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On a mathematical model of journal bearing lubrication

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Abstract

We consider the steady motion of an incompressible fluid whose viscosity depends on the pressure and the shear rate. The system is completed by suitable boundary conditions involving non-homogeneous Dirichlet, Navier's slip and inflow/outflow parts. We prove the existence of weak solutions and show that the resulting level of the pressure is fixed by the boundary conditions. The problem is motivated by particular applications from tribology.

Key words: existence, weak solutions, incompressible fluids, non-Newtonian fluids, pressure dependent viscosity, shear dependent viscosity, inflow/outflow boundary conditions, pressure boundary conditions, filtration boundary conditions
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1. Introduction

When mathematically describing the flow of an incompressible viscous fluid, a common hypothesis is that the viscous forces are a linear function of the velocity gradient and are independent of other variables, namely the pressure. Such assumption is inherent with the model of the so-called Newtonian fluid governed by the Navier–Stokes system. A number of generalizations have been made in order to capture phenomena that are observed in various fluids at various operating conditions and cannot be covered by the Newtonian model. Two such features are addressed in this paper: the shear-thinning, where the viscosity decreases with the shear rate, and the pressure-thickening, where it increases with the pressure. It is worth noting that the assumption of pressure-independent viscous forces was recognised already by Stokes [42] to be valid only within a limited range of pressures. Note also that while changes of the viscosity due to the pressure can be severe, density variations can remain insignificant by comparison, so that the assumption of incompressibility is not violated.

There is a particular distinction of the pressure-thickening models, which partly inspired our study. It is a well-known property of the equations describing the motion of incompressible fluids that the pressure is determined to within a constant. As fas as only the pressure gradient is involved in the governing equations, this constant does not play important role. However, as soon as the

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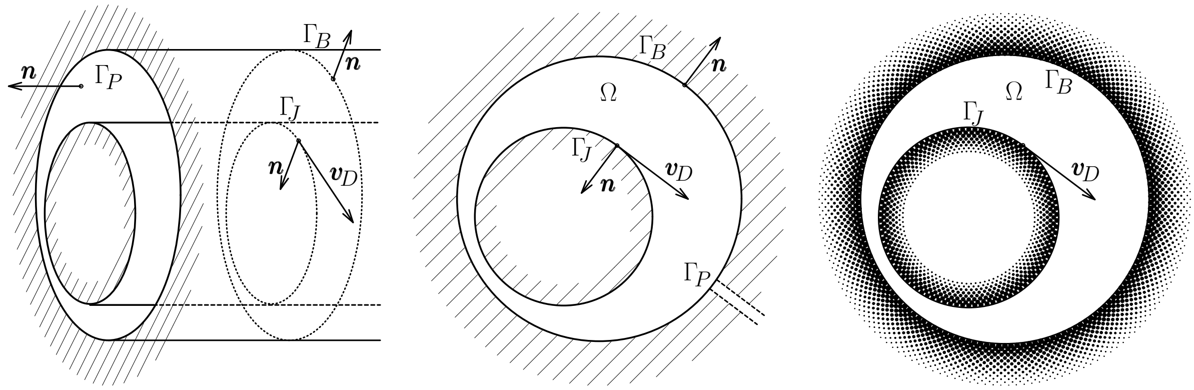

(a) Finite bearing. (b) Planar model. (c) Porous bearing.

Figure 1: Three examples of journal bearing problem setting.

pressure affects the viscosity, the level of the pressure is conjugate to the whole solution, including the velocity field. Therefore, the system of PDEs has to be completed by an additional constraint fixing the level of pressure. Such constraint—usually attracting no particular attention—becomes an important part of the model.

In previous theoretical studies, such as [15, 19, 28], the mean value of the pressure over the domain (or its non-trivial subdomain) was prescribed as an input parameter. A difficulty of this approach lies in the fact that the pressure mean value is not a proper quantity from the practical point of view, i.e. there is no hint on what value should be prescribed for a particular application. In [29], we showed that the pressure level is fixed in a natural way in the case that suitable boundary conditions, allowing flow through the boundary, are given. In the present paper we keep this approach.

To motivate the model presented hereafter, let us advert to the following applications, all concerned with the flow of a lubricant inside a journal bearing. Both the shear-thinning and the pressure-thickening are of particular importance to the lubrication theory,[1] see e.g. [1, 5, 6, 22, 30, 40] or the book by Szeri [43] and the references given therein. The problem setting can follow several situations; we took the liberty to choose three examples.

Finite journal bearing

A three-dimensional setting is depicted in Fig. 1a. The fluid is enclosed in between the cylindrical journal and the inner surface of cylindrical bearing; the flow is induced by the rotation of the journal around its axis, while the bearing is held steady. Usually, the *no-slip* (non-homogeneous Dirichlet) boundary condition is prescribed on the solid surfaces:

$$\boldsymbol{v} = \boldsymbol{0} \;\text{ on } \Gamma_B \qquad \text{and} \qquad \boldsymbol{v} = \boldsymbol{v}_D\,,\; \boldsymbol{v}_D \cdot \boldsymbol{n} = 0 \;\text{ on } \Gamma_J, \tag{1.1a}$$

denoting by $\boldsymbol{v}$ the velocity of the fluid, $\boldsymbol{v}_D$ the velocity of the solid surface and $\boldsymbol{n}$ the unit outward normal vector to the boundary. The ends of the bearing are immersed in a lubricant pool and it

[1]The theory presented below, however, is far from aiming the full complexity of the journal bearing lubrication problem: we do not consider thermal effects, cavitation of the fluid, elastic response of the solid boundaries, to name some of the phenomena that are deliberately neglected.

is assumed that the entire area Ω between the cylinders is filled with the lubricant (such that no free-boundary is involved). The flow at the interface between the domain Ω and the reservoir can be approximated by prescribing

$$-\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n}+\tfrac{1}{2}\,|\boldsymbol{v}|^2=h \quad \text{ and } \quad -(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}}=\mathbf{0} \qquad \text{on } \Gamma_P. \tag{1.1b}$$

Here $\boldsymbol{w}_{\boldsymbol{n}}:=(\boldsymbol{w}\cdot\boldsymbol{n})\boldsymbol{n}$ and $\boldsymbol{w}_{\boldsymbol{\tau}}:=\boldsymbol{w}-\boldsymbol{w}_{\boldsymbol{n}}$ for any vector $\boldsymbol{w}$ defined on the boundary $\partial\Omega$. $\mathbf{T}$ stands for the Cauchy stress tensor, h is the value of *total pressure* at the boundary Γ_P. As will be shown later, $h\equiv h(\boldsymbol{x})$ determines the level of pressure in the resulting flow. Other formulas than $(1.1\text{b})_1$ could be considered[2].

Slip flow and a supply channel

In Fig. 1b, a two-dimensional setting (established as the long-bearing approximation) is illustrated. Here we relax the assumption of that the fluid adheres to the solid boundary and—instead of (1.1a)—we prescribe the non-homogeneous Navier slip boundary condition

$$\boldsymbol{v}\cdot\boldsymbol{n}=0 \quad \text{and} \quad -(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}}=\alpha\,(\boldsymbol{v}-\boldsymbol{v}_D)_{\boldsymbol{\tau}}\,,\ \alpha\geq 0 \qquad \text{on } \Gamma_J\cup\Gamma_B, \tag{1.2a}$$

$\boldsymbol{v}_D$ being again the velocity of the solid surface, $\boldsymbol{v}_D\cdot\boldsymbol{n}=0$. The assumption of slip or no-slip at the boundary is a complex issue in the continuum mechanics of viscous fluids and the precise circumstances determining the validity of these assumptions are subject to an unceasing concern, see e.g. [21, 34].

Boundary Γ_P approximates a thin channel through the bearing body (supplying the area by the lubricant); similarly as in the previous example we prescribe

$$-\,\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n}+\tfrac{1}{2}\,|\boldsymbol{v}|^2=h(\boldsymbol{x}) \quad \text{ and } \quad -(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}}=\alpha\,\boldsymbol{v}_{\boldsymbol{\tau}}\,,\ \alpha\geq 0 \qquad \text{on } \Gamma_P\,. \tag{1.2b}$$

Again, h represents the *total pressure* on Γ_P and determines the resulting pressure level.

Porous bearing

Slightly different setting is shown in Fig. 1c. Both the journal and the bearing bodies consist of a porous material, permitting some flow both through the surface and in the tangential direction, see e.g. [20, 38]. Here, avoiding to couple the flow in Ω with the flow in the porous media, we merely approximate the porous wall by a suitable boundary condition. Keeping the example even more simple, we imagine a reservoir with a known pressure $h(\boldsymbol{x})$ on the opposite side of the porous wall. We consider a boundary condition of the following type:

$$\left.\begin{aligned} -\,\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n} &= h(\boldsymbol{x})+(c_1+c_2\,|\boldsymbol{u}|+c_3\,|\boldsymbol{u}|^2)\,\boldsymbol{u}\cdot\boldsymbol{n}\,, && (1.3\text{a})\\ -\,(\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}} &= (\tilde{c}_1+\tilde{c}_2\,|\boldsymbol{u}|+\tilde{c}_3\,|\boldsymbol{u}|^2)\,\boldsymbol{u}_{\boldsymbol{\tau}}\,, && (1.3\text{b}) \end{aligned}\right\}\ \text{on } \Gamma_J\cup\Gamma_B,$$

[2]Let us note that a special assumption **(B2)** will be imposed (see page 6) in order to show the mathematical self-consistency of the model. It excludes for instance

$$-\mathbf{T}\boldsymbol{n}\cdot\boldsymbol{n}=h(\boldsymbol{x})$$

to be considered, see also [29]. Since Γ_P is an *artificial boundary*, the choice of proper boundary condition is not obvious in general, see e.g. the discussion in [23].

with $c_i, \tilde{c}_i \geq 0$, $i = 1, 2, 3$. Here $\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{v}_D$ denotes the velocity relative to the motion of the wall. The constants reflect the geometrical and permeability characteristics of the porous wall.

Eq. (1.3a) can be found in the literature as the *filtration boundary condition*, usually with $c_1 > 0$ and $c_2 = c_3 = 0$; this corresponds to the Darcy equation, which governs the flow in the porous media (see [10, 39]) under the assumption of small velocity—the assumption well met in the most of practical applications. Since the theory presented below is not restricted to slow flows, the boundary conditions have to reflect that as well. Due to **(B2)** (page 6), either $c_2, \tilde{c}_2 \geq \frac{1}{2}$ or $c_3 > 0$ will be required in (1.3) for our analysis. This is, however, perfectly consistent with the physics as these terms correspond to well estabished corrections to Darcy equation due to inertial effects, namely the Forchheimer equation, see [10, 18].

Similarly, the relation (1.3b) with only the linear term present is well known as *Beavers–Joseph(–Saffman–Jones) condition* for flows past porous media based on experimental observations, see [11, 25, 35, 41]. This is, again, used for relatively slow flows, see e.g. [37]. Here, the higher-order terms in (1.3b) appear consistently with (1.3a). Note that a permeable boundary condition complemented with the no-slip assumption in the tangential direction is sometimes formulated; that case is not covered in this paper, cf. [29].

Boundary conditions modelling the presence of porous wall, which would involve shear-thinning and pressure-thickening fluids as well as the inertial effects are, up to our best knowledge, not treated in the literature. Concerning the flow within the porous medium, we refer to the recent works [26, 39], where both the fluids with shear rate- and pressure-dependent viscosity are considered; but only slow velocities relative to the porous medium are assumed, however.

The assumption **(B2)**, requiring the presence of higher-order terms in $(1.2\text{b})_1$ and (1.3), allows to control the kinetic energy coming to the system due to flow through its boundary. In practical applications, the modeller might know a priori (e.g. due to a special setting of the problem) that the overall added kinetic energy is limited. In such case, **(B2)** seems not to be required any more.

The focus of this paper is in the question of mathematical self-consistency of the model describing the flow of a lubricant, namely the existence of a weak solution to the governing system of equations. The paper is organised as follows. In Section 2 we specify the governing equations, briefly review previous results and present the necessary tools. The existence of a weak solution—the main result of the paper—is then proved in Section 3. The uniqueness is finaly discussed in Section 4.

2. Definition of the problem and the main result

We investigate the steady flow of an incompressible homogeneous viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^d$ with the Lipschitz boundary, $d = 2$ or 3, governed by the following system of PDEs:

$$\left.\begin{aligned} \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \mathbf{T} &= \boldsymbol{f} \\ \operatorname{div} \boldsymbol{v} &= 0 \end{aligned}\right\} \qquad \text{in } \Omega, \tag{2.1}$$

where $\boldsymbol{v}$, $\boldsymbol{f}$, $\mathbf{T}$ is the velocity, the body force and the Cauchy stress tensor, respectively. The following constitutive relation is considered:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \text{where} \qquad \mathbf{S} \equiv \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) = 2\,\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)\mathbf{D}(\boldsymbol{v}), \tag{2.2}$$

with p the kinematic pressure, $\nu(p, |\mathbf{D}(\boldsymbol{v})|^2)$ the kinematic viscosity and $\mathbf{D}(\boldsymbol{v}) = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ the symmetric part of the velocity gradient.

While the mathematical consistency of models with shear rate-dependent viscosity—in particular those linked to the power-law model—have been studied systematically for some decades, a theory involving the pressure-thickening fluids has been developed only recently. For a thorough survey and more references see [32, 33] (for unsteady flows) and [19] (for steady flows), more recent accomplishments can be found in [15–17, 24]. The theory is based on the monotone operators approach and is bottomed on the structure of the viscous stress described below.

We use the following notation: $\mathrm{W}^{1,r}(\Omega)$, $\mathrm{L}^p(\Omega)$ for the Sobolev and the Lebesgue space, respectively, $\|\cdot\|_{1,r}$ and $\|\cdot\|_p$ for their standard norms. $\mathrm{L}^p_0(\Omega)$ denotes the subspace of $\mathrm{L}^p(\Omega)$ of functions with zero mean value. Bold symbols stand for their vector-valued analogues. The Hölder conjugate index is denoted $p' := \frac{p}{p-1}$, while $p^* := \frac{(d-1)p}{d-p}$ relates to the space of traces imbedding: $\mathrm{tr}(\mathbf{W}^{1,p}(\Omega)) \hookrightarrow \mathbf{L}^{p^*}(\partial\Omega)$. We often omit the trace operator, writing e.g. $\boldsymbol{v} = \boldsymbol{v}_D$ on $\partial\Omega$.

2.1. Structural assumptions

We consider $\mathbf{S}$ with the following properties:

(A1) For a given $r \in (1, 2)$, there are positive constants C_1 and C_2 such that for all $\mathbf{B}$, $\mathbf{D} \in \mathbb{R}^{d\times d}_{sym}$ and all $p \in \mathbb{R}$:

$$C_1(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2 \le \frac{\partial \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))}{\partial \mathbf{D}} \cdot (\mathbf{B} \otimes \mathbf{B}) \le C_2(1 + |\mathbf{D}|^2)^{\frac{r-2}{2}}|\mathbf{B}|^2 \,,$$

where $(\mathbf{B} \otimes \mathbf{B})_{ijkl} = \mathbf{B}_{ij}\mathbf{B}_{kl}$.

(A2) For all $\mathbf{D} \in \mathbb{R}^{d\times d}_{sym}$ and for all $p \in \mathbb{R}$:

$$\left|\frac{\partial \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))}{\partial p}\right| \le \gamma_0(1 + |\mathbf{D}|^2)^{\frac{r-2}{4}} \le \gamma_0 \,,$$

with $\gamma_0 > 0$ specified in $(3.1)_2$ below.

We state some implications of **(A1)** and **(A2)**. It was proven in [31, Lemma 1.19 of Chapter 5], that for every $p \in \mathbb{R}$ and $\mathbf{D} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}$:

$$|\mathbf{S}(p, \mathbf{D})| \le \frac{C_2}{r-1}(1 + |\mathbf{D}|)^{r-1} \,, \tag{2.3a}$$

$$\mathbf{S}(p, \mathbf{D}) : \mathbf{D} \ge \frac{C_1}{2r}(|\mathbf{D}|^r - 1) \,. \tag{2.3b}$$

Next, defining

$$I^{1,2} := |\mathbf{D}^1 - \mathbf{D}^2|^2 \int_0^1 \left(1 + |\mathbf{D}^1 + s(\mathbf{D}^2 - \mathbf{D}^1)|^2\right)^{\frac{r-2}{2}} \,\mathrm{d}s \,,$$

one can show that (see e.g. [15, Lemma 1.4])

$$\frac{C_1}{2} I^{1,2} \le \left(\mathbf{S}(p^1, \mathbf{D}^1) - \mathbf{S}(p^2, \mathbf{D}^2)\right) : (\mathbf{D}^1 - \mathbf{D}^2) + \frac{\gamma_0^2}{2C_1}|p^1 - p^2|^2 \,, \tag{2.4a}$$

$$|\mathbf{S}(p^1, \mathbf{D}^1) - \mathbf{S}(p^2, \mathbf{D}^2)| \le C_2\sqrt{I^{1,2}} + \gamma_0|p^1 - p^2| \,, \tag{2.4b}$$

$$\|1 + |\mathbf{D}^1| + |\mathbf{D}^2|\|_r^{r-2} \, \|\mathbf{D}^1 - \mathbf{D}^2\|_r^2 \le \int_\Omega I^{1,2} \,\mathrm{d}\boldsymbol{x} \tag{2.4c}$$

for all $p^1, p^2 \in \mathbb{R}$ and $\mathbf{D}^1, \mathbf{D}^2 \in \mathbb{R}^{d\times d}_{sym}$.

The structure of **(A1)** and **(A2)** manifests a dominant (for large $\mathbf{D}$ and p) shear-thinning behavior, while changes of the viscosity due to the pressure are restrained. This appears as a natural requirement of the approach we use; cf. [16].

Most of the engineering literature relies on the exponential pressure–viscosity relation by Barus [9]

$$\nu = \nu_0 \exp(\alpha_0 \, p), \qquad \nu_0,\ \alpha_0 > 0 \tag{2.5}$$

based on the experimental evidence; see [3, 13] or [4, 7, 8]. It is worth mentioning that no existence theory that would cover the viscosity of the above form is available at the moment. However, one can consider for example a model of the following type

$$\nu = \nu_0 \left(\nu_1 + \nu_2 |\mathbf{D}(\boldsymbol{v})|^2 + \exp\!\left(\tfrac{2\,\alpha_0}{r-2}\, p^+\right)\right)^{\frac{r-2}{2}}, \quad p^+ := \max(0, p)\,,$$

which approximates (2.5) in some limited range of parameters p, $\mathbf{D}(\boldsymbol{v})$. For suitable constants $\nu_i > 0$, $i = 0, 1, 2$, the latter is covered by **(A1)**–**(A2)**; see e.g. [32].

2.2. Boundary conditions

This paper generalizes the result by Franta et al. [19], which was formulated for flows subject to the homogeneous Dirichlet boundary condition; note that in such setting a non-trivial flow can only be induced by non-potential body forces $\boldsymbol{f}$. The non-homogeneous Dirichlet boundary condition was studied in [28]. In [29], the homogeneous Dirichlet and inflow/outflow conditions were considered and it was shown that such setting makes the level of pressure fixed. Here we show that the latter two studies can be extended to cover the three examples from Section 1.

Let the domain boundary consist of three parts: $\partial\Omega = \Gamma_D \cup \Gamma_N \cup \Gamma_P$, $|\Gamma_P| > 0$, on which we prescribe (cf. (1.1)–(1.3)):

$$\boldsymbol{v} = \boldsymbol{v}_D\,, \quad \boldsymbol{v}_D \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_D\,, \tag{2.6a}$$

$$\left.\begin{aligned} &\boldsymbol{v}\cdot\boldsymbol{n} = 0 \\ &(-\mathbf{T}\boldsymbol{n})_{\boldsymbol{\tau}} = \alpha\,(\boldsymbol{v} - \boldsymbol{v}_D)_{\boldsymbol{\tau}}\,, \quad \alpha \ge 0 \end{aligned}\right\} \qquad \text{on } \Gamma_N\,, \tag{2.6b}$$

$$-\mathbf{T}\boldsymbol{n} = \boldsymbol{g}(\boldsymbol{v} - \boldsymbol{v}_D) \qquad \text{on } \Gamma_P\,. \tag{2.6c}$$

The following assumptions concerning[3] (2.6a) and (2.6c) are made:

(B1) There exists a constant $\gamma \ge 3$ such that the mapping $\boldsymbol{g}(\cdot) : \mathbf{L}^\gamma(\Gamma_P) \to \mathbf{L}^{\gamma'}(\Gamma_P)$ is continuous and bounded.

(B2) With some[4] $B_1, B_2 \in \mathbb{R}$ and $B_c \ge 0$,

$$\langle \boldsymbol{g}(\boldsymbol{u}), \boldsymbol{u}\rangle_{\Gamma_P} \ge -\frac{1}{2}\int_{\Gamma_P} (\boldsymbol{u}\cdot\boldsymbol{n})|\boldsymbol{u} + \boldsymbol{v}_D|^2 \,\mathrm{d}\boldsymbol{x} - B_1 \|\boldsymbol{u}\|_{r^*,\Gamma_P} - B_2 + B_c \|\boldsymbol{u}\|^\gamma_{\gamma,\Gamma_P} \tag{2.7}$$

for all $\boldsymbol{u} \in \mathbf{L}^\gamma(\Gamma_P) \cap \mathbf{L}^{r^*}(\Gamma_P)$; remind that $r^* := \frac{(d-1)r}{d-r}$. Moreover, if $\gamma > r^*$ then we require the coercivity: $B_c > 0$.

[3]We could also consider more general form of $(2.6\text{b})_2$ and pose assumptions analogous to **(B1)-(B3)**. For simplicity of notation we confine ourselves to the Navier slip $(2.6\text{b})_2$.

[4]Terms with B_1, B_2 represent any terms of lower order than $\|\boldsymbol{u}\|^r_{1,r}$. Similarly, $B_c \|\boldsymbol{u}\|^\gamma_{\gamma,\Gamma_P}$ could be replaced by any coercive function of $\|\boldsymbol{u}\|_{\gamma,\Gamma_P}$. See also Lemma 2.3.

(B3) If $\gamma \geq r^*$, then $\boldsymbol{g}$ is uniformly[5] monotone:

$$\langle \boldsymbol{g}(\boldsymbol{w}) - \boldsymbol{g}(\boldsymbol{z}), \boldsymbol{w} - \boldsymbol{z} \rangle_{\Gamma_P} \geq m(\|\boldsymbol{w} - \boldsymbol{z}\|_{\gamma,\Gamma_P}) \,, \tag{2.8}$$

for all $\boldsymbol{w} \neq \boldsymbol{z} \in \mathbf{L}^\gamma(\Gamma_P)$. Here $m : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function such that $\lim\limits_{x \searrow 0} m(x) = 0$.

(BD) The function $\boldsymbol{v}_D$ in (2.6) is realized by the trace of a function $\boldsymbol{u}_0$ with the following properties:

$$\boldsymbol{u}_0 \in \mathbf{W}^{1,r}(\Omega) \cap \mathbf{L}^\infty(\Omega) \,, \qquad \operatorname{div} \boldsymbol{u}_0 = 0 \quad \text{a.e. in } \Omega,$$
$$\boldsymbol{u}_0 = \boldsymbol{v}_D \,, \; \boldsymbol{v}_D \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega.$$

In the existence proof we will need[6]a specific extension of $\boldsymbol{v}_D$, based on the following lemma, proved in [28, Lemma 3 and Corollary 4].

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a domain with Lipschitz boundary, $r > 2 - \frac{1}{d}$. Let $\boldsymbol{\Phi} \in \mathbf{W}^{1,r}(\Omega) \cap \mathbf{L}^\infty(\Omega)$, $\boldsymbol{\Phi} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Then for each $H > 0$ there exists $\lambda_H \geq 1$ and $\boldsymbol{\Phi}^H \in \mathbf{W}^{1,r}(\Omega)$ such that*

$$\begin{aligned}
\operatorname{div} \boldsymbol{\Phi}^H &= 0 \text{ a.e. in } \Omega, \\
\operatorname{tr} \boldsymbol{\Phi}^H &= \operatorname{tr} \boldsymbol{\Phi} \text{ on } \partial\Omega, \\
\|\boldsymbol{\Phi}^H\|_{1,r} &\leq H\lambda_H, \\
\|\boldsymbol{\Phi}^H\|_q &\leq H\lambda_H^{r-2} \,, \qquad q = \tfrac{dr}{(d+1)r-2d} \,.
\end{aligned}$$

2.3. Weak formulation

We define the following function space

$$\mathbf{W}^{1,r}_{\text{b.c.}}(\Omega) := \Big\{ \boldsymbol{u} \in \mathbf{W}^{1,r}(\Omega) \,;\; \operatorname{tr} \boldsymbol{u} \big|_{\Gamma_D} = \mathbf{0} \,,\; \operatorname{tr} \boldsymbol{u} \cdot \boldsymbol{n} \big|_{\Gamma_N} = 0 \,,\; \operatorname{tr} \boldsymbol{u} \big|_{\Gamma_P} \in \mathbf{L}^\gamma(\Gamma_P) \Big\}$$

and for all $\boldsymbol{u}, \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$ we write:

$$\langle \boldsymbol{b}(\boldsymbol{u}), \boldsymbol{\varphi} \rangle := \alpha \int_{\Gamma_N} \boldsymbol{u}_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi}_{\boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g}(\boldsymbol{u}), \boldsymbol{\varphi} \rangle_{\Gamma_P} .$$

Given $\boldsymbol{f} \in \mathbf{W}^{1,r}(\Omega)^*$, we consider the following weak formulation:

Definition 2.2 (Problem (P)). A pair $(\boldsymbol{v}, p)$ is said to be a weak solution of Problem **(P)** if and only if $\boldsymbol{u} := (\boldsymbol{v} - \boldsymbol{u}_0) \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$, $p \in \mathrm{L}^{r'}(\Omega)$, $\operatorname{div} \boldsymbol{v} = 0$ a.e. in Ω and

$$\int_\Omega \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \int_\Omega \mathbf{S}(p, \mathbf{D}(\boldsymbol{v})) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} - \int_\Omega p \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{u}), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \tag{2.9}$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$.

[5]For the sake of simplicity, the uniform monotonicity is assumed here. The readers can verify themselves, that the monotonicity of $\boldsymbol{g}$ would also allow to show the existence of a weak solution, with help of the Minty trick.

[6]The procedure of finding a suitable extension of $\boldsymbol{v}_D$ and the splitting $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{u}_0$ used below is, in fact, necessary only in order to handle the Dirichlet boundary condition, i.e. only if $|\Gamma_D| > 0$. Indeed, assumption **(BD)** can be reformulated requiring only $\boldsymbol{u}_0 = \boldsymbol{v}_D$ on Γ_D, with $\boldsymbol{u}_0$ arbitrary on $\Gamma_N \cup \Gamma_P$. But then, $\boldsymbol{v} - \boldsymbol{v}_D$ in (2.6b) and (2.6c) would not be the same as $\boldsymbol{u}$ but, instead, would become $\boldsymbol{u} + \boldsymbol{u}_0 - \boldsymbol{v}_D$, cf. (2.9). We prefer the above formulation of **(BD)** so as to simplify the notation in what follows.

Note that, in accordance with [19, 28], we will make the restriction $r > \frac{3d}{d+2}$ (see $(3.1)_1$ below), so that the equation (namely the convective term) can be tested by the solution; due to (2.3a), **(B1)** and **(BD)**, all integrals in (2.9) are finite. The previous theory has been extended to include smaller values of r, see [14] (and the short note [27], eventually). Note that $r > \frac{3d}{d+2} \Leftrightarrow r^* > 3$.

Before stating the main theorem, we present the following variant of the Korn inequality:

Lemma 2.3 (Korn's inequality). *Remind that $\alpha, B_c \geq 0$ and $\gamma \geq 3$. Let at least one of the following apply:*

i) $|\Gamma_D| > 0$,
ii) $|\Gamma_N| > 0$ *and* Γ_N *is not a part of boundary of any rotational body in* $\mathbb{R}^d$,
iii) $|\Gamma_N| > 0$ *and* $\alpha > 0$,
iv) $|\Gamma_P| > 0$ *and* $B_c > 0$.

Then, with some $c_K \equiv c_K(\Omega, \Gamma_D, \Gamma_N, \Gamma_P, r)$, the following inequality holds for any $\boldsymbol{u} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$:

$$\|\boldsymbol{u}\|_{1,r}^r \leq c_K \|\mathbf{D}(\boldsymbol{u})\|_r^r + \alpha\, \|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{2,\Gamma_N}^2 + B_c \|\boldsymbol{u}\|_{\gamma,\Gamma_P}^\gamma\,. \tag{2.10}$$

Proof. The case i) with $\Gamma_D = \partial\Omega$, namely the inequality

$$c(\Omega, p)\, \|\boldsymbol{u}\|_{1,p} \leq \|\mathbf{D}(\boldsymbol{u})\|_p\,, \quad \text{for any } \boldsymbol{u} \in \mathbf{W}^{1,p}_0(\Omega),\ p \in (1, +\infty),$$

can be found e.g. in [31, Theorem 1.10 on p. 196]. Its proof, in fact, covers even i) and ii) as formulated above; it is merely to notice that a vector field of the form $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x}$ contradicts $\|\boldsymbol{u}\|_p = 1$ under either of the assumptions $\boldsymbol{u} = \mathbf{0}$ on Γ_D, or $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on Γ_N, with Γ_D, Γ_N as above. The inequality

$$c(\Omega, p)\, \|\boldsymbol{u}\|_{1,p} \leq \|\mathbf{D}(\boldsymbol{u})\|_p + \|\boldsymbol{u}\|_{2,\partial\Omega}\,, \quad \text{for any } \boldsymbol{u} \in \mathbf{W}^{1,p}(\Omega),\ p \in (1, +\infty),$$

is then stated e.g. in [15, Lemma 1.11], but its proof again covers[7] also iii) and iv).

We also recall some properties of the Bogovskii operator (see [36, Lemma 3.17] or [2, 12]) and state its corollary (proved in [29]).

Lemma 2.4 (Bogovskii's operator). *Let $t \in (1, \infty)$. Then there exists a continuous linear operator $\mathcal{B} : \mathrm{L}^t_0(\Omega) \to \mathbf{W}^{1,t}_0(\Omega)$ such that for all $f \in \mathrm{L}^t_0(\Omega)$:*

$$\left.\begin{aligned} \operatorname{div}(\mathcal{B}f) &= f \quad \text{a.e. in } \Omega, \\ \|\mathcal{B}f\|_{1,t} &\leq C_{\operatorname{div}}(\Omega, t) \|f\|_t. \end{aligned}\right\} \tag{2.11}$$

Lemma 2.5. *Let $t \in (1, \infty)$, $s \in \langle 1, \infty \rangle$, and $|\Gamma_P| > 0$. Then there exists a continuous bounded linear operator $\widetilde{\mathcal{B}} : \mathrm{L}^t(\Omega) \to \mathbf{W}^{1,t}_{\text{b.c.}}(\Omega)$ such that for all $f \in \mathrm{L}^t(\Omega)$:*

$$\left.\begin{aligned} \operatorname{div}(\widetilde{\mathcal{B}}f) &= f \quad \text{a.e. in } \Omega, \\ \|\widetilde{\mathcal{B}}f\|_{1,t} &\leq \widetilde{C}_{\operatorname{div}}(\Omega, \Gamma_P, t) \|f\|_t, \\ \|\widetilde{\mathcal{B}}f\|_{s,\Gamma_P} &\leq C'_{\operatorname{div}}(\Omega, \Gamma_P, s) \left| \textstyle\int_\Omega f \right|. \end{aligned}\right\} \tag{2.12}$$

[7] And, for any $|\Gamma| > 0$, $\Gamma \subset \partial\Omega$ not lying on boundary of any rotational body, one can also see that $c(\Omega, \Gamma, p)\, \|\boldsymbol{u}\|_{1,p} \leq \|\mathbf{D}(\boldsymbol{u})\|_p + \|\boldsymbol{u} \cdot \boldsymbol{n}\|_{2,\Gamma}$.

3. Existence of a weak solution

Theorem 3.1 (Well-posedness of (P)). *Let* $\boldsymbol{f} \in \mathbf{W}^{1,r}(\Omega)^*$ *and* **(A1)**, **(A2)** *hold for the viscosity,* **(B1)**–**(B3)** *and* **(BD)** *hold for the boundary data, with*

$$\frac{3d}{d+2} < r < 2 \qquad \textit{and} \qquad \gamma_0 < \frac{1}{\widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2)} \frac{C_1}{C_1 + C_2} \tag{3.1}$$

and the Korn's inequality (2.10) *hold. Then there exists a weak solution* $(\boldsymbol{v}, p) := (\boldsymbol{u}+\boldsymbol{u}_0, p)$ *to* ***(P)****. Moreover, the pressure is uniquely determined by the velocity.*

The basic structure of the proof derives from that by Franta et al. [19]: In 3.1, we define an approximate problem $(\mathbf{P}^\varepsilon)$, establish the energy estimates and show the existence of a weak solution to $(\mathbf{P}^\varepsilon)$ via Galerkin approximations. In 3.2, we show estimates for the pressure p^ε which are uniform with respect to ε and find the sequences $\varepsilon_n \searrow 0$, $\{(\boldsymbol{v}^{\varepsilon_n}, p^{\varepsilon_n})\}$ weakly converging to a limit $(\boldsymbol{v}, p)$. In 3.3, the strong convergence of p^{ε_n} and $\mathbf{D}(\boldsymbol{v}^{\varepsilon_n})$ is shown and $(\boldsymbol{v}, p)$ is identified as the weak solution to problem **(P)**.

3.1. Approximate problem $(\boldsymbol{P}^\varepsilon)$

We relax the incompressibility constraint and look for a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon) := (\boldsymbol{u}^\varepsilon + \boldsymbol{u}_0, p^\varepsilon)$ such that $(\boldsymbol{u}^\varepsilon, p^\varepsilon) \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) \times \mathrm{W}^{1,2}(\Omega)$, $\varepsilon > 0$, satisfying

$$\varepsilon \int_\Omega \nabla p^\varepsilon \cdot \nabla \xi \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^\varepsilon \, \xi \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon) \xi \, \mathrm{d}\boldsymbol{x} = 0 \quad \text{for all } \xi \in \mathrm{W}^{1,2}(\Omega) , \tag{3.2a}$$

together with

$$\int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \frac{1}{2} \int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon) \, \boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{u}^\varepsilon), \boldsymbol{\varphi} \rangle = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega) . \tag{3.2b}$$

Due to (2.3a), **(B1)**, **(BD)** and $(3.1)_1$, all integrals are finite. Note that (contrary to the case studied in [19]) equation (3.2a) does not determine the mean value of the pressure $\fint_\Omega p^\varepsilon \, \mathrm{d}\boldsymbol{x}$, since $\boldsymbol{v}^\varepsilon \cdot \boldsymbol{n}|_{\Gamma_P}$ is not prescribed a priori.

We show that $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ can be found as a limit of the Galerkin approximations $(\boldsymbol{v}^N, p^N)$ defined as follows:

$$p^N := \sum_{k=1}^N c_k^N \alpha_k \qquad \text{and} \qquad \boldsymbol{v}^N := \boldsymbol{u}_0 + \boldsymbol{u}^N, \; \boldsymbol{u}^N := \sum_{k=1}^N d_k^N \boldsymbol{a}_k \qquad \text{for } N = 1, 2, \ldots ,$$

where $\{\alpha_k\}_{k=1}^\infty$ and $\{\boldsymbol{a}_k\}_{k=1}^\infty$ are bases of $\mathrm{W}^{1,2}(\Omega)$ and $\mathbf{W}^{1,r}_{\mathrm{b.c.}}(\Omega)$, respectively, and where $\boldsymbol{c}^N := (c_1^N, \ldots, c_N^N)$ and $\boldsymbol{d}^N := (d_1^N, \ldots, d_N^N)$ solve the equation

$$P(\boldsymbol{c}^N, \boldsymbol{d}^N) = \mathbf{0} . \tag{3.3}$$

The mapping $P : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ is defined as follows:

$$P_k(\boldsymbol{c}^N, \boldsymbol{d}^N) := \varepsilon \int_\Omega \nabla p^N \cdot \nabla \alpha_k \, \mathrm{d}\boldsymbol{x} + \varepsilon \int_\Omega p^N \, \alpha_k \, \mathrm{d}\boldsymbol{x} + \int_\Omega (\operatorname{div} \boldsymbol{v}^N) \alpha_k \, \mathrm{d}\boldsymbol{x} , \qquad k = 1, \ldots, N , \tag{3.4a}$$

$$P_{N+l}(\boldsymbol{c}^N,\boldsymbol{d}^N) := \int_\Omega \operatorname{div}(\boldsymbol{v}^N\otimes\boldsymbol{v}^N)\cdot\boldsymbol{a}_l\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega(\operatorname{div}\boldsymbol{v}^N)\,\boldsymbol{u}^N\cdot\boldsymbol{a}_l\,\mathrm{d}\boldsymbol{x} - \int_\Omega p^N\operatorname{div}(\boldsymbol{a}_l)\,\mathrm{d}\boldsymbol{x}$$
$$+\int_\Omega \mathbf{S}(p^N,\mathbf{D}(\boldsymbol{v}^N)):\mathbf{D}(\boldsymbol{a}_l)\,\mathrm{d}\boldsymbol{x} + \langle\boldsymbol{b}(\boldsymbol{u}^N),\boldsymbol{a}_l\rangle - \langle\boldsymbol{f},\boldsymbol{a}_l\rangle\,,\qquad l=1,\dots,N\,. \tag{3.4b}$$

We realize that

$$P(\boldsymbol{c}^N,\boldsymbol{d}^N)\cdot(\boldsymbol{c}^N,\boldsymbol{d}^N) = \varepsilon\|p^N\|_{1,2}^2 + \overbrace{\int_\Omega \operatorname{div}(\boldsymbol{v}^N\otimes\boldsymbol{v}^N)\cdot\boldsymbol{u}^N\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega(\operatorname{div}\boldsymbol{v}^N)\,|\boldsymbol{u}^N|^2\,\mathrm{d}\boldsymbol{x}}^{=:I_{\mathrm{conv}}}$$
$$+\int_\Omega \mathbf{S}(p^N,\mathbf{D}(\boldsymbol{v}^N)):\mathbf{D}(\boldsymbol{u}^N)\,\mathrm{d}\boldsymbol{x} + \langle\boldsymbol{b}(\boldsymbol{u}^N),\boldsymbol{u}^N\rangle - \langle\boldsymbol{f},\boldsymbol{u}^N\rangle\,. \tag{3.5}$$

Using Green's theorem, we observe:

$$I_{\mathrm{conv}} = \frac{1}{2}\int_{\Gamma_P}(\boldsymbol{u}^N\cdot\boldsymbol{n})|\boldsymbol{v}^N|^2\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_{\Gamma_P}(\boldsymbol{u}^N\cdot\boldsymbol{n})|\boldsymbol{v}_D|^2\,\mathrm{d}\boldsymbol{x} - \int_\Omega(\boldsymbol{v}^N\otimes\boldsymbol{u}_0):\nabla\boldsymbol{u}^N\,\mathrm{d}\boldsymbol{x}. \tag{3.6}$$

Due to the imbeddings and trace operator properties and $r>\frac{3d}{d+2}$

$$\int_\Omega|\boldsymbol{v}^N||\boldsymbol{u}_0||\nabla\boldsymbol{u}^N|\,\mathrm{d}\boldsymbol{x} \le c_1\,(\|\boldsymbol{u}^N\|_{1,r}^2\|\boldsymbol{u}_0\|_q + \|\boldsymbol{u}^N\|_{1,r}\|\boldsymbol{u}_0\|_q^2)\,,\qquad \text{with } q:=\tfrac{dr}{(d+1)r-2d},$$
$$\frac{1}{2}\int_{\Gamma_P}|\boldsymbol{u}^N||\boldsymbol{v}_D|^2\,\mathrm{d}\boldsymbol{x} \le \frac{1}{2}\|\boldsymbol{u}^N\|_{r^*,\Gamma_P}\|\boldsymbol{v}_D\|^2_{\frac{2r^*}{r^*-1},\Gamma_P} \le c_2\,\|\boldsymbol{u}^N\|_{1,r}\,,$$

with c_1 and c_2 depending only on Ω, r and $\boldsymbol{v}_D$. From **(B2)** we obtain (with $c_3\equiv c_3(B_1,B_2,\Omega,r)$)

$$\frac{1}{2}\int_{\Gamma_P}(\boldsymbol{u}^N\cdot\boldsymbol{n})|\boldsymbol{v}^N|^2\,\mathrm{d}\boldsymbol{x} + \langle\boldsymbol{b}(\boldsymbol{u}^N),\boldsymbol{u}^N\rangle \ge \alpha\,\|\boldsymbol{u}_{\boldsymbol{\tau}}\|_{2,\Gamma_N}^2 + B_c\,\|\boldsymbol{u}^N\|_{\gamma,\Gamma_P}^\gamma - c_3\,(\|\boldsymbol{u}^N\|_{1,r}+1)\,.$$

Using (2.3a) and (2.3b), we observe

$$\int_\Omega \mathbf{S}(p^N,\mathbf{D}(\boldsymbol{v}^N)):\mathbf{D}(\boldsymbol{u}^N)\,\mathrm{d}\boldsymbol{x} \ge \frac{C_1}{2r}\|\mathbf{D}(\boldsymbol{u}^N)\|_r^r - c_4\,(\|\boldsymbol{u}_0\|_{1,r}^r + \|\boldsymbol{u}_0\|_{1,r}\|\boldsymbol{u}^N\|_{1,r}^{r-1} + \|\boldsymbol{u}_0\|_{1,r}^{r-1}\|\boldsymbol{u}^N\|_{1,r}+1)\,,$$

the constant c_4 depending on C_1,C_2,Ω,r. Finally, using Korn's inequality (2.10), we conclude

$$P(\boldsymbol{c}^N,\boldsymbol{d}^N)\cdot(\boldsymbol{c}^N,\boldsymbol{d}^N) \ge \varepsilon\|p^N\|_{1,2}^2 + \tfrac{1}{2}\,B_c\|\boldsymbol{u}^N\|_{\gamma,\Gamma_P}^\gamma + d\,\|\boldsymbol{u}^N\|_{1,r}^r - c_5\,\big(\|\boldsymbol{u}^N\|_{1,r}+1\big)$$
$$- c_6\,\Big(\|\boldsymbol{u}_0\|_q\|\boldsymbol{u}^N\|_{1,r}^2 + \|\boldsymbol{u}_0\|_q^2\|\boldsymbol{u}^N\|_{1,r} + \|\boldsymbol{u}_0\|_{1,r}^r + \|\boldsymbol{u}_0\|_{1,r}\|\boldsymbol{u}^N\|_{1,r}^{r-1} + \|\boldsymbol{u}_0\|_{1,r}^{r-1}\|\boldsymbol{u}^N\|_{1,r}\Big)\,,$$

Note that d, c_5 and c_6 depend neither on N, ε nor on the choice of the extension $\boldsymbol{u}_0$. The difficultylies in the negative term involving $\|\boldsymbol{u}^N\|_{1,r}^2$, while the positive term $d\,\|\boldsymbol{u}^N\|_{1,r}^r$ is of lower order. Therefore, we set $E>0$ (large enough) and $H>0$ (small enough) arbitrary numbers satisfying

$$\tfrac{d}{2}\,e^r - c_5\,(e+1)\ge 0\quad\text{for any } e\ge E, \tag{3.7}$$
$$\tfrac{d}{2}\,E^r - c_6\,\underbrace{\left(HE^2+H^2E+H^r+HE^{r-1}+H^{r-1}E\right)}_{=:F}\ge 0 \tag{3.8}$$

From Lemma 2.1 it follows that there is a function $\boldsymbol{\Phi}^H$ that can be used instead of $\boldsymbol{u}_0$, satisfying $\|\boldsymbol{\Phi}^H\|_{1,r}\le H\lambda_H$ and $\|\boldsymbol{\Phi}^H\|_q\le H\lambda_H^{r-2}$ for certain $\lambda_H\ge 1$. In the sequel we use this function and denote it again by $\boldsymbol{u}_0$. We distinguish two cases:

(i) If $\|\boldsymbol{u}^N\|_{1,r} = E\lambda_H$, then we have:

$$P(\boldsymbol{c}^N,\boldsymbol{d}^N)\cdot(\boldsymbol{c}^N,\boldsymbol{d}^N) \geq \varepsilon\|p^N\|_{1,2}^2 + \tfrac{1}{2}B_c\,\|\boldsymbol{u}^N\|_{\gamma,\Gamma_P}^{\gamma} + \tfrac{d}{2}\,(E\lambda_H)^r - c_5\,(E\lambda_H+1) + \lambda_H^r\,\left(\tfrac{d}{2}\,E^r - c_6\,F\right) \geq 0\,.$$

(ii) If $\|\boldsymbol{u}^N\|_{1,r} < E\lambda_H$, then

$$P(\boldsymbol{c}^N,\boldsymbol{d}^N)\cdot(\boldsymbol{c}^N,\boldsymbol{d}^N) \geq \varepsilon\,\|p^N\|_{1,2}^2 + \tfrac{1}{2}B_c\,\|\boldsymbol{u}^N\|_{\gamma,\Gamma_P}^{\gamma} - c_5\,(E\lambda_H+1) - \lambda_H^r\,c_6\,F\,.$$

There certainly exists a number $G>0$ such that the last expression is non-negative provided that $\varepsilon\,\|p^N\|_{1,2}^2 \geq G$.

From (i) and (ii) we see that

$$P(\boldsymbol{c}^N,\boldsymbol{d}^N)\cdot(\boldsymbol{c}^N,\boldsymbol{d}^N) \geq 0$$

for any $(\boldsymbol{c}^N,\boldsymbol{d}^N)\in\mathbb{R}^{2N}$ such that $\max\{\frac{\|\boldsymbol{u}^N\|_{1,r}}{E\lambda_H},\frac{\varepsilon\,\|p^N\|_{1,2}^2}{G}\} = 1$. Consequently, due to the Brouwer theorem, there exists a solution $(\boldsymbol{c}^N,\boldsymbol{d}^N)$ to (3.3) such that

$$\varepsilon\|p^N\|_{1,2}^2 + \|\boldsymbol{u}^N\|_{\gamma,\Gamma_P} + \|\boldsymbol{u}^N\|_{1,r} \leq C\,. \tag{3.9}$$

Note that either $\gamma<r^*$, allowing to use $\mathbf{W}^{1,r}(\Omega)\hookrightarrow\mathbf{L}^{\gamma}(\Gamma_P)$, or $B_c>0$. Here and in what follows, $C>0$ stands for a generic constant, independent of N and ε. Using (2.3a) we obtain the estimate

$$\|\mathbf{S}(p^N,\mathbf{D}(\boldsymbol{v}^N))\|_{r'} \leq C. \tag{3.10}$$

Due to (3.9), (3.10) and the boundedness of $\boldsymbol{g}$, there is a subsequence of $\{(\boldsymbol{v}^N,p^N)\}$ (denoted by the same symbol) and a pair $(\boldsymbol{v}^\varepsilon,p^\varepsilon)$ such that

$$\boldsymbol{u}^N \rightharpoonup \boldsymbol{u}^\varepsilon =: \boldsymbol{v}^\varepsilon - \boldsymbol{u}_0 \qquad \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^{\gamma}(\Gamma_P)\,, \tag{3.11a}$$

$$p^N \rightharpoonup p^\varepsilon \qquad \text{weakly in } \mathrm{W}^{1,2}(\Omega)\,, \tag{3.11b}$$

$$\mathbf{S}(p^N,\mathbf{D}(\boldsymbol{v}^N)) \rightharpoonup \overline{\mathbf{S}^\varepsilon} \qquad \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d\times d}\,, \tag{3.11c}$$

$$\boldsymbol{g}(\boldsymbol{u}^N) \rightharpoonup \overline{\boldsymbol{g}^\varepsilon} \qquad \text{weakly in } \mathbf{L}^{\gamma'}(\Gamma_P)\,,\ N\to\infty. \tag{3.11d}$$

Moreover, the compact embeddings yield:

$$\boldsymbol{u}^N \to \boldsymbol{u}^\varepsilon \qquad \text{strongly in } \mathbf{L}^{s}(\Omega) \text{ for all } s\text{: } 1\leq s<\tfrac{rd}{d-r}\,, \tag{3.12a}$$

$$p^N \to p^\varepsilon \qquad \text{strongly in } \mathrm{L}^{2}(\Omega)\,,\ N\to\infty. \tag{3.12b}$$

If $\gamma<r^*$, **(B1)**, (3.11a) and the compact imbedding give immediately

$$\boldsymbol{g}(\boldsymbol{u}^N) \to \boldsymbol{g}(\boldsymbol{u}^\varepsilon) \qquad \text{strongly in } \mathbf{L}^{\gamma'}(\Gamma_p)\,,\ N\to\infty.$$

In the following, we will treat only the case $\gamma\geq r^*$. The fact that $r>\frac{3d}{d+2}$, (3.11a) and (3.12a) are sufficient to show that

$$\int_\Omega \operatorname{div}(\boldsymbol{v}^N\otimes\boldsymbol{v}^N)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega(\operatorname{div}\boldsymbol{v}^N)\,\boldsymbol{u}^N\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} \longrightarrow \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon\otimes\boldsymbol{v}^\varepsilon)\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega(\operatorname{div}\boldsymbol{v}^\varepsilon)\,\boldsymbol{u}^\varepsilon\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}\,, \qquad N\to\infty,$$

for all $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$. Thus, we can pass to the limit in the Galerkin system (3.3) and obtain (3.2a) together with

$$\int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon)\, \boldsymbol{u}^\varepsilon \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} - \int_\Omega p^\varepsilon \operatorname{div} \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x}$$
$$+ \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} + \alpha \int_{\Gamma_N} \boldsymbol{u}^\varepsilon_{\boldsymbol{\tau}} \cdot \boldsymbol{\varphi} \, \mathrm{d}\boldsymbol{x} + \langle \overline{\boldsymbol{g}^\varepsilon}, \boldsymbol{\varphi} \rangle_{\Gamma_P} = \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)\,. \quad (3.13)$$

Similarly as in [19], we prove the strong convergence of $\boldsymbol{u}^N$. From (2.4a) and (2.4c) with $(p^1, p^2, \mathbf{D}^1, \mathbf{D}^2) := (p^N, p^\varepsilon, \mathbf{D}(\boldsymbol{v}^N), \mathbf{D}(\boldsymbol{v}^\varepsilon))$, it follows that

$$\frac{C_1}{2} \|\mathbf{D}(\boldsymbol{u}^N) - \mathbf{D}(\boldsymbol{u}^\varepsilon)\|_r^2 \le \int_\Omega [\mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) - \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))] : (\mathbf{D}(\boldsymbol{v}^N) - \mathbf{D}(\boldsymbol{v}^\varepsilon)) \, \mathrm{d}\boldsymbol{x}$$
$$+ \frac{\gamma_0^2}{2C_1} \|p^N - p^\varepsilon\|_2^2\,. \qquad (3.14)$$

Using (3.14), (2.8), (3.11) and (3.12b) we observe:

$$\limsup_{N\to\infty} \left(\frac{C_1}{2} \|\mathbf{D}(\boldsymbol{u}^N) - \mathbf{D}(\boldsymbol{u}^\varepsilon)\|_r^2 + m(\|\boldsymbol{u}^N - \boldsymbol{u}^\varepsilon\|_{\gamma,\Gamma_P}) \right)$$
$$\le \limsup_{N\to\infty} \left(\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{v}^N) \, \mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g}(\boldsymbol{u}^N), \boldsymbol{u}^N \rangle_{\Gamma_P} \right) - \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{v}^\varepsilon) \, \mathrm{d}\boldsymbol{x} - \langle \overline{\boldsymbol{g}^\varepsilon}, \boldsymbol{u}^\varepsilon \rangle.$$

This can be further estimated from above, with help of (3.3), (3.12), weak lower semi-continuity of $\|p^N\|_{1,2}$ and (3.13), by

$$\langle \boldsymbol{f}, \boldsymbol{u}^\varepsilon \rangle - \varepsilon \|p^\varepsilon\|_{1,2}^2 - \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon \otimes \boldsymbol{v}^\varepsilon) \cdot \boldsymbol{u}^\varepsilon \, \mathrm{d}\boldsymbol{x} + \frac{1}{2}\int_\Omega (\operatorname{div} \boldsymbol{v}^\varepsilon) |\boldsymbol{u}^\varepsilon|^2 \, \mathrm{d}\boldsymbol{x}$$
$$- \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{u}^\varepsilon) \, \mathrm{d}\boldsymbol{x} - \langle \overline{\boldsymbol{g}^\varepsilon}, \boldsymbol{u}^\varepsilon \rangle_{\Gamma_P} = 0\,.$$

Therefore, and due to (3.12b), there hold the almost everywhere convergence

$$\mathbf{D}(\boldsymbol{u}^N) \to \mathbf{D}(\boldsymbol{u}^\varepsilon) \text{ a.e. in } \Omega, \qquad \boldsymbol{u}^N \to \boldsymbol{u}^\varepsilon \text{ a.e. on } \Gamma_P$$
$$\text{and} \qquad p^N \to p^\varepsilon \text{ a.e. in } \Omega, \qquad N \to \infty.$$

Vitali's theorem and the continuity of $\boldsymbol{g}$ allow us to identify the limits as follows:

$$\int_\Omega \mathbf{S}(p^N, \mathbf{D}(\boldsymbol{v}^N)) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} \to \int_\Omega \mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon)) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} = \int_\Omega \overline{\mathbf{S}^\varepsilon} : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x}\,,$$
$$\langle \boldsymbol{g}(\boldsymbol{u}^N), \boldsymbol{\varphi} \rangle_{\Gamma_P} \to \langle \boldsymbol{g}(\boldsymbol{u}^\varepsilon), \boldsymbol{\varphi} \rangle_{\Gamma_P} = \langle \overline{\boldsymbol{g}^\varepsilon}, \boldsymbol{\varphi} \rangle_{\Gamma_P}\,, \qquad N \to \infty,$$

for every $\boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\text{b.c.}}(\Omega)$.

3.2. Uniform estimates for the pressure p^ε and weak convergence

We proved that for every $\varepsilon > 0$ there is a pair $(\boldsymbol{v}^\varepsilon, p^\varepsilon)$ which satisfies (3.2) and the following estimates:

$$\varepsilon \|p^\varepsilon\|_{1,2}^2 + \|\boldsymbol{u}^\varepsilon\|_{\gamma,\Gamma_P} + \|\boldsymbol{u}^\varepsilon\|_{1,r} + \|\mathbf{S}(p^\varepsilon, \mathbf{D}(\boldsymbol{v}^\varepsilon))\|_{r'} \le C\,. \qquad (3.15)$$

Let us recall Lemma 2.5 and test (3.2b) with $\boldsymbol{\varphi}^\varepsilon := \widetilde{\mathcal{B}}(|p^\varepsilon|^{r'-2}p^\varepsilon)$. Note that

$$\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} \le \widetilde{C}_{\mathrm{div}}(\Omega,\Gamma_P,r)\|p^\varepsilon\|_{r'}^{r'/r},$$
$$\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma,\Gamma_P} \le C'_{\mathrm{div}}(\Omega,\Gamma_P,\gamma)\|p^\varepsilon\|_{r'/r}^{r'/r}.$$

Then, using (2.3a), Hölder's inequality, boundedness of $\boldsymbol{b}$ and (3.15), we obtain:

$$\begin{aligned}\|p^\varepsilon\|_{r'}^{r'} &= \int_\Omega \operatorname{div}(\boldsymbol{v}^\varepsilon\otimes\boldsymbol{v}^\varepsilon)\cdot\boldsymbol{\varphi}^\varepsilon, \mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\operatorname{div}\boldsymbol{v}^\varepsilon)\boldsymbol{u}^\varepsilon\cdot\boldsymbol{\varphi}^\varepsilon\,\mathrm{d}\boldsymbol{x}\\ &\quad + \int_\Omega \mathbf{S}(p^\varepsilon,\mathbf{D}(\boldsymbol{v}^\varepsilon)):\mathbf{D}(\boldsymbol{\varphi}^\varepsilon)\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{b}(\boldsymbol{u}^\varepsilon),\boldsymbol{\varphi}^\varepsilon\rangle - \langle \boldsymbol{f},\boldsymbol{\varphi}^\varepsilon\rangle\\ &\le C\,\|\boldsymbol{v}^\varepsilon\|_{1,r}^2\,\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \frac{C_2}{r-1}\|1+|\mathbf{D}(\boldsymbol{v}^\varepsilon)|\|_r^{r-1}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r} + \|\boldsymbol{f}\|_{\mathbf{W}^{1,r}(\Omega)^*}\|\boldsymbol{\varphi}^\varepsilon\|_{1,r}\\ &\quad + \|\boldsymbol{b}(\boldsymbol{u}^\varepsilon)\|_{\gamma',\Gamma_P}\|\boldsymbol{\varphi}^\varepsilon\|_{\gamma,\Gamma_P} \;\le\; C\,\|p^\varepsilon\|_{r'}^{r'/r}\,.\end{aligned}$$

Since $r>1$, this implies

$$\|p^\varepsilon\|_{r'} \le C\,. \tag{3.16}$$

Again, we find a sequence $\varepsilon_n \searrow 0$ and a pair $(\boldsymbol{v},p)$ such that

$$\begin{aligned}
\boldsymbol{u}^{\varepsilon_n} &\rightharpoonup \boldsymbol{u} := \boldsymbol{v}-\boldsymbol{u}_0 && \text{weakly in } \mathbf{W}^{1,r}(\Omega) \text{ and in } \mathbf{L}^\gamma(\Gamma_P)\,, && (3.17a)\\
p^{\varepsilon_n} &\rightharpoonup p && \text{weakly in } \mathrm{L}^{r'}(\Omega)\,, && (3.17b)\\
\mathbf{S}(p^{\varepsilon_n},\mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) &\rightharpoonup \overline{\mathbf{S}} && \text{weakly in } \mathrm{L}^{r'}(\Omega)^{d\times d}\,, && (3.17c)\\
\boldsymbol{g}(\boldsymbol{u}^{\varepsilon_n}) &\rightharpoonup \overline{\boldsymbol{g}} && \text{weakly in } \mathbf{L}^{\gamma'}(\Gamma_P)\,, && (3.17d)\\
\boldsymbol{u}^{\varepsilon_n} &\to \boldsymbol{u} && \text{strongly in } \mathbf{L}^s(\Omega) \text{ for all } s\text{: } 1\le s<\frac{dr}{d-r}. && (3.17e)
\end{aligned}$$

Note that (3.17a) and (3.15) together with (3.2a) yield:

$$\operatorname{div}\boldsymbol{v} = 0 \quad \text{a.e. in } \Omega\,. \tag{3.18}$$

We can then pass to the limit in (3.2b), obtaining

$$\begin{aligned}\int_\Omega \operatorname{div}(\boldsymbol{v}\otimes\boldsymbol{v})\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \int_\Omega \overline{\mathbf{S}}:\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} - \int_\Omega p\operatorname{div}\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x}&\\ + \alpha\int_{\Gamma_N}\boldsymbol{u}\cdot\boldsymbol{\varphi}\,\mathrm{d}\boldsymbol{x} + \langle\overline{\boldsymbol{g}},\boldsymbol{\varphi}\rangle_{\Gamma_P} = \langle \boldsymbol{f},\boldsymbol{\varphi}\rangle \quad \text{for all } \boldsymbol{\varphi}\in\mathbf{W}^{1,r}_{\mathrm{b.c}}(\Omega)\,.&\end{aligned} \tag{3.19}$$

Finally, we use Vitali's theorem and the continuity of $\boldsymbol{g}$ to show that

$$\int_\Omega \mathbf{S}(p^{\varepsilon_n},\mathbf{D}(\boldsymbol{v}^{\varepsilon_n})):\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} \;\to\; \int_\Omega \mathbf{S}(p,\mathbf{D}(\boldsymbol{v})):\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x} = \int \overline{\mathbf{S}}:\mathbf{D}(\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x},$$
$$\langle\boldsymbol{g}(\boldsymbol{u}^{\varepsilon_n}),\boldsymbol{\varphi}\rangle_{\Gamma_P} \;\to\; \langle\boldsymbol{g}(\boldsymbol{u}),\boldsymbol{\varphi}\rangle_{\Gamma_P} = \langle\overline{\boldsymbol{g}},\boldsymbol{\varphi}\rangle_{\Gamma_P}, \qquad \varepsilon_n\searrow 0$$

for all $\boldsymbol{\varphi}\in\mathbf{W}^{1,r}_{\mathrm{b.c}}(\Omega)$. In this respect we are going to prove the convergence

$$\mathbf{D}(\boldsymbol{u}^{\varepsilon_n}) \to \mathbf{D}(\boldsymbol{u}) \text{ a.e. in } \Omega, \quad \boldsymbol{u}^{\varepsilon_n}\to\boldsymbol{u} \text{ a.e. on } \Gamma_P \quad \text{and} \quad p^{\varepsilon_n}\to p \text{ a.e. in } \Omega \tag{3.20}$$

in the next part.

3.3. The almost everywhere convergence

Taking $\boldsymbol{\varphi} := \boldsymbol{u}^{\varepsilon_n} - \boldsymbol{u}$ in (3.2b), $\xi := p^{\varepsilon_n}$ in (3.2a), using (3.17) and (3.18), we observe that

$$\begin{aligned}
&\limsup_{\varepsilon_n \searrow 0} \Big(\int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : (\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v}))\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g}(\boldsymbol{u}^{\varepsilon_n}) - \boldsymbol{g}(\boldsymbol{u}), \boldsymbol{u}^{\varepsilon_n} - \boldsymbol{u}\rangle_{\Gamma_P} \Big) \\
&\quad = \limsup_{\varepsilon_n \searrow 0} \left(\int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{u}^{\varepsilon_n})\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g}(\boldsymbol{u}^{\varepsilon_n}), \boldsymbol{u}^{\varepsilon_n}\rangle_{\Gamma_P} \right) - \int_\Omega \overline{\mathbf{S}} : \mathbf{D}(\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} - \langle \overline{\boldsymbol{g}}, \boldsymbol{u}\rangle_{\Gamma_P} \leq 0\,,
\end{aligned}$$

which together with (2.4a) and (2.8) yields (denoting by $o(1)$ a sequence vanishing as $\varepsilon_n \searrow 0$):

$$m(\|\boldsymbol{u}^{\varepsilon_n} - \boldsymbol{u}\|_{\gamma,\Gamma_P}) + \frac{C_1}{2} Y^n \leq \frac{\gamma_0^2}{2C_1}\|p^{\varepsilon_n} - p\|_2^2 + o(1)\,. \tag{3.21}$$

Here we denote

$$Y^n := \int_\Omega \int_0^1 \left(1 + |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) + s(\mathbf{D}(\boldsymbol{v}) - \mathbf{D}(\boldsymbol{v}^{\varepsilon_n}))|^2\right)^{\frac{r-2}{2}} |\mathbf{D}(\boldsymbol{v}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{v})|^2\,\mathrm{d}s\,\mathrm{d}\boldsymbol{x}\,.$$

Next, we set $\boldsymbol{\varphi}^n := \widetilde{\mathcal{B}}(p^{\varepsilon_n} - p)$, so that $\|\boldsymbol{\varphi}^n\|_{1,2} \leq \widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2)\|p^{\varepsilon_n} - p\|_2$. Since $(p^{\varepsilon_n} - p) \rightharpoonup 0$ weakly in $\mathrm{L}^{r'}(\Omega)$, it follows that $\boldsymbol{\varphi}^n \rightharpoonup 0$ weakly in $\mathbf{W}^{1,r}(\Omega)$ and $\boldsymbol{\varphi}^n \to 0$ strongly in $\mathbf{L}^\gamma(\Gamma_P)$. Testing (3.2b) with $\boldsymbol{\varphi}^n$ we obtain:

$$\begin{aligned}
\int_\Omega p^{\varepsilon_n}(p^{\varepsilon_n} - p)\,\mathrm{d}\boldsymbol{x} &= \int_\Omega \mathrm{div}(\boldsymbol{v}^{\varepsilon_n} \otimes \boldsymbol{v}^{\varepsilon_n}) \cdot \boldsymbol{\varphi}^n\,\mathrm{d}\boldsymbol{x} - \frac{1}{2}\int_\Omega (\mathrm{div}\,\boldsymbol{v}^{\varepsilon_n})\,(\boldsymbol{u}^{\varepsilon_n} \cdot \boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} \\
&\quad + \int_\Omega \mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) : \mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} + \alpha \int_{\Gamma_N} \boldsymbol{u}^{\varepsilon_n} \cdot \boldsymbol{\varphi}^n\,\mathrm{d}\boldsymbol{x} + \langle \boldsymbol{g}(\boldsymbol{u}^{\varepsilon_n}), \boldsymbol{\varphi}^n\rangle_{\Gamma_P} - \langle \boldsymbol{f}, \boldsymbol{\varphi}^n\rangle\,,
\end{aligned}$$

from which it follows that

$$\|p^{\varepsilon_n} - p\|_2^2 = \int_\Omega [\mathbf{S}(p^{\varepsilon_n}, \mathbf{D}(\boldsymbol{v}^{\varepsilon_n})) - \mathbf{S}(p, \mathbf{D}(\boldsymbol{v}))] : \mathbf{D}(\boldsymbol{\varphi}^n)\,\mathrm{d}\boldsymbol{x} + o(1)\,.$$

This implies, using (2.4b), (3.17) and (3.21), that

$$\begin{aligned}
\|p^{\varepsilon_n} - p\|_2^2 &\leq C_2\sqrt{Y^n}\|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + \gamma_0\|p^{\varepsilon_n} - p\|_2\|\mathbf{D}(\boldsymbol{\varphi}^n)\|_2 + o(1) \\
&\leq \gamma_0 \widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2)\left(1 + \frac{C_2}{C_1}\right)\|p^{\varepsilon_n} - p\|_2^2 + o(1)\,,
\end{aligned}$$

which leads to:

$$\left(1 - \gamma_0 \widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2)\left(1 + \frac{C_2}{C_1}\right)\right)\|p^{\varepsilon_n} - p\|_2^2 \leq o(1)\,.$$

Due to assumption $(3.1)_2$, (3.21) and (2.4c), we finally observe that

$$\|p^{\varepsilon_n} - p\|_2 \to 0, \qquad \|\mathbf{D}(\boldsymbol{u}^{\varepsilon_n}) - \mathbf{D}(\boldsymbol{u})\|_r \to 0 \qquad \text{and} \qquad \|\boldsymbol{u}^{\varepsilon_n} - \boldsymbol{u}\|_{\gamma,\Gamma_P} \to 0\,,$$

which implies (3.20) and completes the proof of the existence part of Theorem 3.1.

4. Remarks on uniqueness

Remark 4.1 (Pressure is fixed by velocity). Let $(\boldsymbol{v}, p^1)$ and $(\boldsymbol{v}, p^2)$ be weak solutions to **(P)**. Then, under the assumptions of Theorem 3.1, $p^1 = p^2$.

PROOF. From (2.4b) we observe that

$$\left| \int_\Omega (\mathbf{S}^1 - \mathbf{S}^2) : \mathbf{D}(\boldsymbol{\varphi}) \, \mathrm{d}\boldsymbol{x} \right| \leq \gamma_0 \|p^1 - p^2\|_2 \|\mathbf{D}(\boldsymbol{\varphi})\|_2 \qquad \text{for all } \boldsymbol{\varphi} \in \mathbf{W}^{1,r}_{\mathrm{b.c}}(\Omega).$$

Then we take the weak formulation (2.9) for the pairs $(\boldsymbol{v}, p^1)$, $(\boldsymbol{v}, p^2)$, subtract them, take a test function $\boldsymbol{\varphi} := \widetilde{\mathcal{B}}(p^1 - p^2)$ and obtain:

$$\|p^1 - p^2\|_2^2 \leq \gamma_0 \widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2) \|p^1 - p^2\|_2^2.$$

Since by asumption $\gamma_0 \widetilde{C}_{\mathrm{div}}(\Omega, \Gamma_P, 2) < 1$, we conclude that $p^1 = p^2$.

Remark 4.2. Note that, in contrast to the homogeneous Dirichlet case, Theorem 3.1 does not guarantee boundedness of *every* weak solution to **(P)**. This insufficiency is due to the presence of the convective term, the non-homogeneous Dirichlet contidion together with the fact that $r < 2$. Consequently, we are not able to prove uniqueness of solutions (but for small solutions and small data).

Remark 4.3 (Uniqueness for Stokes-like system). However, the main difficulty does not lie in the structure of $\mathbf{S}$ itself: Let us consider the system

$$-\operatorname{div} \mathbf{S} + \nabla p = \boldsymbol{f}, \qquad \operatorname{div} \boldsymbol{v} = 0, \qquad \text{in } \Omega \tag{$\mathbf{P}_\mathrm{S}$}$$

and the boundary term

$$\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}) \qquad \text{on } \Gamma_P.$$

The readers can verify themselves, that the weak solution to $(\mathbf{P}_\mathrm{S})$ exists and is unique even for large data.

5. Conclusion

The class of fluids whose viscosities depend on pressure and shear rate was studied, supplemented by mixed non-homogeneous boundary conditions. Existence of weak solutions was shown under suitable assumptions, without restriction of the size of the data. The proof follows the ideas of Franta et al. [19], extending the theory to the non-homogeneous Dirichlet, Navier and inflow/outflow boundary conditions. Compared to the homogeneous Dirichlet case, the presented setting proves much more useful for modelling real world problems—applications to the journal bearing lubrication problem were discussed in particular. Moreover the choice of boundary conditions provides a natural constraint that determines the level of pressure, which otherwise has to be fixed by an additional input parameter, inconvenient for practical use.

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