [0,T). \tag{5.1}$$

At this stage, we introduce the Helmholtz decomposition on R^3,

$$\mathbf{v} = \mathbf{H}[\mathbf{v}] + \mathbf{H}^\perp[\mathbf{v}],$$

where $\mathbf{H}^\perp \approx \nabla_x \Delta^{-1} {\rm div}_x$ can be determined in terms of the Fourier symbols as

$$\mathbf{H}^\perp[\mathbf{v}] = \mathcal{F}^{-1}_{\xi \to x}\Big[\frac{\xi \otimes \xi}{|\xi|^2} \mathcal{F}_{x \to \xi}[\mathbf{v}] \Big],$$

where $\mathcal{F}$ denotes the Fourier transform in the $x-$variable.

Applying $\mathbf{H}$ to the second equation in (4.23) we deduce easily that

$$\mathbf{H}[\ [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta] \to \mathbf{H}[\ [\overline{\varrho}\mathbf{u}]_\delta] = \overline{\varrho}[\mathbf{u}]_\delta \ \mbox{in} \ L^2(0,T;L^2(B;R^3)) \tag{5.2}$$

for any fixed $\delta > 0$. Consequently, in order to complete the proof of (4.22), it is enough to handle the gradient component

$$\mathbf{H}^\perp[\ [\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta] = \mathbf{H}^\perp[\ [\mathbf{V}_\varepsilon]_\delta\ ] = \nabla_x \Delta^{-1} {\rm div}_x [\mathbf{V}_\varepsilon]_\delta.$$

Introducing

$$\Psi_\varepsilon \equiv \Delta^{-1} {\rm div}_x [\mathbf{V}_\varepsilon]_\delta, \ z_\varepsilon = - [Z_\varepsilon]_\delta$$

we arrive at a "classical" wave equation in the form

$$\varepsilon \partial_t z_\varepsilon - A \Delta \Psi_\varepsilon = \varepsilon (g^1_\varepsilon + g^2_\varepsilon) \tag{5.3}$$

$$\varepsilon \partial_t \Psi_\varepsilon - z_\varepsilon = \varepsilon (h^1_\varepsilon + h^2_\varepsilon), \tag{5.4}$$

supplemented with the initial conditions

$$\Psi_\varepsilon(0, \cdot) = \Psi_{0,\varepsilon}, \ z_\varepsilon(0,\cdot) = z_{0,\varepsilon}, \tag{5.5}$$

where, in accordance with (4.24 - 4.26),

$$\{ \Psi_{0,\varepsilon} \}_{\varepsilon > 0}, \ \{ z_{0,\varepsilon} \}_{\varepsilon > 0} \ \mbox{are bounded in} \ W^{k,2}(R^3), \tag{5.6}$$

$$\int_{R^3} g^i_\varepsilon \ {\rm d}\mathbf{x} = \int_{R^3} h^i_\varepsilon \ {\rm d}\mathbf{x} = 0, \ i = 1,2, \tag{5.7}$$

$$\{ g^1_\varepsilon \}_{\varepsilon > 0}, \ \{ h^1_\varepsilon \}_{\varepsilon > 0} \ \mbox{are bounded in} \ L^2(0,T;W^{k,1}(R^3)), \tag{5.8}$$

and

$$\{ g^2_\varepsilon \}_{\varepsilon > 0}, \ \{ h^2_\varepsilon \}_{\varepsilon > 0} \ \mbox{are bounded in} \ L^2(0,T;W^{k,2}(R^3)) \tag{5.9}$$

for any $k = 0,1, \dots$

Since $[\mathbf{V}_\varepsilon]_\delta$ coincides with $[\varrho_\varepsilon \mathbf{u}_\varepsilon]_\delta$ on the set $(0,T) \times B$, and since we have already shown (5.2), relation (4.22) follows as soon as we are able to verify that

$$\Psi_\varepsilon \to 0 \ \mbox{in} \ L^2(0,T;W^{1,2}(B)). \tag{5.10}$$

Any solution of (5.3 - 5.5) can be expressed by means of Duhamel's formula

$$\left[\begin{array}{c} z_\varepsilon \\ \Psi_\varepsilon \end{array}\right](t) = S\Big(\frac{t}{\varepsilon}\Big)\left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array}\right] + \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} (g^1_\varepsilon + g^2_\varepsilon)(s) \\ (h^1_\varepsilon + h^2_\varepsilon)(s) \end{array}\right] \mathrm{d}s, \tag{5.11}$$

where

$$S(t)\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right] = \left[\begin{array}{c} z(t) \\ \Psi(t) \end{array}\right] \tag{5.12}$$

is the unique solutions of the homogeneous problem

$$\partial_t z - A\Delta\Psi = 0,\ \partial_t\Psi - z = 0,\ z(0) = z_0,\ \Psi(0) = \Psi_0. \tag{5.13}$$

As we need only a local bound, the component

$$\int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} g^1_\varepsilon(s) \\ h^1_\varepsilon(s) \end{array}\right] \mathrm{d}s$$

is easily controlled by means of the classical $L^1 - L^\infty$ dispersive estimates for the wave equation (see Strauss [14, Chapter 1]).

Let us introduce the homogeneous Sobolev space $D^{1,2}(R^3)$ defined as a completion of functions from $\mathcal{D}(R^3)$ with respect to the gradient norm $\|\nabla_x \cdot\|_{L^2(R^3)}$. In order to handle the L^2-terms, we use the following result by Burq [1, Theorem 3] (see also Metcalfe [13, Lemma 4.1]).

Proposition 5.1 *For any function $\chi \in \mathcal{D}(R^3)$, there is a constant $c = c(\chi)$ such that*

$$\int_{-\infty}^{\infty} \left\| \chi\ S(t)\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}(R^3)} \mathrm{d}t \le c \left\|\left[\begin{array}{c} z_0 \\ \Psi_0 \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}(R^3)}. \tag{5.14}$$

Rescaling (5.14) in t we get

$$\int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big)\left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array}\right]\right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t \le \varepsilon c \left\|\left[\begin{array}{c} z_{0,\varepsilon} \\ \Psi_{0,\varepsilon} \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}(R^3)}. \tag{5.15}$$

Finally, by the same token

$$\int_0^T \left\| \int_0^t S\Big(\frac{t-s}{\varepsilon}\Big)\left[\begin{array}{c} g^2_\varepsilon(s) \\ h^2_\varepsilon(s) \end{array}\right] \mathrm{d}s \right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t \tag{5.16}$$

$$\le c(T) \int_0^T \int_{-\infty}^{\infty} \left\| S\Big(\frac{t}{\varepsilon}\Big) S\Big(\frac{-s}{\varepsilon}\Big)\left[\begin{array}{c} g^2_\varepsilon(s) \\ h^2_\varepsilon(s) \end{array}\right]\right\|^2_{L^2(B)\times D^{1,2}(B)} \mathrm{d}t\ \mathrm{d}s$$

$$\le \varepsilon c(T) \int_0^T \left\| S\Big(\frac{-s}{\varepsilon}\Big)\left[\begin{array}{c} g^2_\varepsilon(s) \\ h^2_\varepsilon(s) \end{array}\right]\right\|^2_{L^2(R^3)\times D^{1,2}(R^3)} \mathrm{d}s$$

$$= \varepsilon c(T) \int_0^T \left\| \left[\begin{array}{c} g_\varepsilon^2(s) \\ h_\varepsilon^2(s) \end{array} \right] \right\|^2_{L^2(R^3)\times D^{1,2}(R^3)} \mathrm{d}s,$$

where we have used the fact that $(S(t))_{t\in R}$ is a group of isometries on $L^2(R^3)\times D^{1,2}(R^3)$.

Combining (5.9), (5.15), (5.16) we obtain (5.10).

6 Conclusion - main result

We have proved the following result.

Theorem 6.1 *Let $\{\Omega_\varepsilon\}_{\varepsilon>0}$ be a family of domains in R^3 enjoying Property L introduced in Section 1. Assume that the thermodynamics functions p, e, s as well as the transport coefficients μ, κ satisfy the structural hypotheses (2.24 - 2.31). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ be a distributional solution of the Navier-Stokes-Fourier system (1.1 - 1.5) in $(0,T)\times\Omega_\varepsilon$ satisfying (2.3 - 2.7) and emanating from the initial data (1.8), (1.9), (2.13) satisfying (2.14).*
Then, at least for a suitable subsequence,

$$\mathbf{u}_\varepsilon \to \mathbf{u} \text{ in } L^2(0,T;L^2(B;R^3)) \text{ for any bounded ball } B\subset R^3,$$

where $\mathrm{div}_x\mathbf{u} = 0$.

Remark 6.1 The presence of the radiation terms in the system is not necessary. The same result can be obtained if $a = 0$ in (2.24).

Remark 6.2 Exactly as in [5], we can show that the solutions $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$ of the complete Navier-Stokes-Fourier system tend for $\varepsilon\to 0$ to the corresponding solution of the Oberbeck-Boussinesq system (locally in space). The details are left to the reader.

Remark 6.3 Proposition 5.1 holds if R^3 is replaced by an exterior domain generated by a non-trapping (convex) obstacle (see [1]). Accordingly, the conclusion of Theorem 6.1 remains valid in this case as well.

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