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Gagliardo-Nirenberg inequalities in regular Orlicz spaces involving nonlinear expressions

Agnieszka Kałamajska* and Miroslav Krbec†

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Abstract

We consider a triple of N-functions (M, H, J) that satisfy the Δ'-condition, $\mu = |x|^\alpha\, dx$ and suppose that an additive variant of interpolation inequality holds

$$\int_{\mathbf{R}^n} M(|\nabla u|)\,\mu(dx) \le C\left(\int_{\mathbf{R}^n} H(|u|)\,\mu(dx) + \int_{\mathbf{R}^n} J(|\nabla^{(2)}u|)\,\mu(dx)\right),$$

where $u \in \mathcal{R} \subseteq W^{2,1}_{loc}(\mathbf{R}^n)$, $\mathcal{R}$ is an arbitrary set invariant with respect to external and internal dilations. We show that the above inequality implies its certain nonlinear variant involving the expressions $\int_{\mathbf{R}^n} H(|u|)\,\mu(dx)$ and $\int_{\mathbb{R}^n} J(|\nabla^{(2)}u|)\,\mu(dx)$. Various generalizations of this inequality to the more general class of N-functions, measures and to higher order derivatives are also discussed and the examples are presented.

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1 Introduction and statement of results

The purpose of this paper is to study an Orlicz variant of the classical Gagliardo-Nirenberg inequality, [13, 30]

$$\|\nabla^{(k)}u\|_q \le C\|u\|_r^{1-\frac{k}{m}}\|\nabla^{(m)}u\|_p^{\frac{k}{m}}, \quad \tfrac{1}{q} = (1-\tfrac{k}{m})\tfrac{1}{r} + \tfrac{k}{m}\tfrac{1}{p}, \quad u \in W^{m,1}_{loc}(\mathbf{R}^n). \tag{1.1}$$

Gagliardo-Nirenberg inequalities have been studied in a large number of papers, starting with the celebrated classical paper by Nirenberg [30]. Inequalities of this type can be traced back to [25], which deals with the case of supremum norms in (1.1), and inequalities obtained earlier by

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Hadamard, Landau and others. It is impossible to give a representative list of relevant references here, let us recall at least monographs [7, 28, 29].

The case of Orlicz spaces is somewhat difficult because of non-homogeneity of N-functions and a rather indirect definition of the norm. It is usually impossible to transfer simply the L^p-technique to the Orlicz setting. Therefore the progress here is slower and at the moment there are many topical unsolved problems. We refer to the papers [3, 4, 5, 6, 20, 21, 22, 23, 24].

The study of Orlicz case is partially motivated by possible applications in linear and nonlinear PDEs and in calculus of variations, arising from mathematical physics, see e.g. [1, 2, 10, 14, 15, 27, 32].

Let us recall what is presented here. As proved in [22, 23] one still has inequalities like

$$\|\nabla^{(k)}u\|_{L^M} \le C\|u\|_{L^{M_1}}^{1-\frac{k}{m}}\|\nabla^{(m)}u\|_{L^{M_2}}^{\frac{k}{m}},$$

within certain class of Orlicz spaces L^M, L^{M_1}, L^{M_2}, but in some cases we cannot expect such inequalities to hold (see e.g. [24]). Sometimes we may expect only an additive variant of those inequalities $\|\nabla^{(k)}u\|_{L^M} \le C(\|u\|_{L^{M_1}} + \|\nabla^{(m)}u\|_{L^{M_2}})$, deduced as a consequence of the additive inequality

$$\int_{\mathbf{R}^n} M(\nabla^{(k)}u)\,\mu(dy) \le C\left(\int_{\mathbf{R}^n} M_1(|u(y)|)\,\mu(dy) + \int_{\mathbf{R}^n} M_2(|\nabla^{(m)}u|)\,\mu(dy)\right), \tag{1.2}$$

see e.g. [24]. On the other hand, there are another inequalities which are expressed in terms of modulars. For example we show that

$$\left(\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)\,d\mu\right)^{\frac{1}{q}} \le C\left(\int_{\mathbb{R}^n} H(|u|)\,d\mu\right)^{\frac{1}{p}(1-\frac{k}{m})}\left(\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)\,d\mu\right)^{\frac{1}{r}\frac{k}{m}}\cdot$$
$$\cdot\left(\ln\left(2+\frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)\,d\mu}{\int_{\mathbb{R}^n} H(|u|)\,d\mu}\right)\right)^{\frac{\beta}{p}(1-\frac{k}{m})}\left(\ln\left(2+\frac{\int_{\mathbb{R}^n} H(|u|)\,d\mu}{\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)\,d\mu}\right)\right)^{\frac{\gamma}{r}\frac{k}{m}},$$

where M, H, J are Orlicz functions like $M_{s,\ell} = t^s\left(\ln(2+t)\right)^{\ell}$, under certain constraints on the involved parameters. Some other nonlinear inequalities dealing with N-functions like $t^s\left(\ln(1+t)\right)^{\ell}$ were obtained in [20]. Even within L^p-setting but with more general measures one cannot expect general inequalities of the form (1.1), see e.g. [17].

Our concern is to study inequalities

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)w(x)\,dx \le \tilde{C}\Psi\left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)w(x)\,dx}{\int_{\mathbb{R}^n} H(|u|)w(x)\,dx}\right)\cdot\int_{\mathbb{R}^n} H(|u|)w(x)\,dx,$$

holding for some M, J, H, Ψ. They are extension of (1.1).

We present a tool to deduce such a nonlinear variant of multiplicative inequality from simpler additive inequality (1.2) directly, or from its more precise variant

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)\,d\mu \le C\left(\int_{\mathbb{R}^n} H(s_1|u|)\,d\mu + \int_{\mathbb{R}^n} J(s_2|\nabla^{(m)}u|)\,d\mu\right) \text{ where } s_1^{1-\frac{k}{m}}s_2^{k/m} = 1, s_i > 0.$$

It seems that this is the first approach to study systematically nonlinear variants of interpolation inequalities involving modulars.

We suppose that the N-functions M_1 and M_2 satisfy the $\Delta^{'}$-condition (see Definition 2.1). Examples of admissible N-functions can be found among logarithmic Zygmund-type functions. In particular our analysis is supported by inequalities holding in such spaces, they seem to be of particular interest, see, e.g. [9, 11, 16, 18, 19].

They might find use in proving apriori estimates in the regularity theory for nonlinear PDEs.

2 Notation and preliminaries

Notation. By $C_0^\infty(\mathbb{R}^n)$ we denote as standard smooth compactly supported functions defined on $\mathbb{R}^n$. The symbol $W^{m,p}(\mathbf{R}^n)$ and $W^{m,p}_{loc}(\mathbf{R}^n)$ denotes Sobolev spaces. By R^{-1} we denote the inverse function to the given function R when it is well defined. If M is an N-function, then $M^*(t) := \sup_{\tau>0}(t\tau - M(\tau))$ is the complementary N-function (see [26]). Having two functions M, R we will write $M \sim R$ if there exist constants $C_1, C_2 > 0$ such that $C_1 M(\lambda) \leq R(\lambda) \leq C_2 M(\lambda)$. In the same way we will also compare functions for arguments near zero and near infinity respectively.

Definition 2.1. We say that the function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the Δ'-condition ($\Phi \in \Delta^{'}$) if there exists the constant $C > 0$ such that for every $\lambda_1, \lambda_2 > 0$ we have

$$\Phi(\lambda_1\lambda_2) \leq C\Phi(\lambda_1)\Phi(\lambda_2). \tag{2.1}$$

We refer e.g. to [26] for details about this family of Orlicz spaces. Let us note that the $\Delta^{'}$-condition is stronger than the usual Δ_2-condition, which asserts that there exists the constant $C > 0$ such that $\Phi(2\lambda) \leq C\Phi(\lambda)$, for every $\lambda > 0$ (we write $\Phi \in \Delta_2$).

We have the following easy observation.

Fact 2.1. *Let $\mathcal{M}_{\Delta'} := \{\Phi : [0,\infty) \to [0,\infty) : \Phi \in \Delta^{'}\}$. The family $\mathcal{M}_{\Delta'}$ is invariant with respect to multiplications and compositions.*

Using Fact 2.1 it is easy to generate elements of $\mathcal{M}_{\Delta'}$. The typical examples among N-functions can be found among Zygmund type logarithmic functions. This is illustrated on the following example. For the proof of part 2 and 3 see [21], similar arguments as to get (4.5).

Example 2.1. *The following N-functions are elements of $\mathcal{M}_{\Delta'}$;*

1. $\Phi(\lambda) = \lambda^p$, $1 < p < \infty$,

2. $M_{p,\alpha}(\lambda) = \lambda^p(\ln(2+\lambda))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$

3. $M^1_{p,\alpha}(\lambda) = \lambda^p(\ln(1+\lambda))^\alpha$, $1 < p < \infty$, $\alpha \geq 0$

4. $\Phi(\lambda) = M_{p_1,\alpha_1} \circ M_{p_2,\alpha_2} \circ \cdots \circ M_{p_k,\alpha_k}(\lambda)$, $\alpha_1, \ldots, \alpha_k \geq 0$, $p_i > 1$ *for* $i = 1, \ldots, k$.

We consider triples of N functions (M, H, J) and the measures μ which are absolutely continuous with respect to the Lebesgue measure and satisfy an additive variant of interpolation inequality

$$\int_{\mathbb{R}^n} M(|\nabla u|)\,\mu(dx) \leq C\left(\int_{\mathbb{R}^n} H(|u|)\,\mu(dx) + \int_{\mathbb{R}^n} J(|\nabla^{(2)}u|)\,\mu(dx)\right), \tag{2.2}$$

or its stronger variant, namely, the one parameter family of inequalities

$$\int_{\mathbb{R}^n} M(|\nabla u|)\,\mu(dx) \leq C\left(\int_{\mathbb{R}^n} H\left(\frac{1}{s}|u|\right)\,\mu(dx) + \int_{\mathbb{R}^n} J(s|\nabla^{(2)}u|)\,\mu(dx)\right). \tag{2.3}$$

This should be satisfied with a constant $C > 0$ independent on u and (in second case) arbitrary $s > 0$. In both cases we assume that u belongs to some set $\mathcal{R} \subseteq W^{2,1}_{loc}(\mathbb{R}^n)$.

If (2.2) holds with the triple (M, H, J), the measure μ and set $\mathcal{R}$, we will say that this objects support (2.2). Analogous concept will be used for (2.3) and in some other places.

3 Homogeneous measure and modeling inequality

Our goal here is to present the most representative technique illustrating our issue. It will be successively developed in next sections.

Our first result reads as follows.

Proposition 3.1. *Suppose that N-functions (M, H, J), the measure $\mu(dx) = |x|^\kappa dx$ and set $\mathcal{R}$ support an additive inequality (2.2). Assume that set $\mathcal{R}$ is invariant with respect to internal and external dilations, i.e. for every $t, s \in \mathbb{R}$ and $u \in \mathcal{R}$ the mapping $u_{t,s}(x) = tu(sx)$ also belongs to $\mathcal{R}$.*

Moreover, assume that functions H and J satisfy the Δ'-condition. Then for every $u \in \mathcal{R}$, $u \not\equiv 0$, we have

$$\int_{\mathbb{R}^n} M(|\nabla u|)\, \mu(dx) \le 2C\Psi\left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(2)} u|)\, \mu(dx)}{\int_{\mathbb{R}^n} H(|u|)\, \mu(dx)}\right) \cdot \int_{\mathbb{R}^n} H(|u|)\, \mu(dx), \tag{3.1}$$

where $\Psi(\lambda) = H \circ R^{-1}(\lambda)$, $R(\lambda) = \frac{H(\lambda)}{J(\frac{1}{\lambda})}$, C is the same constant as in (2.2).

Proof.
We apply (2.2) to the function $u_s(x) = \frac{1}{s}u(sx)$ and compute directly that

$$\int_{\mathbb{R}^n} M(|\nabla u_s(x)|)\, \mu(dx) = s^{-(\kappa+n)} \int_{\mathbb{R}^n} M(|\nabla u(y)|)\, \mu(dy),$$
$$\int_{\mathbb{R}^n} H(|u_s(x)|)\, \mu(dx) = s^{-(\kappa+n)} \int_{\mathbb{R}^n} H(\frac{1}{s}|u(y)|)\, \mu(dy),$$
$$\int_{\mathbb{R}^n} J(|\nabla^{(2)} u_s(y)|)\, \mu(dy) = s^{-(\kappa+n)} \int_{\mathbb{R}^n} J(s|\nabla^{(2)} u(y)|)\, \mu(dy).$$

Therefore (2.2) implies

$$\int_{\mathbb{R}^n} M(|\nabla u(x)|)\, \mu(dx) \le C\left(\int_{\mathbb{R}^n} H\left(\frac{1}{s}|u(x)|\right) \mu(dx) + \int_{\mathbb{R}^n} J(s|\nabla^{(2)} u(x)|)\, \mu(dx)\right),$$

holding for every $u \in \mathcal{R}$, $s > 0$, with the constant independent on u and s. Using the Δ'-condition (2.1) we obtain the one parameter family of inequalities:

$$\int_{\mathbb{R}^n} M(|\nabla u(x)|)\, \mu(dx) \le C\left(H\left(\frac{1}{s}\right) \int_{\mathbb{R}^n} H(|u(x)|)\, \mu(dx) + J(s) \int_{\mathbb{R}^n} J(|\nabla^{(2)} u(x)|)\, \mu(dx)\right),$$

holding with C independent of u and s. In other terms

$$a \le H\left(\frac{1}{s}\right) b + J(s)c, \tag{3.2}$$

where

$$a = \int M(|\nabla u(x)|)\,\mu(dx), \quad b = C\int H(|u(x)|)\,\mu(dx), \quad c = C\int J(|\nabla^{(2)}u(x)|)\,\mu(dx). \qquad (3.3)$$

Let us choose s_0 such that $H(\frac{1}{s_0})b = J(s_0)c$, i.e. according to our notation $R(\frac{1}{s_0}) = \frac{c}{b}$, equivalently $\frac{1}{s_0} = R^{-1}(\frac{c}{b})$ (note that R^{-1} is well defined).
Inequality (3.2) implies

$$a \le 2H\left(\frac{1}{s_0}\right)b = 2H \circ R^{-1}\left(\frac{c}{b}\right)\cdot b,$$

which is exactly what we have claimed. □

Remark 3.1. Inequality (3.1) looks stronger than (2.2) at first glance. Indeed, if $\int_{\mathbb{R}^n} J(|\nabla^{(2)}u_n(x)|)\,\mu(dx) \to 0$ as $n \to \infty$, for some $\{u_n\} \subseteq C_0^\infty(\mathbb{R}^n)$, while $\int_{\mathbb{R}^n} H(|u_n(x)|)\,\mu(dx)$ remains to be bounded and bounded away from 0, we observe from (3.1) that $\int M(|\nabla u(x)|)\,\mu(dx)$ converges to 0. This is not readily seen from (2.2).

Remark 3.2. Proposition 3.1 shows that inequality (2.2) implies (3.1). Let us show that inequality (3.1) implies (2.2). Hence those inequalities are equivalent, possibly with different constants.

To prove the implication "(3.1) $\Longrightarrow$ (2.2)", we use the notation (3.3) and observe that inequality (3.1) reads:

$$a \le 2H \circ R^{-1}\left(\frac{c}{b}\right)\cdot b = 2H\left(\frac{1}{s_0}\right)\cdot b, \qquad (3.4)$$

where we put $R^{-1}(\frac{c}{b}) = \frac{1}{s_0}$. From the very definition of R we have $H(\frac{1}{s_0})b = J(s_0)c$. Moreover,

$$H\left(\frac{1}{s_0}\right)b \le H\left(\frac{1}{s}\right)b + J(s)c, \quad \text{for every } s > 0.$$

Indeed, for $s > s_0$ we have $J(s)c > J(s_0)c = H(\frac{1}{s_0})b$, while for $s \le s_0$ we have $\frac{1}{s_0} \le \frac{1}{s}$, therefore $H(\frac{1}{s_0})b \le H(\frac{1}{s})b$. Therefore (3.4) implies

$$a \le 2\mathrm{inf}\left\{H\left(\frac{1}{s}\right)b + J(s)c,\ s > 0\right\} \le \widetilde{C}(b+c), \qquad \widetilde{C} = 2\mathrm{max}(H(1), J(1)).$$

This implies (2.2).

4 Inequalities with Lebesgue measure

4.1 More general inequalities

We will now discuss inequalities which can be proved when one considers the Lebesgue measure. It turns that in such a case one obtains more general inequalities, taking into account the choice of admissible Orlicz spaces.

Before we formulate the result, let us introduce the following auxiliary function $M_n : (0,\infty) \to (0,\infty)$:

$$M_n(\lambda) = \frac{|\lambda M'(\lambda) - M(\lambda)| + \sqrt{n-1}\, M(\lambda)}{\lambda^2} = \left| \left(\frac{M(\lambda)}{\lambda} \right)' \right| + \frac{\sqrt{n-1}\, M(\lambda)}{\lambda^2}, \tag{4.1}$$

and a notation of a suitable compatibility. Note that if M satisfies Δ_2-condition, then $M_n \sim M(\lambda)/\lambda^2$ as in such a case $M^{'}(\lambda) \sim M(\lambda)/\lambda$.

Definition 4.1. *A couple of continuous functions* $\Psi_1, \Psi_2 : \mathbb{R}^n \times [0,\infty) \times [0,\infty) \to [0,\infty)$ *will be called compatible if* $\Psi_1(x,\lambda_1,\lambda_2)\Psi_2(x,\lambda_1,\lambda_2) = \lambda_1\lambda_2$ *for every* $x \in \mathbb{R}^n, \lambda_1, \lambda_2 \geq 0$.

Remark 4.1. As typical examples of a compatible couple we consider $\Psi_1(x,\lambda_1,\lambda_2) = sw(x)\lambda_1^{\theta_1}\lambda_2^{\theta_2}$ and $\Psi_2(x,\lambda_1,\lambda_2) = s^{-1}w(x)^{-1}\lambda_1^{1-\theta_1}\lambda_2^{1-\theta_2}$, with parameters $(\theta_1,\theta_2) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$, $s > 0$, and an arbitrary measurable function $w > 0$ a.e. The simplest case is $\Psi_1(x,\lambda_1,\lambda_2) = \lambda_1$, $\Psi_2(x,\lambda_1,\lambda_2) = \lambda_2$.

The following result is a special case of Theorem 3.1 in [22].

Theorem 4.1. *Let* M *be an N-function and suppose that* $M^{'}(\lambda)/\lambda$ *is bounded in some neighborhood of* 0, M_n *is given by* (4.1), *and that* $H, J : [0,\infty) \to [0,\infty)$ *are continuous functions satisfying inequality*

$$\forall\, x,y,z \geq 0 \quad M_n(x)yz \leq M(x) + H(y) + J(z). \tag{4.2}$$

Let (Ψ_1,Ψ_2) *be a couple of continuous compatible functions. Then for an arbitrary* $u \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned} &\int M(|\nabla u(x)|)\, dx \\ &\leq \int H(\sqrt{2}\cdot \Psi_1(x,|u(x)|,|\nabla^{(2)}u(x)|))\, dx + \int J(\sqrt{2}\cdot \Psi_2(x,|u(x)|,|\nabla^{(2)}u(x)|))\, dx. \end{aligned} \tag{4.3}$$

Remark 4.2. In [22] the authors considered inequality (4.3) with Ψ_1 and Ψ_2 independent of x. The proof given there works in the general case as well without changes.

Remark 4.3. In other words we deduce that under the assumptions of Theorem 4.1 the triple (M,H,J), the Lebesgue measure and set $\mathcal{R} = C_0^\infty(\mathbb{R}^n)$ support (4.3), for any compatible couple (Ψ_1,Ψ_2).

Remark 4.4. Replacing (Ψ_1,Ψ_2) by $(\frac{1}{s}\Psi_1, s\Psi_2)$ one obtains the following inequality

$$\int M(|\nabla u|)\, dx \leq C\left(\int H\left(\frac{1}{s}\Psi_1(x,|u|,|\nabla^{(2)}u|)\right) dx + \int J(s\Psi_2(x,|u|,|\nabla^{(2)}u|))\, dx \right), \tag{4.4}$$

where $s > 0$ can be an arbitrary given parameter. The constant $C > 0$ is such that $H(\sqrt{2}\lambda) \leq CH(\lambda)$, $J(\sqrt{2}\lambda) \leq CJ(\lambda)$ for every $\lambda > 0$.

The variant of Proposition 3.1 (considering Lebesgue measure) reads as follows.

Proposition 4.1. *Suppose that N-function* M *is such that* $M^{'}(\lambda)/\lambda$ *is bounded next to* 0, M_n *is given by* (4.1) *and assume that N-functions* H, J *satisfy* (4.2) *and the* Δ'*-condition. Let* (Ψ_1,Ψ_2) *be a couple of compatible functions. Then for every* $u \in C_0^\infty(\mathbb{R}^n)$, $u \not\equiv 0$ *we have*

$$\int_{\mathbb{R}^n} M(|\nabla u(x)|)\, dx \leq 2C\Psi\left(\frac{\int_{\mathbb{R}^n} J(w_2(x))\, dx}{\int_{\mathbb{R}^n} H(w_1(x))\, dx} \right) \cdot \int_{\mathbb{R}^n} H(w_1(x))\, dx,$$

where $w_1 = \Psi_1(x, |u|, |\nabla^{(2)}u|)$, $w_2 = \Psi_2(x, |u|, |\nabla^{(2)}u|)$, $\Psi(\lambda) = H \circ R^{-1}(\lambda)$, $R(\lambda) = \frac{H(\lambda)}{J(\frac{1}{\lambda})}$, C *is a constant satisfying* $H(\sqrt{2}\lambda) \le CH(\lambda)$ *and* $J(\sqrt{2}\lambda) \le CJ(\lambda)$.

Proof. We start with inequality (4.4) and repeat the same arguments as in the proof of Proposition 3.1. The difference is that now we deal with $\Psi_1(x, |u|, |\nabla^{(2)}u|)$ instead of $|u|$ and $\Psi_2(x, |u|, |\nabla^{(2)}u|)$ instead of $|\nabla^{(2)}u|$. $\square$

Various methods for construction of triples (M, H, J) supporting (4.2) are discussed in [22].

4.2 The case of logarithmic functions

Set $M_{s,l} = t^s \left(\ln(2+t)\right)^l$, and consider $M(\lambda) = M_{q,\alpha}(\lambda)$, $H(\lambda) = M_{p,\beta}(\lambda)$, $J(\lambda) = M_{r,\gamma}(\lambda)$, where

$$\frac{2}{q} = \frac{1}{p} + \frac{1}{r}, \qquad \frac{2\alpha}{q} = \frac{\beta}{p} + \frac{\gamma}{r}, \quad p \ge 2, q, r > 1, \quad \alpha, \beta, \gamma \ge 0. \tag{4.5}$$

It is proved in [21], Theorem 1.1 that the triple (H, M, J), the measure $\mu = dx$, and $\mathcal{R} = C_0^\infty(\mathbb{R}^n)$ support (2.2).
Direct computation (see (4.10) in [21]) gives

$$R^{-1}(\lambda) \sim \left(\lambda \frac{(\ln(2+\lambda^{-1}))^\gamma}{(\ln(2+\lambda))^\beta}\right)^{\frac{1}{p+r}}$$

and (see (4.12) in [21])

$$\Psi(\lambda) \sim \lambda^{\frac{p}{p+r}} \left(\ln(2+\lambda^{-1})\right)^{\frac{\gamma p}{p+r}} \left(\ln(2+\lambda)\right)^{\frac{\beta r}{p+r}}.$$

Therefore

$$\Psi(\frac{c}{b}) \cdot b \sim b^{\frac{r}{p+r}} c^{\frac{p}{p+r}} \cdot \left(\ln\left(2 + \frac{b}{c}\right)\right)^{\frac{\gamma p}{p+r}} \cdot \left(\ln\left(2 + \frac{c}{b}\right)\right)^{\frac{\beta r}{p+r}}.$$

This leads to the following multiplicative inequality obtained in [21] (in the slightly more general version).

Theorem 4.2 ([21], Theorem 4.2)**.** *Suppose that* $p, q, r, \alpha, \beta, \gamma$ *are given real numbers satisfying* (4.5)*. Let* (Ψ_1, Ψ_2) *be the pair of compatible functions,*
$w_1(x) = \Psi_1(x, |u(x)|, |\nabla^{(2)}u(x)|), w_2(x) = \Psi_2(x, |u(x)|, |\nabla^{(2)}u(x)|)$, $u \in C_0^\infty(\mathbb{R}^n)$.

Then there exists a constant $C = C(p, r, \beta, \gamma) > 0$ *such that:*

$$\left(\int M_{q,\alpha}(|\nabla f(x)|)\, dx\right)^{\frac{2}{q}} \le C \left(\int M_{p,\beta}(w_1(x))\, dx\right)^{\frac{1}{p}} \left(\int M_{r,\gamma}(w_2(x))\, dx\right)^{\frac{1}{r}} \cdot$$
$$\cdot \left(\ln\left(2 + \frac{\int M_{p,\beta}(w_1(x))\, dx}{\int M_{r,\gamma}(w_2(x))\, dx}\right)\right)^{\frac{\gamma}{r}} \left(\ln\left(2 + \frac{\int M_{r,\gamma}(w_2(x))\, dx}{\int M_{p,\beta}(w_1(x))\, dx}\right)\right)^{\frac{\beta}{p}}.$$

Remark 4.5. In the case $\alpha = \beta = \gamma = 0$ the above statement reduces to the classical Gagliardo-Nirenberg inequality.

5 Higher order derivatives and more general measures

Our next goal is to generalize inequalities like

$$\int_{\mathbb{R}^n} M(|\nabla u|)\,dx \le 2C\Psi\left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(2)}u|)\,dx}{\int_{\mathbb{R}^n} H(|u|)\,dx}\right)\cdot \int_{\mathbb{R}^n} H(|u|)\,dx,$$

with a suitable choice of functions M, J, H and Ψ, to the more general ones having the form

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)w(x)\,dx \le 2C\Psi_{m,k}\left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)w(x)\,dx}{\int_{\mathbb{R}^n} H(|u|)w(x)\,dx}\right)\cdot \int_{\mathbb{R}^n} H(|u|)w(x)\,dx, \tag{5.1}$$

holding for some $M, J, H, \Psi_{m,k}$, depending on $m, k \in \mathbf{N}$. Note that we take $\nabla^{(m)}u$ and $\nabla^{(k)}u$ instead of $\nabla^{(2)}u$ and ∇u, respectively, and we replace the Lebesgue measure by a weighted measure $\mu = w(x)\,dx$. The measure needs to be sufficiently regular, see Remark 5.1 below.

Contrary to the approach using Lebesgue measure now we cannot deduce inequalities like (5.1), where u and $\nabla^{(m)}u$ are substituted by more general expressions like $\Psi_1(x, |u|, |\nabla^{(m)}u|)$, $\Psi_2(x, |u|, |\nabla^{(m)}u|)$ as in Proposition 4.1.

Our first result in this direction reads as follows.

Proposition 5.1. *Suppose that N-functions (M, H, J), the measure $\mu = w(x)\,dx$ and set $\mathcal{R} \subseteq W^{m,1}_{loc}(\mathbb{R}^n)$ support inequalities*

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)\,d\mu \le C\left(\int_{\mathbb{R}^n} H(s_1|u|)\,d\mu + \int_{\mathbb{R}^n} J(s_2|\nabla^{(m)}u|)\,d\mu\right), \tag{5.2}$$

holding for every $s_1, s_2 > 0$ such that $s_1^{1-\frac{k}{m}} s_2^{k/m} = 1$, with some general constant C independent of u, s_1, s_2. Moreover, assume that functions H and J satisfy the Δ'-condition. Then for every $u \in \mathcal{R}$, $u \not\equiv 0$, we have

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)\,\mu(dx) \le 2C\Psi_{m,k}\left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)\,\mu(dx)}{\int_{\mathbb{R}^n} H(|u|)\,\mu(dx)}\right)\cdot \int_{\mathbb{R}^n} H(|u|)\,\mu(dx), \tag{5.3}$$

where $\Psi_{m,k}(\lambda) = H \circ R^{-1}_{m,k}(\lambda)$, $R_{m,k}(\lambda) = \frac{H(\lambda)}{J(\lambda^{-(\frac{m-k}{k})})}$, the constant C is the same as in (5.2).

Proof. Inequality (5.2) and the Δ'-condition imply

$$a = \int_{\mathbb{R}^n} M(|\nabla^{(k)}u|)\,d\mu \le H(s)b + J(s^{-(\frac{m-k}{k})})c := I(s) + II(s), \quad s > 0,$$
$$b = C\int_{\mathbb{R}^n} H(|u|)\,d\mu, \quad c = C\int_{\mathbb{R}^n} J(|\nabla^{(m)}u|)\,\mu(dx).$$

Choosing $s = s_0$ such that $I(s) = II(s)$, i.e. $s_0 = R^{-1}_{m,k}(\frac{c}{b})$, we get $a \le 2\Psi_{m,k}(\frac{c}{b})b$, which is the same as (5.3). □

To proceed further we recall some useful definitions.

Definition 5.1 (the class W_Φ,[8])**.** *Suppose that $\Phi : [0,\infty) \to [0,\infty)$ is an N-function. We say that a weight $w : \mathbb{R}^n \to [0,\infty)$ belongs to the class W_Φ if and only if for all cubes $S \subset \mathbb{R}^n$ and all $\lambda > 0$*

$$\int_S \Phi^* \left(\frac{\Phi(\lambda)\mu(S)}{c\lambda |S| w(x)} \right) w(x)\, dx \le \Phi(\lambda)\mu(S) < +\infty,$$

with the constant $c > 0$ independent of S, where $\mu(A) = \int_A \omega(x)dx$.

In the particular case of $\Phi(\lambda) = \lambda^p$, $p > 1$, the class W_Φ coincides with the class of A_p-weights, see e.g. [31].

The following result was obtained in [23], Theorem 4.3.

Theorem 5.1. *Suppose that $M : [0,\infty) \to [0,\infty)$ and $F : [0,\infty) \to [0,\infty)$ are two N-functions. Let $\mu(dx) = w(x)\, dx$, where w is a nonnegative weight on $\mathbb{R}^n$. For $k, m \in \mathbb{Z}_+$, $0 < k < m$, define*

$$H(\lambda) = M(F(\lambda^{1-\frac{k}{m}})), \quad J(\lambda) = M(F^*(\lambda^{\frac{k}{m}})).$$

When the functions H, J are N-functions, $H^*, J^* \in \Delta_2$, and $w \in W_H \cap W_J$, then for every $u \in C_0^m(\mathbb{R}^n)$, and arbitrary positive numbers s_1, s_2 such that $1 = s_1^{1-\frac{k}{m}} s_2^{\frac{k}{m}}$, one has

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)} u|)\, d\mu \le \int_{\mathbb{R}^n} H(s_1 B_1 |u|)\, d\mu + \int_{\mathbb{R}^n} J(s_2 B_2 |\nabla^{(m)} u|)\, d\mu;$$

with some constants B_1 and B_2 independent of u and s_1, s_2.

Taking into account Theorem 5.1 and Proposition 5.1 we obtain the following result.

Theorem 5.2. *Let the assumptions of Theorem 5.1 be satisfied and additionally let $H, J \in \Delta'$. Then for every $u \in C_0^m(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)} u|)\, \mu(dx) \le C \Psi_{m,k} \left(\frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)} u|)\, \mu(dx)}{\int_{\mathbb{R}^n} H(|u|)\, \mu(dx)} \right) \cdot \int_{\mathbb{R}^n} H(|u|)\, \mu(dx),$$

where $\Psi_{m,k}(\lambda) = H \circ R_{m,k}^{-1}(\lambda)$, $R_{m,k}(\lambda) = \frac{H(\lambda)}{J(\lambda^{-(\frac{m-k}{k})})}$, and the constant C is independent of u.

If H^* or J^* does not satisfy the Δ_2-condition we use another approach.

We recall the following definitions.

Definition 5.2. *We say that a weight function $w : \mathbb{R}^n \to [0,\infty)$ belongs to the A_1-class ($w \in A_1$), if there exists a constant $C > 0$ such that for every cube $S \subseteq \mathbb{R}^n$ we have*

$$\frac{1}{|S|} \int_S w(y) dy \le C \operatorname{ess\,inf}_{x \in S} w(x).$$

Definition 5.3. *We say that a weight function $w : \mathbb{R}^n \to [0,\infty)$ belongs to the A'_∞-class ($w \in A'_\infty$) if there exists a constant $C > 0$ such that for every cube $S \subseteq \mathbb{R}^n$ we have*

$$\frac{1}{|S|} \int_{2S} w(y) dy \ge C \operatorname{ess\,sup}_{x \in S} w(x).$$

The following theorem holds true.

Theorem 5.3 ([23], Theorem 4.4)**.** *Let $k, m \in \mathbb{Z}_+$, $0 < k < m$ and $\mu(dx) = w(x)\,dx$, where $w \in A_1 \cap A'_\infty$. Suppose that $M : [0, +\infty) \to [0, +\infty)$ is an increasing function of class $C^1((0, \infty))$ such that $M(0) = (M)'_+(0) = 0$, and that $F : [0, +\infty) \to [0, +\infty)$ is an N-function of class C^1. Set*

$$H(\lambda) = M(F(\lambda^{1-\frac{k}{m}})), \;\; J(\lambda) = M(F^*(\lambda^{\frac{k}{m}})).$$

Assume further that $\int_0^1 \frac{R(v)}{v^2}\,dv < \infty$ for $R \in \{M, H, J\}$ and define

$$\widetilde{R}(\lambda) = \int_0^1 \frac{R(\lambda v)}{v^2}\,dv, \quad R \in \{M, H, J\}. \tag{5.4}$$

Then there exist constants $C, K > 0$ such that for every $u \in C_0^m(\mathbb{R}^n)$ and for every positive numbers s_1, s_2 such that $1 = s_1^{1-\frac{k}{m}} s_2^{\frac{k}{m}}$,

$$\int_{\mathbb{R}^n} \widetilde{M}(C|\nabla^{(k)}u|)\,d\mu \le K\left(\int_{\mathbb{R}^n} \widetilde{H}(s_1|u|)\,d\mu + \int_{\mathbb{R}^n} \widetilde{J}(s_2|\nabla^{(m)}u|)\,d\mu\right)$$

As direct consequence of Theorem 5.3 and Proposition 5.1 we obtain the following theorem.

Theorem 5.4. *Let the assumptions of Theorem 5.3 be satisfied and additionally $\widetilde{H}, \widetilde{J} \in \Delta'$. Then for every $u \in C_0^m(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} \widetilde{M}(|\nabla^{(k)}u|)\,\mu(dx) \le C\Psi_{m,k}\left(\frac{\int_{\mathbb{R}^n} \widetilde{J}(|\nabla^{(m)}u|)\,\mu(dx)}{\int_{\mathbb{R}^n} \widetilde{H}(|u|)\,\mu(dx)}\right) \cdot \int_{\mathbb{R}^n} \widetilde{H}(|u|)\,\mu(dx),$$

where $\Psi_{m,k}(\lambda) = \widetilde{H} \circ R^{-1}_{m,k}(\lambda)$, $R_{m,k}(\lambda) = \frac{\widetilde{H}(\lambda)}{\widetilde{J}\left(\lambda^{-(\frac{m-k}{k})}\right)}$, the constant C is independent on u.

Remark 5.1. It follows that for every weight function ω of class W_Φ and for every N-function Φ, the measure $\mu = \omega\,dx$ necessarily satisfies doubling property: $\mu(B(x, 2r)) \le C\mu(B(x, r))$, with constant C independent on r. The same property holds for $\omega \in A_1$ and for $\omega \in A'_\infty$, see [31]. In particular every such a measure is rather regular.

More comments concerning admissible weights and functions can be found in [23], Section 5. For example when the N-function R in (5.4) is strictly monotone and $R^* \in \Delta_2$, then functions R and $\widetilde{R}$ are equivalent (see Proposition 5.1 in [23], the statement (6)).

5.1 Logarithmic case revisited

As in Subsection 4.2 we deal with $M_{s,l} = t^s\,(\ln(2 + t))^l$ and consider

$$M(\lambda) = M_{q,\alpha}(\lambda), \quad H(\lambda) = M_{p,\beta}(\lambda), \quad J(\lambda) = M_{r,\gamma}(\lambda). \tag{5.5}$$

We have the following result.

Theorem 5.5. *Let $k, m \in \mathbb{Z}_+$ be given and such that $0 < k < m$. Suppose that the parameters p, q, r and α, β, γ satisfy the conditions*

$$\frac{1}{q} = \left(1 - \frac{k}{m}\right)\frac{1}{p} + \frac{k}{m}\frac{1}{r}, \quad \frac{\alpha}{q} = \left(1 - \frac{k}{m}\right)\frac{\beta}{p} + \frac{k}{m}\frac{\gamma}{r}, \quad p, q, r > 1, \quad \alpha, \beta, \gamma \geq 0$$

and let $\mu(dx) = w(x)\,dx$ be a weighted measure with weight belonging to the class $W_{M_{p,\beta}} \cap W_{M_{r,\gamma}}$. Then for every function $u \in C_0^m(\mathbb{R}^n)$ we have

$$\left(\int_{\mathbb{R}^n} M(|\nabla^{(k)} u|)\,d\mu\right)^{\frac{1}{q}} \leq C\left(\int_{\mathbb{R}^n} H(|u|)\,d\mu\right)^{\frac{1}{p}(1-\frac{k}{m})}\left(\int_{\mathbb{R}^n} J(|\nabla^{(m)} u|)\,d\mu\right)^{\frac{1}{r}\frac{k}{m}} \cdot$$
$$\cdot\left(\ln\left(2 + \frac{\int_{\mathbb{R}^n} J(|\nabla^{(m)} u|)\,d\mu}{\int_{\mathbb{R}^n} H(|u|)\,d\mu}\right)\right)^{\frac{\beta}{p}(1-\frac{k}{m})}\left(\ln\left(2 + \frac{\int_{\mathbb{R}^n} H(|u|)\,d\mu}{\int_{\mathbb{R}^n} J(|\nabla^{(m)} u|)\,d\mu}\right)\right)^{\frac{\gamma}{r}\frac{k}{m}}.$$

with a constant C independent of u.

Proof. Let M, H, J be as in (5.5) and consider $F(\lambda) = M_{s,\kappa}(\lambda)$, where we choose $s = \frac{p}{q}\frac{m}{m-k}$, $\kappa = \frac{\beta-\alpha}{q}$. It is proved in [23], the proof of Theorem 6.1 that for our choice of parameters we have

$$H \sim M(F(\lambda^{1-\frac{k}{m}})) \quad J \sim M(F(\lambda^{\frac{k}{m}})).$$

Theorem 5.1 implies

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)} u|)\,d\mu \leq C\left(\int_{\mathbb{R}^n} H(s_1|u|)\,d\mu + \int_{\mathbb{R}^n} J(s_2|\nabla^{(m)} u|)\,d\mu\right),$$

with some universal constant C (depending on the Δ_2-condition for H and J only), $s_1, s_2 > 0$ are arbitrary parameters such that $s_1^{1-\frac{k}{m}} s_2^{\frac{k}{m}} = 1$. By Proposition 5.1

$$\int_{\mathbb{R}^n} M(|\nabla^{(k)} u|)\,d\mu \leq 2C\Psi_{m,k}(\frac{c}{b})b, \quad \text{where} \quad b = \int_{\mathbb{R}^n} H(|u|)\,d\mu, \ c = \int_{\mathbb{R}^n} J(|\nabla^{(m)} u|)\,d\mu. \tag{5.6}$$

Now it suffices to compute $\Psi_{m,k}$. We have

$$R_{m,k}(\lambda) = \frac{H(\lambda)}{J(\lambda^{-\frac{m-k}{k}})} \sim \frac{M_{p,\beta}(\lambda)}{M_{r,\gamma}(\lambda^{-\frac{m-k}{k}})}.$$

For λ close to 0 we have

$$R_{m,k}(\lambda) \sim \frac{\lambda^p}{\lambda^{-r(\frac{m}{k}-1)}} \cdot \frac{1}{(\ln(\lambda^{-(\frac{m}{k}-1)}))^\gamma} \sim \frac{\lambda^{p+r(\frac{m}{k}-1)}}{(\ln(\lambda^{-1}))^\gamma} \sim \lambda^{p+r(\frac{m}{k}-1)}\frac{(\ln(2+\lambda))^\beta}{(\ln(2+\lambda^{-1}))^\gamma}.$$

By similar arguments, for λ tending to ∞, we have $R_{m,k}(\lambda) \sim \lambda^{p+r(\frac{m}{k}-1)}\frac{(\ln(2+\lambda))^\beta}{(\ln(2+\lambda^{-1}))^\gamma}$. Therefore

$$R_{m,k}(\lambda) \sim \lambda^{p+r(\frac{m}{k}-1)}\frac{(\ln(2+\lambda))^\beta}{(\ln(2+\lambda^{-1}))^\gamma}.$$

One readily checks that

$$R_{m,k}^{-1}(\lambda) \sim \left(\lambda \frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\right)^{\frac{1}{p+r(\frac{m}{k}-1)}}. \tag{5.7}$$

Let us compute $\Psi_{m,k}(\lambda)$. Using (5.7) and (5.5) we get

$$\Psi_{m,k}(\lambda) = \left[\lambda^{\frac{p}{p+r(\frac{m}{k}-1)}}\right] \cdot \left[\left(\frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\right)^{\frac{p}{p+r(\frac{m}{k}-1)}}\right] \cdot \left[\left(\ln\left(2+\left\{\lambda \frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\right\}^{\frac{1}{p+r(\frac{m}{k}-1)}}\right)\right)^{\beta}\right]$$

$$:= \left[\lambda^{\frac{p}{p+r(\frac{m}{k}-1)}}\right] \cdot [A(\lambda)] \cdot [B(\lambda)]\,.$$

For λ close to 0 we have $\frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}} \sim (\ln \lambda^{-1})^{\gamma}$, therefore for $\lambda \sim 0$,

$$\left(\lambda \frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}}\right)^{\frac{1}{p+r(\frac{m}{k}-1)}} \sim \lambda^{\frac{1}{p+r(\frac{m}{k}-1)}} \left(\ln \lambda^{-1}\right)^{\frac{\gamma}{p+r(\frac{m}{k}-1)}} \overset{\lambda \to 0}{\longrightarrow} 0.$$

Therefore $B(\lambda) \sim C$ for λ close to 0.
For λ near ∞,

$$\lambda \frac{(\ln(2+\lambda^{-1}))^{\gamma}}{(\ln(2+\lambda))^{\beta}} \sim \frac{\lambda}{(\ln \lambda)^{\beta}} \overset{\lambda \to \infty}{\longrightarrow} \infty.$$

Hence in such a case

$$B(\lambda) \sim \left[\ln\left(\frac{\lambda}{(\ln \lambda)^{\beta}}\right)\right]^{\beta} \sim (\ln \lambda)^{\beta} \sim (\ln(2+\lambda))^{\beta}.$$

Therefore for λ close to 0,

$$A(\lambda) \cdot B(\lambda) \sim \left(\ln(2+\lambda^{-1})\right)^{\frac{\gamma p}{p+r(\frac{m}{k}-1)}},$$

while for λ close to ∞,

$$A(\lambda) \cdot B(\lambda) \sim \frac{(\ln(2+\lambda))^{\beta}}{\left(\ln(2+\lambda)\right)^{\frac{\beta p}{p+r(\frac{m}{k}-1)}}} = (\ln(2+\lambda))^{\frac{\beta r(\frac{m}{k}-1)}{p+r(\frac{m}{k}-1)}}.$$

In both cases

$$\Psi_{m,k}(\lambda) \sim \left[\lambda^{\frac{p}{p+r(\frac{m}{k}-1)}}\right] \cdot [A(\lambda)] \cdot [B(\lambda)] \sim \lambda^{\frac{p}{p+r(\frac{m}{k}-1)}} \cdot \left(\ln(2+\lambda^{-1})\right)^{\frac{\gamma p}{p+r(\frac{m}{k}-1)}} \cdot (\ln(2+\lambda))^{\frac{\beta r(\frac{m}{k}-1)}{p+r(\frac{m}{k}-1)}}.$$

Therefore

$$\Psi_{m,k}\left(\frac{c}{b}\right) b \sim b^{\frac{r(\frac{m}{k}-1)}{p+r(\frac{m}{k}-1)}} c^{\frac{p}{p+r(\frac{m}{k}-1)}} \left(\ln\left(2+\frac{b}{c}\right)\right)^{\frac{\gamma p}{p+r(\frac{m}{k}-1)}} \cdot \left(\ln\left(2+\frac{c}{b}\right)\right)^{\frac{\beta r(\frac{m}{k}-1)}{p+r(\frac{m}{k}-1)}}$$

$$= b^{\frac{q}{p}(1-\frac{k}{m})} c^{\frac{q}{r}\frac{k}{m}} \left(\ln\left(2+\frac{b}{c}\right)\right)^{q\frac{\gamma}{r}\frac{k}{m}} \cdot \left(\ln(2+\frac{c}{b})\right)^{q\frac{\beta}{p}(1-\frac{k}{m})}.$$

This and (5.6) implies the thesis. □

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Agnieszka Kałamajska
Institute of Mathematics
Warsaw University
ul. Banacha 2
02-097 Warszawa, Poland
email: kalamajs@mimuw.edu.pl

Mirosław Krbec
Institute of Mathematics
Academy of Sciences of the Czech Republic,
Žitná 25
CZ-115 67 Praha 1, Czech Republic
e-mail: krbecm@math.cas.cz