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On the hierarchies of universal predicates

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Abstract

We investigate a hierarchy of arithmetical structures obtained by transfinite addition of a canonic universal predicate, where the canonic universal predicate for $\mathcal{M}$ is defined as a minimum universal predicate for $\mathcal{M}$ in terms of definability. We determine the upper bound of the hierarchy and give a characterisation for the sets definable in the hierarchy.

Keywords: Universal predicate; Canonic universal predicate
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1 Introduction

In his fundamental works, Bernard Bolzano develops the idea that all natural languages are approximations of a single *universal language;* a language in which we can describe anything that exists or that could exist. The idea was not new at Bolzano's time and it still persists, at least as a call in the hearts of logicians. However, if there is a general conclusion which can be drawn from the results related to the Gödel theorem then it is this: *there is no universal language.* For any language L, there exists at least one thing that cannot be fully described in the language: the semantics of L itself. In this paper, we shall be working with first-order languages interpreted over natural numbers, but we believe that some of the results are valid in general. Let $\mathcal{M}$ be a first-order structure with a countable language interpreted over ω. According to the well-known theorem of Tarski, the truth predicate for $\mathcal{M}$ (i.e., the set of Gödel numbers of sentences true in $\mathcal{M}$) is not definable in $\mathcal{M}$. It must be noted that this proposition does not merely assert that there is a set which cannot be defined in $\mathcal{M}$, but it gives an example of such a set; and moreover, the set, as a description of the semantics of $\mathcal{M}$, is presupposed in the structure $\mathcal{M}$ itself. If we take a structure $\mathcal{M}_0$ and the truth predicate T_0 for $\mathcal{M}_0$ then the structure $\mathcal{M}_1 := M_0 + T_0$ will be stronger then $\mathcal{M}_0$. Moreover, it is an extension in a sense presupposed already in the structure $\mathcal{M}_0$. Similarly we can define $\mathcal{M}_2 := \mathcal{M}_1 + T_1$ etc. and we can even imagine that we iterate the process transfinitely and obtain an infinite hierarchy of structures $\{M_\alpha\}$. The structures in the hierarchy are natural extensions of $\mathcal{M}_0$ and it makes sense to ask what are the properties of such a hierarchy, which sets are definable at some stage of the sequence etc. The notion of the hierarchy, however, is a reminiscence of the idea of the universal language, and it must inevitably lead into difficulties. The first and main problem is that the notion 'truth for $\mathcal{M}$' is not determined uniquely. More generally, we want the structure $\mathcal{M}_{\alpha+1}$ to be obtained as *'$\mathcal{M}_\alpha$+'the description of the semantics of $\mathcal{M}_\alpha$'*. However, the notion *'the description of the semantics of $\mathcal{M}_\alpha$'* is not unambivalent, as there may exist infinitely many sets which may be said to describe the semantics of $\mathcal{M}_\alpha$. It is an obvious move to try to chose a particular, *canonic,* description of semantics of $\mathcal{M}_\alpha$ and define the hierarchy in terms of adding the canonic description. Two alternative definitions of such a description will be given below under the headings *canonic universal predicate* and *proper canonic universal predicate.* Of course, we must then answer the question whether such a canonic choice is possible, i.e., we must determine whether a (proper) canonic universal predicate for a given $\mathcal{M}$ exists, and this problem will form the major part of the present paper.

For rather technical reasons (explained on page 7) the truth predicate itself is not exactly suitable for the purpose of defining a hierarchy, and we shall thus define the hierarchy in a related but different way. Moreover, we shall investigate two kinds of hierarchies, one obtained using *a proper universal predicate,* and

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the other using *a universal predicate for* $\mathcal{M}$. Let $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{P}(\omega)$ denote the set of all (first-order) definable sets in $\mathcal{M}$, then[1]

1. $G \subseteq \omega^2$ is *a universal predicate for* $\mathcal{M}$ iff $\mathcal{D}(\mathcal{M}) \subseteq \{G(n,.); n \in \omega\}$
2. $P \subseteq \omega^2$ is *a proper universal predicate for* $\mathcal{M}$ iff $\mathcal{D}(\mathcal{M}) = \{P(n,.); n \in \omega\}$.

Evidently, a (proper) universal predicate for $\mathcal{M}$ is not definable in $\mathcal{M}$; but it must be observed that neither P nor G are unique for a given $\mathcal{M}$. Moreover, P or G can be chosen in such a way that the structures $\mathcal{M}+P$ and $\mathcal{M}+G$ can have an arbitrary strength. In order to avoid the problem we introduce a *canonic (proper) universal predicate* as a minimum (proper) universal predicate in the following sense

P_0 is *a canonic (proper) universal predicate for* $\mathcal{M}$ iff

1. P_0 is a (proper) universal predicate for $\mathcal{M}$ and
2. for every (proper) universal predicate P for $\mathcal{M}$, P_0 is definable in $\mathcal{M}+P$.

The obvious question is whether the canonic universal or canonic proper universal predicates exist. The answer, which is partially given in this paper, is non-trivial: there are structures which have a canonic (proper) universal predicate and there are countable structures which do not. Further, the two concepts are not equivalent and there are structures which possess a canonic proper universal predicate but do not have a canonic universal predicate.

For a given countable structure $\mathcal{N}$ we shall define the *Tarski hierarchy*[2] *over* $\mathcal{N}$ to be a sequence of structures $\{M_\alpha\}_{\alpha \leq \lambda(\mathcal{N})}$ such that

1. $\mathcal{M}_0 = \mathcal{N}$,
2. for every $\alpha < \lambda(\mathcal{N})$, $\mathcal{M}_\alpha$ has a canonic universal predicate and $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha \cup \{P_\alpha\}$,
3. $\mathcal{M}_\beta = \bigcup_{\beta<\alpha} M_\alpha$ for every limit ordinal $\beta \leq \lambda(\mathcal{N})$,
4. $\lambda(\mathcal{N})$ is the maximum ordinal satisfying 1)-3).

An analogous hierarchy obtained by replacing the notion *canonic universal predicate* by that of *canonic proper universal predicate* and $\lambda(\mathcal{N})$ by $\lambda^p(\mathcal{N})$ will be called *proper Tarski hierarchy over* $\mathcal{N}$. In essence, proper Tarski hierarchy can be viewed as a sequence of truth predicates.

The key characteristic of the Tarski hierarchy is the ordinal $\lambda(\mathcal{N})$. A priori, we know that $\lambda(\mathcal{N}) \leq \aleph_1$, as an uncountable structure cannot have a universal predicate. If $\lambda(\mathcal{N}) = \aleph_1$ then every countable structure obtained by the process of adding a canonic universal predicate does possess a canonic proper universal predicate. If, on the other hand, we have $\lambda(\mathcal{N}) < \aleph_1$ then the structure $\mathcal{M}_\lambda$ does not have a canonic universal predicate.

The basic properties of the proper Tarski hierarchy can be obtained from [1]. The authors define the hierarchy as a sequence of Turing degrees $\{H_\alpha\}_{\alpha<\xi}$. On the isolated steps they take for $H_{\alpha+1}$ simply the Turing jump for H_α and the minimum-proper-universal-predicate question enters on the limit steps. But in essence, their definition is equivalent to the notion of Tarski hierarchy adopted here.[3] In particular, using the techniques developed in their paper, it can be shown that for every countable $\mathcal{N}$ the ordinal $\lambda^p(\mathcal{N})$ is a countable limit ordinal. An alternative approach to proper Tarski hierarchy and a comparison of Tarski and proper Tarski hierarchy can be found in [3].

In this paper we shall be concerned mainly with the Tarski hierarchy. We shall prove the following main results

[1] If $X \subseteq \omega^2$ and $n \in \omega$, we set $X(n,.) := \{m \in \omega; X(n,m)\}$.

[2] We use the name of Tarski because he was the first one to state the undefinability of truth. I am not aware that he would ever attempt to iterate the process of adding the truth predicate finitely or transfinitely.

[3] Let $\{H_\alpha\}$ be the sequence of Turing degrees in the sense of [1] and let $\{\mathcal{M}_\alpha\}$ be a proper Tarski hierarchy over $\mathbf{A}$. The interrelation between the hierarchies is based on the following fact: if $\mathcal{M}_\beta \doteq H_\gamma$, where $\doteq$ is understood in terms of Turing reducibility, then $\mathcal{M}_{\beta+1} \doteq H_{\gamma+\omega}$. Since every element in H_α is finite, there is no counterpart in $\{H_\alpha\}$ to $\mathcal{M}_\beta$ if β is a limit. But it is trivial to find some γ such that $\mathcal{M}_{\beta+1} \doteq H_\gamma$.

Theorem 1 *Let* $\{\mathcal{M}_\alpha\}_{\alpha\le\lambda(\mathcal{N})}$ *be a Tarski hierarchy over a L-finite structure* $\mathcal{N}$*. Then* $\lambda(\mathcal{N})$ *is countable. Furthermore,* $\lambda(\mathcal{N}) = Ord(\mathcal{N})$*, the first undefinable (i.e., non-recursive) ordinal in* $\mathcal{N}$*, and the structure* $\mathcal{M}_{\lambda(\mathcal{N})}$ *is the minimal structure containing all sets implicitly definable in* $\mathcal{N}$*.*

We shall note that the structure $\mathcal{M}_{\lambda(\mathcal{N})}$ is at the same time the smallest structure containing all sets Δ^1_1-definable in $\mathcal{N}$, ie. it coincides with the sets hyperarithmetical in $\mathcal{N}$.

Theorem 2 *Let* $\{\mathcal{M}_\alpha\}_{\alpha\le\lambda(\mathcal{N})}$ *be a Tarski hierarchy over a L-finite structure* $\mathcal{N}$ *and let* $\{\mathcal{M}^p_\alpha\}_{\alpha\le\lambda^p(\mathcal{N})}$ *be a proper Tarski hierarchy over* $\mathcal{N}$*. Then* $\lambda^p(\mathcal{N}) > \lambda(\mathcal{N})$ *and for every* $\alpha \le \lambda(\mathcal{N})$*,* $M^p_\alpha \sim M_\alpha$*. Hence the structure* $M_{\lambda(\mathcal{N})}$ *does not have a canonic universal predicate but does have a canonic proper universal predicate.*

The part of Theorem 1 asserting that $\lambda(\mathcal{N}) \ge Ord(\mathcal{N})$ is proved as Theorem 20. That the structure $\mathcal{M}_{Ord(\mathcal{N})}$ is the smallest structure containing all implicitly definable sets in $\mathcal{N}$ is claimed in Theorem 21 and proved on page 19. Finally, the fact that $\mathcal{M}_{Ord(\mathcal{N})}$ does not have a canonic universal predicate and hence $\lambda(\mathcal{N}) = Ord(\mathcal{N})$ is claimed in Theorem 22 and we prove it on page 21. Theorem 2 is contained in Theorem 37 and Corollary 2 of Theorem 20.

We must emphasize that in the case of proper Tarski hierarchy the ordinal $\lambda^p(\mathcal{N})$ *is much larger than the first non-recursive ordinal in* $\mathcal{N}$*.* Consequently, the proper Tarski hierarchy over $\mathcal{N}$ does not coincide with the sets hyperarithmetical in $\mathcal{N}$. Though true, it is not therefore evident that the Tarski hierarchy does stop at the first non-recursive ordinal.

2 General notions

In this paper, We take *a structure* to be a set of predicates and function symbols where we assume predicates and function symbols to be inherently interpreted. In addition, we assume predicates and function symbols to be interpreted on the natural numbers ω, i.e. the standard model of natural numbers. Finally, we shall deal only with structures of basic strength, i.e. those in which all the usual arithmetical operations are definable.

Definition 1 *1. Let* $n > 0$*. Then* $P = \langle A, n\rangle$ *is* (n-ary) predicate *iff* $A \subseteq \omega^n$*; n will be called* the arity of P *and A its* extension *.* $\langle A, n\rangle$ *will also be denoted by* ${}^nA$*.*

2. Let $n \ge 0$*. Then* $F = \langle f, n\rangle$ *is* (n-ary) function symbol *iff* $f : \omega^n \to \omega$ *is a total n-ary function from* ω^n *to* ω*; if* $n = 0$ *we assume* $f \in \omega$*; n will be called* the arity of F *and n its* extension *.* $\langle f, n\rangle$ *will also be denoted by* ${}^nf$*.*

Definition 2 *1.* The arithmetic*,* **A***, is the set of predicates and function symbols,* $\{=, <, S, +, .\}$*, interpreted in the usual way over* ω*.*

2. M is a structure *iff* $\mathcal{M}$ *is a set of predicates and function symbols and* $\mathbf{A} \subseteq \mathcal{M}$

$\mathcal{P}$, $\mathcal{F}$ will denote the set of all predicate resp. function symbols. For a structure $\mathcal{M}$, we set $\mathcal{P}(\mathcal{M}) := \mathcal{P} \cap \mathcal{M}$ and $\mathcal{P}(\mathcal{F}) := \mathcal{F} \cap \mathcal{M}$. $\mathcal{P}_n$, $\mathcal{F}_n$ denotes the set of n-ary predicates resp. of function symbols. $\mathcal{P}_n(\mathcal{M})$, $\mathcal{F}_n(\mathcal{M})$ is defined in a similar fashion.

If Y is a set of predicates and function symbols then $\mathcal{M} + Y$ denotes the structure $\mathcal{M} \cup Y$. If $Y = \{A_1 \dots A_s\}$, we shall write simply $\mathcal{M} + A_1 \dots A_s$

The first order variables(or simply just *variables*) are the elements of the set $\{x_i; i \in \omega\}$. The elements of the set $\{X^k_i; i, k \in \omega\}$ are the *second-order variables*, k is the *arity* of the variable X^k_i. For binary *logical connectives* we shall take $\wedge, \vee, \equiv, \Rightarrow$ and $\neg$ is the unary connective. The symbols for quantifiers are $\exists$ and $\forall$.

Syntactical concepts, *terms*, *formulae* etc are defined in the usual way. *Formula scheme* is simply a second-order formula with no second-order quantifications; we shall never need formulae of higher order. A formula scheme ψ will be written as

$$\psi = \psi[Y_1, \dots Y_k](y_1, \dots y_n) = \psi[Y_1, \dots Y_k] = \psi(y_1, \dots y_n),$$

where $Y_1, \dots Y_k$ are the second and $y_1, \dots y_n$ are the first-order variables occuring in ψ. If $H_1, \dots H_k$ are second-order variables or predicates of arities corresponding to Y_i's, then $\psi[H_1, \dots H_k]$ denotes the result of substituting H_i for Y_i in ψ, $i = 1, \dots k$. We may also write only $\psi[Y_s/H_s]$.In an obvious way, we define the $\psi[Y_s/\phi]$ where ϕ has n free variables and the arity of Y_s is $\leq n$[4]; the extra variables in ϕ will serve as parameters.

The class of all formulae with n free variables (resp. the class of formulae with n free variables of a structure $\mathcal{M}$) will be denoted by Fle_n (resp. $Fle_n(\mathcal{M})$) If $\psi = \psi[Y_1, \dots Y_k](y_1, \dots y_m)$ where Y_i is of arity n_i, $i = 1, \dots k$, then we write $\psi \in Fle_m^{n_1, \dots n_k}$. or alternatively $\psi \in Fle_m^{n_1, \dots n_k}(\mathcal{M})$.

Definition 3 *Let $\mu \in \omega^{<\omega}$, $\mu = \langle k_1, \dots k_s \rangle$.*

1. *we say that $\psi \in Fle_k^{\mu}$ iff $\psi \in Fle_m^{k_1, \dots k_s}$.*
2. *we say that $X \in P(\omega^{\mu})$ iff $X = X_1, \dots X_s$ and for every $i = 1, \dots s$, $X_i \subseteq \omega^{k_i}$.*
3. *If $X \in P(\omega^{\mu})$ then ${}^{\mu}X$ will denote the list of predicates ${}^{k_1}X_1, \dots {}^{k_s}X_s$.*

The definitions of a formula being *true* or *satisfied* by a sequence of natural numbers will be left to the reader. [5]

In the obvious manner we introduce partial function

$$Val : Term \times \omega^{<\omega} \to \omega$$

such that if $t = t(x_{i_1}, \dots x_{i_k})$, $i_1 < \dots i_k$, then $Val(t, \langle a_1, \dots a_k \rangle) = a$ iff $t(\overline{a_1}, \dots \overline{a_k}) = \overline{a}$ is true. [6].

Definition 4 *Let $\mathcal{M}, \mathcal{N}$ be structures.*

1. *Let $\psi = \psi(x_{i_1}, \dots x_{i_n}) \in Fle^n$, $i_1 < i_2 < \dots i_n$. Then*
$$Ext(\psi) = \{\langle a_1, \dots a_n \rangle \in \omega^n; \langle a_1, \dots a_n \rangle \ satisfies \ \psi\}$$
2. *We say that $X \subseteq \omega^k$* is defined by *$\psi \in Fle^k$ iff $X = Ext(\psi)$. X* is definable in *$\mathcal{M}$ iff there is $\psi \in Fle^k(\mathcal{M})$ which defines X.*
3. *The set of all $X \subseteq \omega$ definable in $\mathcal{M}$ will be denoted by $\mathcal{D}(\mathcal{M})$.*
4. *We say $\mathcal{M} \sim \mathcal{N}$ iff $\mathcal{D}(\mathcal{M}) = \mathcal{D}(\mathcal{N})$. The classes of equivalence of the relation $\sim$ will be called* definability classes.
5. *Let $F : P(\omega^{k_1, \dots k_s}) \to P(\omega^s)$. Then F* is definable in *$\mathcal{M}$ iff there is $\psi \in Fle_n^{k_1, \dots k_s}(\mathcal{M})$ such that for every $X_1, \dots X_s \in P(\omega^{k_1, \dots k_s})$ and $X \in P(\omega^n)$ there is $F(x_1, \dots X_s) = X$ iff $X = Ext(\psi[{}^{k_1}X_1, \dots {}^{k_s}X_s])$.*

Since we assume that structures have at least the strength of arithmetic we can find a simple coding function

$$[\ ] : \omega^{<\omega} \to \omega$$

which enables us to express quantification over finite sets and sequences of numbers. For a sequence $a_1, \dots a_n$ the number $[a_1, \dots a_n]$ will be called *the code* or *the Gödel number* of the sequence $a_1, \dots a_n$. For $A = \{a_1, \dots a_n\}$, $[A]$ will denote the code of the sequence $b_1, \dots b_n$ such that $A = \{b_1, \dots b_n\}$ and $b_1 < b_2 < \dots < b_n$. If $S = s_1, \dots s_n$, where s_i are sequences or finite sets of numbers then $[S] := [[s_1], \dots [s_n]]$.

An important consequence is that inductively specified sets are definable, as we state in the following lemma.

[4]Here, we must make sure that ϕ is substitutable in ψ, i.e. there is no confusion between variables in ψ and ϕ

[5]Note that formulae are taken interpreted in themselves. Hence we do not say that ψ is true in a structure $\mathcal{M}$, but simply that ψ is true.

[6]$\overline{n}$ denotes the n-th numeral

Lemma 3 *Let $k > 0$. There exists S^k a Σ_0-definable function in $\mathbf{A}$, $S^k : P(\omega) \times (P(\omega^2)^k) \to P(\omega)$ with the following property: Let $C \subseteq \omega$. Let $R_1, \ldots R_k \subseteq \omega^2$ be a list of binary relations. Then for every $a \in \omega$ $a \in S^k(C, R_1, \ldots R_k)$ iff a is the code of a sequence $y_0, \ldots y_n \in \omega$ such that for every $j \leq n$ either*

1. *$y_j \in C$, or*
2. *there is $1 \leq l \leq k$ and $i_1, \ldots i_s < j$ such that $R_l([y_{i_1}, \ldots y_{i_s}], y_j)$*

`Proof.` Easy. `QED`

3 Truth and universal predicates

We have introduced notions which describe semantics and syntax of a structure. The notions are set-theoretical and hence they cannot be directly taken as predicates or functions which are assumed to range over natural numbers. In order to be able to define something like 'the jump operator' we must formulate concepts which describe properties of a structure by means of predicates defined on natural numbers. For this purpose we define (proper) universal predicate for $\mathcal{M}$ and the truth predicate for $\mathcal{M}$ under a coding c, $Tr_{\mathcal{M},c}$.

For a relation $R \subseteq \omega^2$ and $a \in \omega$, $R(a, .)$ will denote the set $\{x \in \omega; R(a, x)\}$. For relations of bigger arity similarly.

Definition 5 *Let $\mathcal{M}$ be a structure. $P, G \subseteq \omega^2$.*

1. *G is* a universal set for $\mathcal{M}$ *iff for every $X \in \mathcal{D}(\mathcal{M})$ there exists $n \in \omega$ such that $X = G(n, .)$, i.e., iff $\mathcal{D}(\mathcal{M}) \subseteq \{G(n, .); n \in \omega\}$. ^2G will be called* a universal predicate for $\mathcal{M}$.
2. P is a proper universal set for $\mathcal{M}$ *iff P is a universal set and for every $n \in \omega$ the set $P(n, .)$ is definable in $\mathcal{M}$, i.e. iff $\mathcal{D}(\mathcal{M}) = \{G(n, .); n \in \omega\}$. ^2P will be called* a proper universal predicate for $\mathcal{M}$.
3. *Let G be a universal set for $\mathcal{M}$, let $X \subseteq \omega$. Then $n \in \omega$ will be called* a G-code of X *iff $X = G(n, .)$. If $X \subseteq \omega^k$, $k > 1$, then n is* a G-code of X *iff n is the G-code of the set $\{[a_1, \ldots a_k]; a_1, \ldots a_k \in X\} \subseteq \omega$.*

We can view a universal set as a list of subsets of ω $G(0, .), G(1, .), G(2, .) \ldots$ such that every definable set in $\mathcal{M}$ occurs in this list. If G is a proper universal set then also every member of that list is definable in $\mathcal{M}$. Consequently, a proper universal predicate enables us to express quantifications over definable sets in $\mathcal{M}$, while the universal set enables us to express quantifications over a class containing all definable sets in $\mathcal{M}$.

Proposition 4 *Let $\mathcal{M}$ be a structure. Let G be a universal predicate for $\mathcal{M}$. Then*

1. *every set definable in $\mathcal{M}$ is Δ_1-definable in $\mathbf{A} + G$,*
2. *G is not definable in $\mathcal{M}$.*

`Proof.` 1) is obvious. 2) is well-known. `QED`

A (proper) universal predicate for $\mathcal{M}$ determines what are the definable sets in $\mathcal{M}$, but does not show what is the internal structure of $\mathcal{M}$, what predicates and functions are in $\mathcal{M}$ etc. On the other hand, the notion of truth predicate for $\mathcal{M}$ under a coding c which we introduce below is a complete description of $\mathcal{M}$. Two structures which define the same sets, $\mathcal{M}_1 \sim \mathcal{M}_2$, have the same (proper) universal predicates but in general will possess different truth predicates. This relation between truth predicate and proper universal predicate is expressed in the Proposition 6.

Definition 6 *Let $\mathcal{M}$ be a structure. A one-to-one function $c : \mathcal{M} \to \omega$ will be called* a coding for $\mathcal{M}$.

Let c be a coding for $\mathcal{M}$. Then $\overline{c} : \mathcal{M} \cup ($ logical symbols $) \to \omega$ is the one-to-one function such that: i) if $x \in \mathcal{P}_n(\mathcal{M})$ then $\overline{c}(x) = 2[c(x), n, 0]$ ii) if $x \in \mathcal{F}_n(\mathcal{M})$ then $\overline{c}(x) = 2[c(x), n, 1]$ iii) if $x = X^i_n$ is a second-order variable then $\overline{c}(x) = 2[i, n, 2]$ iv) $\overline{c} : \wedge, \vee, \neg, \Rightarrow, \equiv, \forall, \exists, (,) \to 1, 3, 5, 7, 9, 11, 13, 15$ respectively and if $x = x_i$ is a first-order variable then $\overline{c} : x_i \to 2i + 17$.

Let $\mathcal{M}$ be a structure, c a coding for $\mathcal{M}$. If $s_1, \ldots s_n$ are logical symbols or elements of $\mathcal{M}$ then

$$[s_1, \ldots s_n]_c \in \omega$$

will denote the number $[\overline{c}(s_1), \overline{c}(s_2), \ldots \overline{c}(s_n)]$ and it will be called *the c-Gödel number of* , or simply *the c-code of $S_1, \ldots s_n$.*

Definition 7 *Let $\mathcal{M}$ be a structure. Let c be a coding for $\mathcal{M}$.*

1. *Let X be a set of strings of symbols from $\mathcal{M}$ or logical symbols. Then $X_c := \{[x]_c; x \in X\} \subseteq \omega$.*
2. *$Tr_{\mathcal{M},c} \subseteq \omega$ is the set of c-Gödel numbers of true sentences of $\mathcal{M}$. The predicate ${}^1Tr_{\mathcal{M},c}$, will be called* the truth predicate for $\mathcal{M}$ under the coding c.
3. *$Tr^k_{\mathcal{M},c} \subseteq \omega$ is the set of c-Gödel numbers of true sentences of $\mathcal{M}$ which are in Π_k or Σ_k prenex form. The predicate ${}^1Tr^k_{L,c}$, will be called* the k-truth predicate for $\mathcal{M}$ under the coding c.
4. *$Val_{\mathcal{M},c} \subseteq \omega^2$ is the relation such that $Val_{\mathcal{M},c}(a, b)$ iff a is a c-code of a closed term t and $Val(t) = b$.*
5. *$D_{\mathcal{M},c} \subseteq \omega^2$ is the relation such that*
   (a) *for every $X \in \mathcal{M}$ $D_{\mathcal{M},c}(\overline{c}(X), .) \subseteq \omega$ is the set of codes of n-tuples $\langle a_1, \ldots a_n\rangle$ such that $\langle a_1, \ldots a_n\rangle \in Ext(X)$ (where the arity of X is n if $X \in \mathcal{P}_n$, and $n - 1$ if $X \in \mathcal{F}_{n-1}$).*
   (b) *if $x \notin Rng(\overline{c}\lceil\mathcal{M})$ then $D_{\mathcal{M},c}(x, .) = \{1\}$.*

The relation $D_{\mathcal{M},c}$ determines what are the predicates and functions of $\mathcal{M}$, what are their codes, arities and extensions. We may notice that in $\mathbf{A}+ {}^2D_{\mathcal{M},c}$ we are able to define the truth on the *atomic* propositions in $\mathcal{M}$, while the predicate $Tr_{\mathcal{M},c}$ is not in general definable in $\mathbf{A}+ {}^2D_{\mathcal{M},c}$, as we shall see.

Proposition 5 *Let $\mathcal{M}$ be a structure, c be a coding for $\mathcal{M}$.*

1. *The following are Δ_1-definable in $\mathbf{A}+ {}^1Rng(\overline{c})$: $Term(\mathcal{M})_c$, $Fle(\mathcal{M})_c$, $Fle_n(\mathcal{M})_c$.*
2. *$D_{\mathcal{M},c}$ is Δ_1-definable in $\mathbf{A}+ {}^1Tr_{\mathcal{M},c}$ and $Rng(\overline{c})$ is Δ_1-definable in $\mathbf{A} + D_{\mathcal{M},c}$*
3. *$Val_{\mathcal{M},c}$ is Δ_1-definable in $\mathbf{A}+ {}^2D_{\mathcal{M},c}$.*
4. *$Tr^k_{\mathcal{M},c}$ is Δ_{k+1}-definable in $\mathbf{A}+ {}^2D_{\mathcal{M},c}$.*

`Proof.` 1), 3) and 4) are an easy application of Lemma 3. 2) is immediate. `QED`

Proposition 6 *Let $\mathcal{M}$ be a structure and c a coding for $\mathcal{M}$. Then*

1. *there exists a proper universal predicate for $\mathcal{M}$ which is Δ_1-definable in $\mathbf{A}+ {}^1Tr_{\mathcal{M},c}$.*
2. *Let G be a universal predicate for $\mathcal{M}$. Then $Tr_{\mathcal{M},c}$ is definable in $\mathcal{M} + G + {}^2D_{\mathcal{M},c}$.*

`Proof.` 1) The relation P: $P(n, x)$ iff $n = [\psi]_c$, $\psi \in Fle_1(\mathcal{M})$ and ψ is in Σ_k or Π_k prenex form and $[\psi(\overline{x})]_c \in Tr^k_{\mathcal{M},c}$ is Δ_1-definable and it is a universal set for $\mathcal{M}$.

2) The proof is an application of Lemma 3 and proceeds as follows.

For a formula $\psi \in Fle(\mathcal{M})$, a sequence $a_1, \ldots a_k \in \omega$ will be called a *formula derivation for ψ* iff i) a_i is a c-code of a string ψ_i, $i = 1, \ldots i$, and $a_k = [\psi]_c$ and ii) for every $i \leq k$ either ψ_i is an atomic formula or there are $i_1, i_2 < i$ and $\psi_i = (\psi_{i_1})\triangle(\psi_{i_2})$, where $\triangle$ is a binary logical connective, or $\psi = \triangle(\psi_{i_1})$, where $\triangle$ is $\neg$ or $\exists y$, $\forall y$.

A sequence $a_1, e_1 \dots a_k, e_k \in \omega$ will be called a *truth derivation for* ψ iff i) $a_1, \dots a_k$ is a formula derivation for ψ and ii) if $a_i = [\psi_i]_c$, $\psi_i \in Fle_n$ then e_i is a G-code of the set $\{[s_1, \dots s_l]; l \geq n, \langle s_1, \dots s_n \rangle \in \omega^{<\omega}$ satisfies $\psi\}$.

The proof of Proposition 5,1) requires to show that every formula of $\mathcal{M}$ has a formula derivation and the set of codes of formula derivations is definable in $\mathbf{A} +\ ^2Rng(\overline{c})$. Here, it must be shown that every formula of $\mathcal{M}$ has a truth derivation and that the set of codes of truth derivations is definable in $\mathcal{M} + G +^2 D_{\mathcal{M},c}$. Both parts are straightforward. Finally, a $[\psi]_c \in Tr_{\mathcal{M},c}$ iff $[\psi]_c \in Fle_0(\mathcal{M})_c$ and ψ has a truth derivation $a_1, e_1 \dots a_k, e_k$ such that $G(e_k, .) \neq \emptyset$. QED

Corollary *Let $\mathcal{M}$ be a structure, c a coding for $\mathcal{M}$. Then*

1. *Every set definable in $\mathcal{M}$ is Δ_1-definable in $\mathbf{A} +\ ^1Tr_{\mathcal{M},c}$.*

2. *$Tr_{\mathcal{M},c}$ is not definable in $\mathcal{M}$.*

Proof. Follows from the previous Proposition and Proposition 4. QED

Definition 8 *Let $\mathcal{M}$ be a structure.*

1. *$\mathcal{M}$* is L-finite *iff $|\mathcal{M}| < \omega$, i.e. iff $\mathcal{M}$ is a finite set function symbols and predicates.*

2. *$\mathcal{M}$ is* essentially finite *iff there exists a structure $\mathcal{M}'$ which is finite and $\mathcal{M} \sim \mathcal{M}'$.*

The following lemma expresses the key property of L-finite structures.

Lemma 7 *Let $\mathcal{M}$ be a L-finite structure. Then $D_{\mathcal{M},c}$ is Δ_1-definable in $\mathcal{M}$.*

Proof. Let $\mathcal{P}(\mathcal{M}) = P_1, \dots P_s$, $\mathcal{F}(\mathcal{M}) = F_1, \dots F_k$.

For $P_i \in \mathcal{P}_n(\mathcal{M}), i \leq s$ there is ψ_i a Δ_1-formula in $\mathcal{M}$ such that for very $a \in \omega$, $a \in Ext(\psi_i)$ iff a is a code of n-tuple $a_1, \dots a_n$ and $\langle a_1, \dots a_n \rangle \in Ext(P)$. Analogically, if $F_i \in \mathcal{F}_n(\mathcal{M}), i \leq k$ then there is ψ_{s+i} a Δ_1-formula such that for every $a \in \omega$, $a \in Ext(\psi_{s+i})$ iff $a = [a_1, \dots a_{n+1}]$ and $\langle a_1, \dots a_n, a_{n+1} \rangle \in Ext(F)$.

Let $t_1, \dots t_s$ and $t_{s+1}, \dots t_{s+k}$ denote the numerals corresponding to $\overline{c}(P_1), \dots \overline{c}(P_s)$ and $\overline{c}(F_1), \dots \overline{c}(F_k)$. Then $D_{\mathcal{M},c}(x, y)$ is defined in $\mathcal{M}$ by the following formula

$((x \neq t_1) \wedge \dots (x \neq t_{s+l}) \wedge y = \overline{1}) \vee ((x = t_1 \wedge \psi_1(y) \vee \dots (x = t_{s+l} \wedge \psi_{k+l}(y)))$ QED

Corollary *Let $\mathcal{M}$ be a L-finite structure, c a coding for $\mathcal{M}$, $k \in \omega$. Then $Tr^k_{L,c}$ is Δ_{k+1}-definable in $\mathcal{M}$. The sets $Term(\mathcal{M})_c$, $Fle(\mathcal{M})_c$, $Fle_n(\mathcal{M})_c$ are Δ_1-definable in $\mathcal{M}$.*

Proof. Follows from the previous Lemma and Proposition 5.QED

For a given structure, by different choices of coding c we can obtain different truth predicates, and the structure $\mathcal{M} +\ ^1Tr_{\mathcal{M},c}$ will have different expressive powers. Similarly for (proper) universal predicates; in particular, if $\mathcal{M}$ is a structure and $B \subseteq \omega$ is any given set then we can find a (proper) universal predicate for $\mathcal{M}$ such that B is definable in $\mathcal{M} +\ ^2G$. We see that neither the universal nor the proper universal predicate can have the role of 'the jump operator' for $\mathcal{M}$, for such an operation would not be unique. It is then an expectable move to try to choose a particular (proper) universal predicate which would be in some sense the weakest. This is achived using the concepts of *canonic universal predicate* and *canonic proper universal predicate* which have been defined on page 2.[7]

Lemma 8 *Let $\mathcal{M}$ be a structure.*

[7] Note that we do not introduce the symmetric concept of *canonic truth predicate.* The reason is that if we defined the Tarski hieararchy (see page 2) using the canonic truth predicate then the Theorem 10 is false, ie. there would exist many incomparable hierarchies over $\mathcal{N}$. In particular, for any $B \subseteq \omega$ we could find a Tarski hierarchy $\{M_\alpha\}_{\alpha<\lambda(\mathcal{N})}$ (defined in terms of canonic truth predicate) such that $\omega < \lambda(\mathcal{N})$ and B is definable in $\mathcal{M}_\omega$.

1. *Assume that there is a coding c_0 for $\mathcal{M}$ such that for every (proper) universal predicate P the set $D_{\mathcal{M},c_0}$ is definable in $\mathcal{M}+P$. Then $\mathcal{M}$ has a canonic (proper) universal predicate and if P_0 is a canonic (proper) universal predicate then $\mathbf{A}+P_0 \sim \mathbf{A}+\ ^1Tr_{\mathcal{M},c_0}$.*
2. *Let $\mathcal{M}'$ be a structure such that $\mathcal{M}\sim\mathcal{M}'$. Then $\mathcal{M}$ has a canonic (proper) universal predicate iff $\mathcal{M}'$ has a canonic (proper) universal predicate. If P and P' are canonic (proper) universal predicates for $\mathcal{M}$ and $\mathcal{M}'$ respectively then $\mathbf{A}+P\sim\mathbf{A}+P'$.*

`Proof.` 1) follows from Proposition 6. 2) follows from the fact that $\mathcal{M}$ and $\mathcal{M}'$ have the same (proper) universal predicates. `QED`

Proposition 9 *Let $\mathcal{M}$ be an essentially L-finite structure. Then $\mathcal{M}$ has both a canonic and a canonic proper universal predicate. If P_0 is a canonic (proper) universal predicate and $\mathcal{N}$ is a L-finite structure such that $\mathcal{M}\sim\mathcal{N}$ and c a coding for N then $\mathbf{A}+P_0\sim\mathbf{A}+\ ^1Tr_{\mathcal{N},c}$.*

`Proof.` By Lemma 8 it is sufficient to show that $D_{\mathcal{N},c}$ is definable in $\mathcal{M}$. But that is claimed in Lemma 7. `QED`

Recall the definitions of Tarski and proper Tarski hierarchy given on page 2. Since for a given structure there in general exist infinitely many canonic (proper) universal predicates, neither the Tarski hierarchy nor the proper Tarski hierarchy are defined uniquely. The following Theorem shows that the hierarchies are unique at least up to the equivalence $\sim$.

Theorem 10 *Let $\mathcal{N}$ be a L-finite structure. Let $\{\mathcal{M}_\alpha\}_{\alpha\in\xi_1}$, $\{\mathcal{M}'_\alpha\}_{\alpha\in\xi_1}$ be two Tarski hierarchies over $\mathcal{N}$. Then $\xi_1=\xi_2>0$ and for every $\alpha\in\xi_1$ there is $\mathcal{M}_\alpha\sim\mathcal{M}'_\alpha$. The same is true for two proper tarski hierarchies.*

`Proof.` Since $\mathcal{N}_0=\mathcal{M}_0=\mathcal{N}$ are finite then $\mathcal{N}_0,\mathcal{M}_0$ have canonic proper universal predicates (Corollary of Proposition 9) and therefore $\xi_1,\xi_2>0$. The rest follows from Lemma 8,2). `QED`.

Definition 9 *Let $\mathcal{N}$ be a L-finite structure, $\{\mathcal{M}_\alpha\}_{\alpha<\xi}$ be a Tarski hierarchy. Then $\lambda(\mathcal{N}):=\xi$. If $\{\mathcal{M}^p_\alpha\}_{\alpha<\xi}$ is a proper Tarski hierarchy then $\lambda(\mathcal{N})^p:=\xi$.*

A priori, we see that $\lambda(\mathcal{N})$ and $\lambda^p(\mathcal{N})$ can at most be equal to $\aleph_1$, the first uncountable ordinal. For then the structure $\mathcal{M}_{\aleph_1}$ is uncountable and there exist no truth or proper universal predicate for $\mathcal{M}_{\aleph_1}$ and we cannot hope to extend the hierarchies above $\aleph_1$. The crucial question concerning the Tarski hierarchy and proper Tarski hierarchy is this: *is $\lambda(\mathcal{N})$ countable*? If it is then the structure $\mathcal{M}_{\aleph_1}$ is a countable structure which does not have a canonic proper universal predicate and the proper Tarski hierarchy cannot be extended above $\lambda(\mathcal{N})$. If $\lambda(\mathcal{N})=\aleph_1$ then we may say that the Tarski hierarchy does not have an upper bound.

Theorem 11 *Let $\mathcal{N}$ be a L-finite structure. Let $\{\mathcal{M}_\alpha\}_{\alpha<\lambda(\mathcal{N})}$ be a Tarski hierarchy over $\mathcal{N}$. Let $\alpha<\lambda(\mathcal{N})$. Then $\mathcal{M}_{\alpha+1}$ is essentially finite. Hence $\alpha+1<\lambda(\mathcal{N})$ and $\lambda(\mathcal{N})$ is a limit ordinal. The same is true for the proper Tarski hierarchy.*

`Proof.` $\mathcal{M}_{\alpha+1}=\mathcal{M}_\alpha+P$, where P is a universal predicate. But $\mathcal{M}_\alpha+P\sim_{\mathcal{D}}\mathbf{A}+P$, from Proposition 4, 1). Hence $\mathcal{M}_{\alpha+1}$ is essentially finite, it has a canonic universal predicate and $\alpha+1<\lambda(\mathcal{N})$ `QED`

4 Ordinals and the first part of Theorem 1

In this section we will prove that for a (proper) Tarski hierarchy over $\mathcal{N}$ there is $Ord(\mathcal{N})\le\lambda(\mathcal{N})$ (resp. $Ord(\mathcal{N})\le\lambda^p(\mathcal{N})$).

Definition 10 *Let $\mathcal{M}$ be a structure. Let $\lambda,\mu\in\omega^{<\omega}$.*

1. $\mathcal{A} \subseteq P(\omega^\lambda)$ *is* a system defined by $\psi \in Fle_0^\lambda$ *iff*

$$\mathcal{A} = \{A \in P(\omega^\lambda); A \textit{ satisfies } \psi\}$$

   $\mathcal{A}$ *will be called* a definable system in $\mathcal{M}$ *iff it is defined by some* $\psi \in Fle_n^\lambda(\mathcal{M})$

2. *Let* $A \in P(\omega^\lambda)$*. Then* $\psi \in Fle_n^\lambda$ *is* a proper implicit definition of A *iff* ψ *defines the system* $\{A\}$*.*
3. $B \in P(\omega^\mu)$ *is* implicitly definable in $\mathcal{M}$ *iff there exist* $A \in P(\omega^\lambda)$*,* A *has a proper implicit definition in* $\mathcal{M}$ *and* B *is definable in the structure* $\mathcal{M} + {}^\lambda A$*.*
4. *Let* $F : P(\omega^\lambda) \to P(\omega^\mu)$*,* $\mu = \mu_1, \dots \mu_n$*. We will say that* F is defined by $\psi_1, \dots \psi_n$, $\psi_i \in Fle_{\mu_i}^\lambda$*, iff for every* $X = X_1, \dots X_s \in P(\omega^\lambda)$*,* $F(X) = \langle Y_1, \dots Y_n\rangle$*, we have* $Y_i = Ext(\psi_i({}^\lambda X)), i = 1, \dots n$*. That* F *is* definable in $\mathcal{M}$ *we introduce in the obvious way.*
5. *Let* $F : P(\omega^\lambda) \to P(\omega^\mu)$*. We will say that* $\psi \in Fle_0^{\lambda,\mu}$ *is* a proper implicit definition of F *iff for every* $X \in P(\omega^\lambda)$ *there is a unique* $Y \in P(\omega^\mu)$ *such that* $\psi[{}^\lambda X, {}^\mu Y]$ *is true and for such* Y*,* $F(X) = Y$*.*
6. *Let* $F : P(\omega^\lambda) \to P(\omega^\mu)$*. We will say that* F is implicitly definable in $\mathcal{M}$ *iff there are functions* F_1, F_2*,* $F_1 : P(\omega^\lambda) \to P(\omega^\pi)$*,* $F_2 : P(\omega^\pi) \to P(\omega^\mu)$ *such that* $F(X) = F_2(F_1(X))$*,* $X \in N^\lambda$ *and* F_1 *has a proper implicit definition in* $\mathcal{M}$ *and* F_2 *is definable in* $\mathcal{M}$*.*

We may observe that

1. If $B \in P(\omega^\lambda)$, $F : P(\omega^\lambda) \to P(\omega^\mu)$ are definable in $\mathcal{M}$ then they have a proper implicit definition in $\mathcal{M}$. If they have a proper implicit definition in $\mathcal{M}$ then they are implicitly definable in $\mathcal{M}$.
2. Let $F_1 : P(\omega^\lambda) \to P(\omega^\pi)$, $F_2 : P(\omega^\pi) \to P(\omega^\mu)$ and $F(X) = F_2(F_1(X))$, $X \in \omega^\lambda$. Then
   (a) if F_1, F_2 are definable resp. implicitly definable in $\mathcal{M}$ then F is definable resp. implicitly definable in $\mathcal{M}$.
   (b) if F_1 is definable in $\mathcal{M}$ and F_2 has a proper implicit definition in $\mathcal{M}$ then F has a proper implicit definition in $\mathcal{M}$
3. if $B \in P(\omega^\lambda)$ and $F : P(\omega^\lambda) \to P(\omega^\mu)$ are definable resp. implicitly definable in $\mathcal{M}$ then $F(B)$ is definable resp. implicitly definable in $\mathcal{M}$.

The following statement will not be used in this work but it gives an important characterisation of implicitly definable sets. We therefore do not enter the proof.

Proposition. *Let* $\mathcal{M}$ *be a structure. Then* $B \in P(\omega^k)$ *is implicitly definable in* M *iff it is* Δ_1^1 *in* $\mathcal{M}$ *(i.e. iff* B *is hyperarithmetical in* $\mathcal{M}$*).* `Proof.` The implication '→' is obvious. The other follows from Lemma 32. `QED`

Lemma 12 *Let* $\mathcal{M}$ *be a structure,* $\lambda, \mu \in \omega^{<\omega}$*.*

1. *Let* $B \in P(\omega^\lambda)$ *be implicitly definable in* $\mathcal{M}$*. Then there exists* $A \subseteq \omega$*,* A *has a proper implicit definition in* $\mathcal{M}$ *and* B *is definable in* $\mathcal{M} + {}^1A$
2. *Let* $B \in P(\omega^\lambda)$ *be implicitly definable in* $\mathcal{M}$ *,* $C \in P(\omega^\mu)$ *implicitly definable in* $\mathcal{M} + B$*. Then* C *is implicitly definable in* $\mathcal{M}$*.*

`Proof.` Straightforward. `QED`

Definition 11 *Let* $\mathcal{M}$ *and* $\mathcal{N}$ *be structures. Then*

1. $\mathcal{M}$ *is* implicitly closed *iff every set which is implicitly definable in* $\mathcal{M}$ *is definable in* $\mathcal{M}$*.*
2. $\mathcal{I}(\mathcal{M})$ *is the structure* $\mathcal{M} + \{{}^1X; X \subseteq \omega,\ X \textit{ implicitly definable in } \mathcal{M}\}$*.*

Corollary of Lemma 12 *Let $\mathcal{M}$ be a structure. Let $\mathcal{N} := \mathcal{I}(\mathcal{M})$. Then i) $\mathcal{N}$ is implicitly closed, ii) $\mathcal{D}(\mathcal{M}) \subseteq \mathcal{D}(\mathcal{N})$ and iii) for every $\mathcal{N}'$ if $\mathcal{N}'$ satisfies i) and ii) then $\mathcal{D}(\mathcal{N}) \subseteq \mathcal{D}(\mathcal{N}')$.*
`Proof.` Let $\mathcal{M}$ be given. By Lemma 12, 2) if a set is implicitly definable in $\mathcal{I}(\mathcal{M})$ then it is implicitly definable in $\mathcal{M}$. Hence $\mathcal{I}(\mathcal{M})$ is implicitly closed. The rest is immediate. `QED`

Proposition 13 *There is a function $TR : P(\omega^2) \to P(\omega)$ which has a proper implicit definition in **A** such that for every structure $\mathcal{M}$ and a coding c for $\mathcal{M}$ we have*

$$TR(D_{\mathcal{M},c}) = Tr_{\mathcal{M},c}$$

`Proof.` First, observe that Proposition 5,3) can be strengthened to assert that there exists a function $VAL : P(\omega^2) \to P(\omega^2)$ definable in **A** such that for every structure $\mathcal{M}$ and a coding c for $\mathcal{M}$,

$$VAL(D_{\mathcal{M},c}) = Val_{\mathcal{M},c}$$

Let $\mathcal{M}$ be a structure and c its coding. Then $X = Tr_{\mathcal{M},c}$ iff $X \subseteq Fle_0(\mathcal{M})_c$ and for every $x = [\psi]_c \in Fle_0(\mathcal{M})_c$ the following conditions are satisfied

1. If $\psi = P(\overline{n_1}, \dots \overline{n_i})$ is atomic and $\overline{n_l}$ is the n_l-th numeral then $x \in X$ iff $D_{\mathcal{M},c}(\bar{c}(P), [n_1, \dots n_l])$
2. If $\psi = P(t_1, \dots t_i)$ is a closed atomic formula, where $t_1, \dots t_i$ are terms, then $x \in X$ iff $[P(\overline{Val_{\mathcal{M},c}(t_1)}, \dots \overline{Val_{\mathcal{M},c}(t_i)}]_c \in X$.
3. If $\psi = \neg\xi$ then $x \in X$ iff not $[\xi]_c \in X$. If $\psi = \xi_1 \wedge \xi_2$ then $x \in X$ iff $[\xi_1]_c \in X$ and $[\xi_2]_c \in X$ and so on for the other logical connectives.
4. if $\psi = \exists y \eta$ then $x \in X$ iff there exists $a \in \omega$ such that $[\eta(y/\overline{a})]_c \in X$. If $\psi = \forall y \eta$ then $x \in X$ iff for every $a \in \omega$, $[\eta(y/\overline{a})]_c \in X$.

Let $S \in Fle^{2,1}(\mathbf{A})$ be a formula scheme obtained as a natural translation of the above conditions and by replacing every occurence of ${}^2D_{\mathcal{M},c}$ (including the one in $Val_{\mathcal{M},c} = VAL(D_{\mathcal{M},c})$) by a second-order variable Y. Then we can see that S is a proper implicit definition of a function TR with the desired property. `QED`

Corollary 1. *Let $\mathcal{M}$ be a L-finite structure, c a coding for $\mathcal{M}$. Then the truth predicate $Tr_{\mathcal{M},c}$ has a proper implicit definition in $\mathcal{M}$.*
`Proof.` It must be shown that $D_{\mathcal{M},c}$ is definable in $\mathcal{M}$ if $\mathcal{M}$ is finite. But that has been claimed in Lemma 7. `QED`

Corollary 2. *Let $\mathcal{M}$ be an essentially L-finite structure. Then there is a proper universal predicate for $\mathcal{M}$ which is implicitly definable in $\mathcal{M}$. Hence, $\mathcal{M}$ is not implicitly closed.*
`Proof.` Apply Proposition 9 on L-finite structure $\mathcal{N}$ such that $\mathcal{N} \sim \mathcal{M}$ to show that a universal predicate for $\mathcal{M}$ is implicitly definable in $\mathcal{M}$. That a universal predicate is not definable in $\mathcal{M}$ is claimed in Proposition 6. `QED`

We shall see that one of the important characteristics of a structure is how many ordinals are definable in the structure. We shall say that R is a *linear ordering on X* iff R is reflexive, transitive, and weakly antisymmetric on X and for every $x, y \in X$, $R(x,y)$ or $R(y,x)$. R is *a linear ordering* iff R is a linear ordering on $Rng(R)$. Thus we take a linear ordering to be *non-strict*. In order to avoid confusion, we shall also write $\preceq_R$ instead of R. $x \prec_R y$ is then defined as $x \preceq_R y$ and $x \neq y$. Note that for a linear ordering we have $Rng(R) = Dom(R)$. If $X \subseteq Rng(R)$ then we define $R\lceil X := R \cap X^2$. If $n \in Rng(R)$ then R_n will denote the relation such that

$$R_n(x,y) \text{ iff } R(x,y) \text{ and } R(y,n)$$

Definition 12 *Let $\rho \subseteq \omega^2$ be a linear ordering, let α be a countable ordinal.*

1. *Then ρ is* a representation of ordinal α *iff $\prec_\rho$ is a well-ordering of the order-type α.*
2. *Let $\beta < \alpha$. Then ρ_β will be defined by induction as follows: let $\rho_0 := a$, a the ρ-smallest member of $Rng(R)$. If $\beta > 0$, let ρ_β be the ρ-smallest member of the set $Rng(\rho) \setminus \{\rho_\gamma; \gamma < \beta\}$.*
3. *Let $\beta \leq \alpha$. Then $\rho\lceil\beta$ is the representation of β such that $\rho\lceil\beta = \rho\lceil\{\rho_\gamma; \gamma < \beta\}$.*

Thus $\rho\lceil\beta$ is a representation of β. ρ_β is the ρ-smallest element majorising $Rng(\rho\lceil\beta)$ if some such ρ_β exists (if $\beta = \alpha$ then $\rho\lceil\alpha = \rho$ while ρ_α is not defined).

Definition 13 *Let $\mathcal{M}$ be a structure,*

1. *Let α be a countable ordinal. Then α is* (implicitly) definable in $\mathcal{M}$ *iff there is ρ a representation of α which is (implicitly) definable in $\mathcal{M}$.*
2. *The smallest undefinable ordinal in $\mathcal{M}$ will be denoted by $Ord(\mathcal{M})$.*

We can see the following:

1. every $\alpha > Ord(\mathcal{M})$ is undefinable in $\mathcal{M}$. I.e., the set of definable ordinals in $\mathcal{M}$ is an interval.
2. $0, 1, \ldots \omega$ are definable in $\mathcal{M}$.
3. If α, β are definable in $\mathcal{M}$ then $\alpha + \beta$ and $\alpha.\beta$ are definable in $\mathcal{M}$. Hence $Ord(\mathcal{M})$ is a limit ordinal.

Now we shall define two important concepts: the concept of iterated truth predicate over a well-ordering, $Tr_{\mathcal{M},c,\rho}$, and the notion of iteration of a general operation over a well-ordering.

Definition 14 *Let $n > 0$, $F : P(\omega^{n+1}) \to P(\omega^n)$, let $B \subseteq N^n$. Let ρ be a representation of an ordinal α. Let $Z \subseteq \omega^{n+1}$. For $\beta < \alpha$ we define*

$$Z_{<\beta} := \{\langle a_0, \ldots a_n\rangle \in \omega^{n+1}; Z(a_0, \ldots a_n) \text{ and } a_0 \prec_\rho \rho_\beta\}$$

We will say that $Z = REK_n(B, F, \rho)$ iff Z satisfies the following conditions

1. *If $y \notin Rng(\rho)$ then $Z(y, .) = \emptyset$. If $\rho \neq \emptyset$, let $Z(\rho_0, .) = B$*
2. *If $0 < \beta < \alpha$ then $Z(\rho_\beta, .) = F(Z_{<\beta})$*

We note that

1. $Z = REK(B, F, \rho)$ as defined above exists and is unique,
2. the definition of $REK_n(B, F, \rho)$ can be rewritten as a formula scheme, as we state in the following proposition.

Proposition 14 *Let $F : P(\omega^{n+1}) \to P(\omega^n)$ have a proper implicit definition in a structure $\mathcal{M}$. Then there exists a function $REK_F : P(\omega^{n,2}) \to P(\omega^{n+1})$ which has a proper implicit definition in $\mathcal{M}$ with the following property: for every $B \subseteq N^n$ and ρ a representation of an ordinal*

$$REK_F(B, \rho) = REK_n(B, F, \rho)$$

Corollary *Let $\mathcal{M}$ be a structure, let $\rho, B \subseteq \omega^n, F : P(\omega^n) \to P(\omega^n)$ let $Z := REK_n(B, F, \rho)$. Then if B, F, ρ have a (proper) implicit definition in $\mathcal{M}$ then Z has a (proper) implicit definition in $\mathcal{M}$.*
`Proof.` Straightforward. `QED`

Later, we shall see that every set which is implicitly definable in $\mathcal{M}$ is also definable in terms of some $REK_n(B, F, \rho)$, where all B, F, ρ are definable in $\mathcal{M}$.

Definition 15 *Let R be a linear ordering. Let $\mathcal{M}$ be a structure, and c a coding for $\mathcal{M}$.*

1. *We will say that* R and c are compatible *iff* $Dom(c) \cap Rng(R) = \emptyset$

2. *For* $U \subseteq \omega^2$ *we shall write that* $U \in \overline{Tr}(\mathcal{M}, c, R)$ *iff the following is satisfied:*

   (a) *If* $x \notin Rng(R)$ *then* $U(x, .) = \emptyset$*. If there is* R_0 *the* R*-first element of* $Rng(R)$*, then* $U(R_0, .) = Tr_{\mathcal{M},c}$

   (b) *If* $R_0 <_R n$ *then* $U(n, .) = Tr_{\mathcal{M}_{<n}, c_n}$*, where* $\mathcal{M}_{<n}$ *is the structure* $\mathcal{M} + \{\ {}^1U(m, .); m <_R n\}$ *and* c_n *is the coding induced on* $\mathcal{M}_{<n}$ *(by* U *and* c*).*

3. *If* R *is a well-ordering then* $Tr_{\mathcal{M},c,\rho}$ *is the set such that* $\overline{Tr}(\mathcal{M}, c, R) = \{Tr_{\mathcal{M},c,R}\}$.

Clearly, if ρ is a well-ordering compatible with c then $Tr_{\mathcal{M},c,\rho}$ can be defined as an iteration of adding a truth predicate along the well-ordering ρ. In this case we have $\overline{Tr}(\mathcal{M}, c, \rho) = \{Tr_{\mathcal{M},c,\rho}\}$. We will see in the last section that $\overline{Tr}(\mathcal{M}, c, \rho)$ is non-empty even for linear orderings which are not well-orderings; in that case U will not in general be unique. Here, we shall deal with $\overline{Tr}(\mathcal{M}, c, \rho)$ only in the case ρ is a well-ordering. The main results about $Tr_{\mathcal{M},c,\rho}$ presented below are that i) it is strong enough to define all sets of the form $REK(B, F, \rho)$, for B, F being definable and ii) we can characterise the Tarski hierarchy by sets of the form $Tr_{\mathcal{M},c,\rho}$ with ρ definable in $\mathcal{M}$.

Proposition 15 1. *There is* $TR^\star : P(\omega^2) \to P(\omega)$ *which has a proper implicit definition in* $\mathbf{A}$ *with the following property: let* $\mathcal{M}$ *be a structure,* ρ *a representation of an ordinal* α *and* c *a coding for* $\mathcal{M}$ *compatible with* ρ*. Then* $Tr_{\mathcal{M},c,\rho} = REK_n(Tr_{\mathcal{M},c}, TR^\star, \rho)$*.*

2. *Moreover, there exists a function* $TRO : P(\omega^{2,2}) \to P(\omega^2)$ *with a proper implicit definition in* $\mathbf{A}$ *with the following property: let* $\mathcal{M}$ *be a structure and* c *a coding for* $\mathcal{M}$*. Let* ρ *be a representation of an ordinal such that* ρ *and* c *are compatible. Then*

$$TRO(D_{\mathcal{M},c}, \rho) = Tr_{\mathcal{M},c,\rho}$$

`Proof.` For 1), use Proposition 13 and 2) immediately follows. `QED`

Corollary *Let* $\mathcal{M}$ *be a structure, Let* c *be a coding for* $\mathcal{M}$ *compatible with* ρ*,* ρ *being a representation of an ordinal. Then* $Tr_{\mathcal{M},c,\rho}$ *has a proper implicit definition in* $\mathbf{A} + {}^2D_{\mathcal{M},c} + {}^2\rho$*.*

Lemma 16 *Let* $\mathcal{M}$ *be a structure,* c *a coding for* $\mathcal{M}$*. Let* ρ *be a representation of an ordinal* α*,* ρ *and* c *compatible. Then*

1. *for every* $\beta < \alpha$ *we have*
$$\mathbf{A} + {}^2Tr_{\mathcal{M},c,\rho\lceil(\beta+1)} \sim \mathbf{A} +^1 Tr_{\mathcal{M},c,\rho}(\rho_\beta, .)$$

2. $\rho\lceil\beta$ *is definable in* $\mathbf{A} + {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$*.*

3. *If* $\beta \leq \alpha$ *let us define*
$$\mathcal{M}_{<\beta} := \mathbf{A} + \{\ {}^1Tr_{\mathcal{M},c,\rho}(\rho_\gamma, .); \gamma < \beta\}$$
Let c_β *be the coding for* $\mathcal{M}_{<\beta}$ *induced on* $\mathcal{M}_{<\beta}$*. Assume that* β *is a limit ordinal. Then there is a universal predicate for* $\mathcal{M}_{<\beta}$ *definable in* $A + {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$*.*

`Proof.` 1) and 2) are straightforward. In 3) notice that every set definable in $\mathcal{M}_{<\beta}$ is Δ_1-definable in $A + {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$ and that 1-truth predicate $Tr^1_{M_{<\beta},c_\beta}$ is definable in $A + {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$. `QED`

Lemma 17 *Let* $\mathcal{M}$ *be a structure, let* P *be a universal predicate for* $\mathcal{M}$*. Let* $B \subseteq N^n$*. Let* $F : P(\omega^{n+1}) \to P(\omega^n)$ *have a proper implicit definition in* $\mathcal{M} + P$*. Assume that* β *is a limit ordinal and that* ρ *is a representation of* $\alpha \geq \beta$ *such that* $\rho\lceil\beta$ *is definable in* $\mathcal{M} + P$*. Assume that for every* $\gamma < \beta$*,* $REK_n(B, F, \rho\lceil(\gamma + 1))$ *is definable in* $\mathcal{M}$*. Then* $REK_n(B, F, \rho\lceil\beta)$ *is definable in* $\mathcal{M} + P$*.*

`Proof.` Let θ be a proper implicit definition of F in $\mathcal{M}+P$. Let η be a definition of $\rho\lceil\beta$ in $\mathcal{M}+P$. Let R_θ be a formula scheme in $\mathcal{M}+P$ which is a proper implicit definition of the function REK_F (see Proposition 14). Let $\eta'(x,y,z)$ be the formula $\eta(x,y)\wedge\eta(y,z)$. Then for every $a=\rho_\gamma,\gamma<\beta$, we have $Ext(\eta'(x,y,\overline{a}))=\rho\lceil(\gamma+1)$, and if $a\notin O_{\rho\lceil\beta}$ then $Ext(\eta'(x,y,\overline{a}))=\emptyset$. Since $REK_n(B,F,\rho\lceil\gamma)(\rho_0,.)=B$ then B is definable in $\mathcal{M}$. Let ξ be a definition of B in $\mathcal{M}$. Let S be the scheme

$$S[X^{n+1}](z):=REK_\theta[X^{n+1},\xi,\eta']$$

Then for every $a=\rho_\gamma$ $Z\subseteq\omega^{n+1}$ satisfies $S(\overline{a})$ iff $Z=REK_n[B,F,\rho\lceil(\gamma+1)]$, and if $a\notin O_{\rho\lceil\beta}$ then $S(\overline{a})$ is satisfied by $\emptyset$ only. In $\mathcal{M}$ we can define the relation $Q\subseteq N^2$ such that $Q(a,b)$ iff b is a P-code of a set $Z\subseteq\omega^n$ which satisfies $S(\overline{a})$. Because we assumed that $REK_n(B,F,\rho\lceil(\gamma+1)),\gamma<\beta$ is definable in $\mathcal{M}$ and P is a universal predicate for $\mathcal{M}$ then

i) for every $a=\rho_{\gamma+1},\gamma<\beta$, $Q(a,.)\neq\emptyset$ and furthermore

ii) if $m\in Q(a,.)$ and $a=\rho_\gamma,\gamma<\beta$ then m is a P-code of $REK_n(B,F,\rho_\gamma)$, and if $a\notin O_{\rho\lceil\beta}$ then $P(m,.)=\emptyset$.

Hence the following are equivalent

a) $\langle k_1,\dots k_{n+1}\rangle\in REK_n(B,F,\rho\lceil\beta)$

b) there exist $a,b\in N$ $Q(a,b)$ and $[k_1,\dots k_{n+1}]\in P(b,.)$

But this equivalence can be written as a definition of $REK_n(B,F,\rho\lceil\beta)$ in $\mathcal{M}+P$ `QED`

Proposition 18 *Let $\mathcal{M}$ be a structure. Let $F:P(\omega^{n+1})\to P(\omega^n)$ and $B\subseteq\omega^n$ be definable in $\mathcal{M}$. Let $Z=REK_n(B,F,\rho)$, where ρ is a representation of α. Let c be a coding for $\mathcal{M}$ compatible with ρ. Then Z is definable in $\mathcal{M}+Tr_{\mathcal{M},c,\rho}$.*

`Proof.` Let us prove by induction that for every $\beta\leq\alpha$, $Z_\beta:=REK_n(B,F,\rho\lceil\beta)$ is definable in $\mathcal{M}_\beta:=\mathbf{A}+\ {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$.

Assume that $\alpha>0$, otherwise the proposition is trivial.
We have $Z_0=\emptyset$ and $Z_1=\{\rho_0\}\times B$ which are definable in $\mathbf{A}$ and resp. in $\mathcal{D}(\mathcal{M})\subseteq\mathcal{D}(\mathbf{A}+\ {}^1Tr_{\mathcal{M},c})\sim(\mathbf{A}+\ {}^2Tr_{\mathcal{M},c,\rho\lceil1})$.

Assume the statement holds for every $\gamma<\beta$.

Assume that β is isolated. Then $Z_{\beta-1}$ is definable in $\mathcal{M}_{\beta-1}$. We have

$$Z_\beta=Z_{\beta-1}\cup\{\rho_{\beta-1}\}\times F(Z_{\beta-1})$$

But F is definable in $\mathcal{M}$ and therefore $F(Z_{\beta-1})$ and hence Z_β are definable already in $\mathcal{D}(\mathcal{M}_{\beta-1})\subseteq\mathcal{D}(\mathcal{M}_\beta)$.

Assume that β is a limit. By the assumption, every Z_γ, $\gamma<\beta$ is definable in $\mathcal{N}':=\mathbf{A}+\{\ {}^2Tr_{\mathcal{M},c,\rho\lceil\gamma};\gamma<\beta\}$. By Lemma 16, 2) we have $\mathcal{N}'\sim\mathbf{A}+\{\ {}^1Tr_{\mathcal{M},c,\rho\lceil\beta}(\rho_\gamma,.);\gamma<\beta\}$ and hence every Z_γ, $\gamma<\beta$, is definable in $\mathcal{M}_{<\beta}$. We shall apply Lemma 17. Let us check that the assumptions of the lemma are satisfied. By Lemma 16,1) $\rho\lceil\beta$ is definable in $\mathcal{M}_\beta$. By Lemma 16,3) a universal predicate for $\mathcal{M}_{<\beta}$ is definable in $\mathcal{M}_\beta$. Hence, by Lemma 17, Z_β is definable in $\mathcal{M}_\beta$. `QED`

Lemma 19 *Let $\mathcal{N}$ be a L-finite structure and c a coding for $\mathcal{N}$. Let ρ be a representation of ordinal α compatible with c. Let $\beta\leq\alpha$ and let $\mathcal{M}_{<\beta}$, c_β be as defined in Lemma 16,3). Let P be a universal predicate for $M_{<\beta}$ such that $\rho\lceil\beta$ is definable in $M_{<\beta}+P$. Then $Tr_{M_{<\beta},c_\beta}$ is definable in $\mathcal{M}_{<\beta}+P$.*

`Proof.` Let us first show that $Tr_{\mathcal{M},c,\rho\lceil\beta}$ is definable in $\mathbf{A}+P$.

Assume that β is isolated. Then ${}^1Tr_{\mathcal{N},c,\rho}(\rho_{\beta-1},.)\in\mathcal{M}_{<\beta}$ and hence it is definable in $\mathbf{A}+P$. But from Lemma 16,1) we have $\mathbf{A}+\ {}^1Tr_{J,c,\rho}(\rho_{\beta-1},.)\sim\mathbf{A}+\ {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$ and $Tr_{\mathcal{M},c,\rho}(\rho_\beta,.)$ is definable in $\mathbf{A}+P$.

Assume that β is limit. We shall use Lemma 17 (note that Proposition 15 asserts that $Tr_{\mathcal{M},c,\rho}=REK(Tr_{\mathcal{M},c},TR^\star,\rho)$ where $TR^\star$ has a *proper* implicit definition). From Lemma 16 we have

$$\mathcal{M}_{<\beta}\sim\mathcal{M}+\{\ {}^2Tr_{J,c,\rho\lceil(\gamma+1)};\gamma<\beta\}$$

and hence every $Tr_{\mathcal{M},c,\rho\lceil(\gamma+1)}$, $\gamma < \beta$, is definable in $\mathcal{M}_{<\beta}$. Furthermore, since P is a universal predicate for $\mathcal{M}_{<\beta}$, then by Lemma 17, $Tr_{\mathcal{M},c,\rho\lceil\beta}$ is definable in $\mathcal{M}$.

It is trivial to show that the set $D_{\mathcal{M}_{<\beta},c_\beta}$ is definable in $\mathbf{A} + {}^2Tr_{\mathcal{M},c,\rho\lceil\beta}$ and hence it is definable in $\mathbf{A} + P$. Therefore, by Proposition 6, $Tr_{\mathcal{M}_{<\beta},c_\beta}$ is definable in $\mathbf{A} + P$. QED

Theorem 20 *Let $\mathcal{N}$ be a L-finite structure. Let $\{\mathcal{M}_\alpha\}_{\alpha<\lambda(\mathcal{N})}$ be a Tarski hierarchy over $\mathcal{N}$. Let β be a definable ordinal in $\mathcal{N}$.*

1. *Then $\beta < \lambda(\mathcal{N})$*
2. *Furthermore, let c be a coding for $\mathcal{N}$. Let ρ be a representation of the ordinal $\beta + 1$ compatible with c definable in $\mathcal{N}$. Let P_β be a canonic universal predicate for $\mathcal{M}_\beta$. Then $\mathbf{A} + P_\beta \sim \mathbf{A} + {}^1Tr_{\mathcal{M},c,\rho}(\rho_\beta, .)$.*

The same is true for the proper universal predicate and proper Tarski hierarchy.

Proof. We shall say that c is an *ultracanonic coding for a structure* $\mathcal{M}$ iff $Tr_{\mathcal{M},c}$ is definable in every $\mathcal{M} + P$, where P is a universal predicate for $\mathcal{M}$. From Lemma 8 and Proposition 6 we obtain the following:

Let $\mathcal{N}_1 \sim \mathcal{N}_2$. Assume that $\mathcal{N}_1$ has an ultracanonic coding c_1. Then $\mathcal{N}_2$ has a canonic universal predicate and if P is a canonic universal predicate for $\mathcal{N}_2$ then $\mathbf{A} + Tr_{\mathcal{N}_1,c_1} \sim \mathbf{A} + P$.

Assume that $\beta, \rho, \mathcal{M}, c$ are as in the statement 2). By transfinite induction we shall prove the proposition:

For every $\alpha \leq \beta$ it is the case that $\alpha < \lambda(\mathcal{N})$. Moreover, if P_α denotes the canonic universal predicate for $\mathcal{M}_\alpha$ then $A + P_\alpha \sim \mathbf{A} +^1 Tr_{\rho,\mathcal{M},c}(\rho_\alpha, .)$

First, let $\alpha = 0$. Then $\alpha < \lambda(J)$. From the definition of $Tr_{\mathcal{M},c,\rho}$ we obtain $Tr_{\mathcal{N},c,\rho}(\rho_0, .) = Tr_{\mathcal{M},c}$. Furthermore, since $\mathcal{N}$ is finite then any coding for $\mathcal{N}$ is ultracanonic (Proposition 9). Hence $\mathbf{A} + {}^1Tr_{\mathcal{M},c} \sim \mathbf{A} + P_0$ and so $\mathbf{A} + {}^1Tr_{\mathcal{N},c,\rho}(\rho_0, .) \sim \mathbf{A} + P_0$.

Let $0 < \alpha \leq \beta$ and assume that the proposition is true for every $\gamma < \alpha$. Let $\mathcal{M}'$ be the structure $\mathcal{M}_\alpha = \mathcal{M} + \{P_\gamma; \gamma < \alpha\}$. By the assumption $\mathcal{M}' \sim \mathcal{M} + \{ {}^1Tr_{\rho,\mathcal{M},c}(\rho_\gamma, .), \gamma < \alpha\}$. Let $\mathcal{M}_{<\alpha}$ denote the structure on the right hand side and let c_α be the coding induced on $\mathcal{M}_{<\alpha}$. By the previous Lemma, c_α is an ultracanonic coding for $\mathcal{M}_\alpha$ and hence M_α has a canonic universal predicate and if P_α is a canonic universal predicate for $\mathcal{M}_\alpha$ then $\mathbf{A} + Tr_{\mathcal{M}_{<\alpha},c_\alpha} \sim \mathbf{A} + P_\alpha$. But from the definition of $Tr_{\mathcal{M},c,\rho}$ we have $Tr_{\mathcal{M}_{<\alpha},c_\alpha} = Tr_{\mathcal{N},c,\rho}(\rho_\alpha, .)$; hence $\mathbf{A} + P_\alpha \sim \mathbf{A} + {}^1Tr_{\mathcal{N},c,\rho}(\rho_\alpha, .)$.

For the proper Tarski hierarchy the proof is exactly the same. QED

Corollary 1 *Let $\{\mathcal{M}_\alpha\}_{\alpha<\lambda(J)}$ be a Tarski hierarchy over $\mathcal{N}$. Let β be isolated, let c be a coding for $\mathcal{M}$. Let ρ be a definable representation of the ordinal β in $\mathcal{N}$ compatible with c. Then*

$$\mathcal{M}_\beta \sim \mathbf{A} + {}^1Tr_{\mathcal{N},c,\rho}(\rho_{\beta-1}, .) \sim \mathbf{A} + {}^2Tr_{\mathcal{N},c,\rho\lceil\beta}$$

Proof. Follows from the previous Theorem and Lemma 16. QED

Corollary 2 *Let $\{\mathcal{M}_\alpha\}_{\alpha<\lambda(\mathcal{N})}$ be a Tarski hierarchy over $\mathcal{N}$ and let $\{\mathcal{M}^p_\alpha\}_{\alpha<\lambda^p(\mathcal{N})}$ be a proper Tarski hierarchy over $\mathcal{N}$. Then for every $\beta \leq Ord(\mathcal{N})$, we have $\beta \leq \lambda(\mathcal{N}) \cap \lambda^p(\mathcal{N})$ and $\mathcal{M}_\beta \sim \mathcal{M}^p_\beta$.*

Proof. Immediate. QED

5 Trees and the second half of Theorem 1

We now proceed to prove the rest of Theorem 1, i.e., to show that $\lambda(\mathcal{N}) = Ord(\mathcal{N})$ and that $\mathcal{M}_{\lambda(\mathcal{N})} = \mathcal{I}(\mathcal{N})$. We shall first prove the theorem (see page 19)

Theorem 21 *Let $\mathcal{N}$ be a L-finite structure, let $\{M_\alpha\}_{\alpha<\lambda(\mathcal{N})}$ be a Tarski hierarchy over $\mathcal{N}$. Then $\mathcal{M}_{Ord(\mathcal{N})} \sim \mathcal{I}(\mathcal{N})$.*

Second, we will prove (see page 21)

Theorem 22 *Let $\mathcal{N}$ be a L-finite structure. Then $\mathcal{I}(\mathcal{N})$ does not have a canonic universal predicate.*

For those purposes, we shall use some properties of trees and linear orderings. Trees are a standard tool for proving uniformization results (see for example [4]). Results concerning the definability of well-orderings can be found in [2].

Definition 16 1. *$\tau \subseteq \omega^2$ is* a tree *iff i) for every $x \in Rng(\tau)$ the set $Pred(x) := \{y \in \omega; \tau(y,x)\}$ is finite and $\tau\lceil Pred(x)$ is a linear ordering and ii) there is $0_\tau \in \omega$ such that for every $x \in Rng(\tau)$, $\tau(0_\tau, x)$.*

2. *τ will also be written as $\preceq_\tau$. $x \prec_\tau y$ is defined in the obvious way, and so is $inf(X)$ for $X \subseteq Rng(\tau)$.*

   $x \prec^s_\tau y$ iff $x \prec y$ and there is no z $x \prec_\tau z$, $z \prec_\tau y$.

   $Succ_\tau(x) := \{y; x \preceq_\tau y\}$ $x \in T_\tau$.

   τ_x is a tree such that $\tau_x := \tau\lceil Succ_\tau(x)$

3. *$B \subseteq T_\tau$ is* a chain in τ *iff for every $x, y \in B$ $x \preceq y$ or $y \preceq x$.* A branch in τ *is a maximum chain in τ.*

Definition 17 *Let τ be a tree. On $Rng(\tau)$ we define a binary relation KB_τ in the following way: $x, y \in \omega$ then $KB_\tau(x,y)$ iff $y \preceq_\tau x$ or there are $u, v \in Rng(\tau)$ $inf(x,y) \prec^s_\tau u, v$, $u \preceq_\tau x$, $v \preceq_\tau y$ and $u < v$. We shall refer to KB_τ as* the Kleene-Brouwer ordering.

The following two Propositions give us the basic properties of KB_τ that we shall need.

Proposition 23 *Let τ be a tree. Then*

1. *KB_τ is a linear ordering on $Rng(\tau)$.*
2. *for every $x \in Rng(\tau)$, $KB(\tau_x) = KB_\tau\lceil Succ_\tau(x)$ and $Succ_\tau(x)$ is an interval in KB_τ.*
3. *τ has no infinite branch iff KB_τ is a well-ordering.*
4. *if $A = a_0, a_1, \ldots$ is an infinite decreasing sequence in KB_τ then the set $\{b_i; i \in \omega\}$, where $b_n := hinf\{a_n, a_{n+1}, a_{n+2}\ldots\}$, is an infinite chain in τ.*

Definition 18 *Let R be a linear ordering. $wo(R) \subset Dom(R)$ is the set such that $wo(R)$ is a maximum lower segment in R such that $\prec_R \lceil wo(R)$ is a well-ordering. $wo(R)$ will be called* the well-ordered part of *R. The order-type of $\prec_R \lceil wo(R)$ shall be denoted by $ord(R)$.*

It is evident that $wo(R)$ is defined uniquely; the existence follows from the axiom of choice.

Proposition 24 *Let $\mathcal{M}$ be a structure, let τ be a tree definable in $\mathcal{M}$. Then*

1. *KB_τ is definable $\mathcal{M}$. In addition, if τ is Δ_1 in $\mathcal{M}$ and for every x, y, $x \prec_\tau y$ implies $x < y$ then KB_τ is Δ_1-definable in $\mathcal{M}$.*
2. *If there is a nonempty $X \subseteq Dom(KB_\tau)$definable in $\mathcal{M}$ which does not have a KB_τ-first member then there is an infinite branch of τ definable in $\mathcal{M}$.*
3. *Assume that τ has an infinite branch. Then if $wo(KB_\tau)$ is definable in $\mathcal{M}$ then an infinite branch of τ is definable in $\mathcal{M}$.*

`Proof.` 1) is obtained be translating the definition of KB_τ to the structure $\mathcal{M}$. 2) and 3) follow from Proposition 23. `QED`

We shall now proceed to assign trees to formula schemes. In the following definition, $\mathbf{y}$ will designate a list of variables $y_1, \ldots y_n$. $\forall\mathbf{y}$ will is an abbreviation for $\forall y_1, \ldots \forall y_n$, and similarly in the case of $\exists\mathbf{y}$.

Definition 19 *Let ψ be a closed formula scheme in a prenex form.*

1. *For* $i = 1, \dots n$ *let* $\mathbf{y}_i = {y^i}_1, \dots {y^i}_{l_k}$*. We can assume that the variables are mutually different. Let*

$$\psi = (\forall \mathbf{x}_1)(\exists \mathbf{y}_1)\dots(\forall \mathbf{x}_n)(\exists \mathbf{y}_n)\psi_0,$$

where ψ_0 *is an open formula. Let* $k := \sum_{i=1}^n l_i$*. Then* the Skolem formula for ψ, ψ_s will be the formula $(H^1(\mathbf{x}_1, {y^1}_1) \wedge \dots H^2(\mathbf{x}_1, {y^1}_{l_1}) \wedge \dots H^{k-l_n}(\mathbf{x}_n, {y^n}_1) \wedge \dots H^k(\mathbf{x}_n, {y^n}_{l_n})) \Rightarrow \psi_0,$
where $H^1, \dots H^k$ are second order variables not occuring in ψ of the appropriate arity.

2. Let $\psi = \psi[X_1, \dots X_k] \in Fle^\lambda$, $\lambda \in \omega^{<\omega}$. If ψ_s is as above, we shall write $\psi_s = \psi_s[H^1, \dots H^k]_s = \psi[H] = \psi_s[X_1, \dots X_k][H^1, \dots H^k]_s$, where $H = \langle H^1, \dots H^k\rangle$. If $\mu = n_1, \dots n_k$ and the arity of H^i is n_i then we shall say that $\psi_s \in Fle^{\lambda,\mu}$.

3. Let ψ be a formula in a prenex form, $\psi_s = \psi_s[H^1, \dots H^k]_s$, where the arity of H^i is $n_i + 1$. Functions $g_i : \omega^{n_i} :\to \omega$, $i = 1, \dots k$, will be called the *Skolem functions for* ψ iff $\psi_s[\ ^{n_1+1}g_1, \dots\ ^{n_k+1}g_k]$ is true.

In the third item of the definition we identify n-ary function $g : \omega^n \to \omega$ with the set of $(n+1)$-tuples and hence we can use a function in the place of a predicate. Note that n-ary function f can stand in a place of a function symbol as ${}^n f$, while in a place of a predicate as ${}^{n+1}f$. The following is then obvious:
Let $\psi = \psi(y_1, \dots y_n, z)$ *be a formula scheme. Let* $f : \omega^n \to \omega$*. Then the scheme* $\psi(y_1, \dots y_n,\ {}^n f(y_1, \dots y_n))$ *is equivalent to the scheme* $\forall z\ {}^{n+1}f(y_1, \dots y_n, z) \Rightarrow \psi(y_1, \dots y_n, z)$*.*
We may conclude that if $\psi_s = (H^1(\mathbf{y}_1, z_1) \wedge \dots H^k(\mathbf{y}_k, z_k)) \Rightarrow \psi_0$ then the functions $g_1, \dots g_k$ are Skolem functions for ψ iff the formula $\forall \mathbf{y}_1 \dots \forall \mathbf{y}_k \psi_0(z_1/\ {}^{n_1}g_1(\mathbf{y}_1), \dots z_k/\ {}^{n_k}g_k(\mathbf{y}_k))$ is true. This implies the following lemma:

Lemma 25 *Let* $\mathcal{M}$ *be a structure. Let* $\lambda,\ \mu \in \omega^{<\omega}$*, let* $\psi \in Fle^\lambda$ *be a formula scheme in* $\mathcal{M}$ *in a prenex form. Let* $B \in P(\omega^\lambda)$*. Let* $\psi_s \in Fle^{\lambda,\mu}$*. Then*

1. *B satisfies* ψ *iff there are functions* $g_1, \dots g_l$ *which are the Skolem functions for* $\psi_s[\ {}^\lambda B]$*.*
2. *Assume that B is definable in* $\mathcal{M}$ *and* $g_1, \dots g_l$ *are Skolem functions for* $\psi_s[\ {}^\lambda B]$*. Let* $h_i := g_i\lceil n,\ i = 1, \dots l$*. Then there are Skolem functions* $w_1, \dots w_l$ *for* $\psi_s[B]$ *such that* $h_i \subseteq w_i, i = 1, \dots l$ *and* $w_1, \dots w_l$ *are definable in* $\mathcal{M}$*.*

Definition 20 *Let* $\lambda, \mu \in \omega^{<\omega}$*,* $\lambda = \langle a_1, \dots a_s\rangle$*,* $\mu = \langle b_1 + 1, \dots b_t + 1\rangle$*. Let* $\psi \in Fle^\lambda$ *be a formula scheme in a prenex form. Let* $\psi_s = \psi_s(y_1, \dots y_l) \in Fle^{\lambda,\mu}$*. Then* α *is a* satisfaction system of degree n for ψ *iff*

$$\alpha = \langle B_1 \dots B_s, g_1 \dots g_t\rangle$$

and the following conditions are satisfied[8]*.*

1. $g_i : n^{b_i} \to \omega, i = 1 \dots t$
2. *Let* $e_\alpha := \max\{n, \max(Rng(g_i)) + 1; i = 1 \dots t\}$*. Let F be the set of functions occuring in* ψ*. Let* $m_\alpha := \max\{n, \max Rng(f\lceil e_\alpha); i = 1 \dots t, f \in F\}$*. Then* $B_i \subseteq m_\alpha^{a_i}$*,* $i = 1 \dots s$
3. *The formula* $\psi_s[\ {}^\lambda B][\ {}^\mu g](y_1, \dots y_l)$ *is satisfied by every* $a_1, \dots a_l$ *such that* $a_1, \dots a_l < e_\alpha$*.*

The intuition behind the definition is simple. Assume, for clarity, that ψ_s contains no function symbols and that $\psi_s = \psi = \psi[A_1, \dots A_n]$ (i.e. ψ is an open formula scheme containing no function symbols). Then a satisfaction system of degree n is a sequence of $B_i \subseteq \{0, 1, \dots n-1\}^{a_i}$, $i = 1, \dots s$, such that the formula $\psi_s[\ {}^{b_1}B_1, \dots\ {}^{b_s}B_s]$ is true when we let the variables range over $0, 1, \dots n-1$ only. The sets B_i can be viewed as predicates defined on $\{0, 1 \dots n-1\}^{b_i}$, and we demand they satisfy the formula on the domain of their definition. In general $\psi_s = \psi_s[A_1, \dots A_n][g_1, \dots g_l]$ and we assume that the functions $g_1, \dots g_l$ are defined on $\{0, 1, \dots n-1\}$. However, we must make sure that the predicates $B_1, \dots B_s$ are defined on the ranges of those functions and the other functions occuring in ψ_s restricted on $\{0, 1, \dots n-1\}$. Point 2) reflects the fact that for a Skolem function f and $g \in \mathcal{F}(L)$ the term $g(f)$ may occur in ψ_s but the term $f(g)$ cannot.

[8] Recall that $n = \{0, 1 \dots n-1\}$, $n \in \omega$.

Definition 21 *Let $\psi \in Fle^\lambda$ be a formula scheme in a prenex form.*

1. *Let $\alpha = \langle B_1 \dots B_s, g_1 \dots g_t \rangle$ be a satisfaction system for ψ of degree n and let $\alpha' = \langle B'_1 \dots B'_s, g'_1 \dots g'_t \rangle$ be a satisfaction system for ψ of degree m. Then we let*

$$\alpha \preceq \beta$$

iff $m \leq n$ and i) for every $i = 1, \dots t$ if g_i is a k-ary function then $g_i = g'_i \lceil n^k$ ii) for every $i = 1, \dots s$ if $B_i \subseteq \omega^k$ then $B_i = B'_i \cap e^k_n$.

2. The characteristic tree of ψ *is the $\tau \subseteq \omega^2$ such that for every $a, b \in \omega$, $\tau(a,b)$ iff there are α, β satisfaction systems of ψ, $a = [\alpha]$, $b = [\beta]$[9] and $\alpha \preceq \beta$.*

3. *Let ψ be a formula scheme in a prenex form. Let τ_ψ be the characteristic tree of ψ. The ordinal $ord(KB(\tau_\psi))$ will be called* the characteristic ordinal of ψ *; it will be denoted by $ord(\psi)$.*

The key property of a satisfaction system is expressed in the next Lemma.

Lemma 26 *Let $\psi \in Fle^\lambda$ be in a prenex form. Let $\alpha^i = \langle B^i_1 \dots B^i_s, g^i_1 \dots g^i_t \rangle$, $i \in \omega$ be a sequence of satisfaction systems of ψ such that $\alpha_0 \prec \alpha_1 \prec \alpha_2 \dots$. Then $B := \langle \bigcup_{i\in\omega} B^i_1, \dots \bigcup_{i\in\omega} B^i_s \rangle$ satisfies ψ and $g_l := \bigcup_{i\in\omega} g^i_l$, $l = 1, \dots t$, are the Skolem functions for $\psi[\ ^\lambda B]$.*

`Proof.` Evident `QED`

Proposition 27 *Let $\mathcal{M}$ be a structure. Let ψ be a formula scheme in a prenex form in $\mathcal{M}$, τ_ψ the characteristic tree of ψ. Let $\mathcal{A}$ be the system defined by ψ. Then*

1. *τ is a Δ_1-definable tree in $\mathcal{M}$.*
2. *$\mathcal{A} \neq \emptyset$ iff τ contains an infinite branch.*
3. *There is $B \in \mathcal{A}$ definable in $\mathcal{M}$ iff τ contains a definable infinite branch in $\mathcal{M}$.*
4. *$KB(\tau_\psi)$ is Δ_1-definable in $\mathcal{M}$. If $\mathcal{A} \neq \emptyset$ and $wo(KB(\tau_\psi))$ is definable in $\mathcal{M}$ then some $A \in \mathcal{A}$ is definable in $\mathcal{M}$.*

`Proof.` 1) follows from the definition of τ_ψ and Proposition 5.
2) and 3) follow from Lemmata 26 and 25.
4) follows from Proposition 24, 1) and 3). `QED`

Lemma 28 *Let $\mathcal{M}$ be a structure, ψ a formula scheme in $\mathcal{M}$ in a prenex form. Let ψ be a formula scheme in $\mathcal{M}$ which defines a non-empty system $\mathcal{A}$. Let ρ be an arbitrary representation of ordinal α, $ord(\mathcal{A}) \leq \alpha$. Let c be a coding for the structure $\mathcal{M}$ compatible with ρ. Then there is $B \in \mathcal{A}$ definable in $\mathbf{A} +\ ^2Tr_{\mathcal{M},c,\rho}$.*

`Proof.` Let τ be the characteristic tree of ψ. By Theorem 27, 4) it is sufficient to show that $wo(KB_\tau)$ is definable in $\mathbf{A} +\ ^2Tr_{\mathcal{M},c,\rho}$

Since KB_τ is not a well-ordering, then $wo(KB_\tau) \neq \omega$. Let us chose $a \notin wo(KB_\tau)$.

Let f be a function $f : Dom(\rho) \to Dom(KB_\tau)$ such that

1. if there is z a KB_τ-minimum of $Rng(\tau)$, let $f(\rho_0) := z$ else $f(\rho_0) := a$
2. Let $y \in Rng(\tau) \setminus \{\rho_0\}$. If there exists z which is KB_τ-minimum of $Rng(KB_\tau) \setminus Rng(f\lceil\{x; x \prec_\rho y\})$, then $f(y) := z$. Otherwise $f(y) = a$.

Since KB_τ is definable in $\mathcal{M}$, it is trivial to find B and F definable in $\mathcal{M}$ such that

$$f = REK_2(B, F, \rho)$$

We can apply Proposition 18 to obtain that f is definable in $\mathbf{A} +\ ^2Tr_{\mathcal{M},c,\rho}$. But $wo(KB_\tau) = Rng(f) \setminus \{a\}$ and hence $wo(KB_\tau)$ is definable in $\mathbf{A} +\ ^2Tr_{\mathcal{M},c,\rho}$. `QED`

[9]Here $[\alpha]$ is the Gödel number of the finite set α see page 4.

Lemma 29 *Let $\mathcal{M}$ be a structure. Let R be a linear ordering definable in R. Then*

$$ord(R) \leq Ord(\mathcal{M})$$

If in addition R is Δ_1 in $\mathcal{M}$ then $ord(R) \leq \xi$, the first ordinal not Δ_1-definable in $\mathcal{M}$.

`Proof.` Let R be a linear ordering definable in $\mathcal{M}$. For $n \in wo(R)$, R_n is a representation of an ordinal $\alpha_n \leq ord(R)$[10]. Clearly $ord(R) = sup\{\alpha_i, i \in \omega\}$. If R is (Δ_1-) definable in $\mathcal{M}$ then R_n is (Δ_1-) definable in $\mathcal{M}$ for every $n \in wo(R)$. Hence $ord(R) \leq Ord(\mathcal{M})$ (resp. $ord(R) \leq \xi$). `QED`

Proposition 30 *Let $\mathcal{M}$ be a structure. Let α be a countable ordinal. Then the following conditions are equivalent*

1. *α is implicitly definable in $\mathcal{M}$.*
2. *α is definable in $\mathcal{M}$.*
3. *α is Δ_1-definable in $\mathcal{M}$.*

`Proof.` The implications 3)$\Rightarrow$ 2) $\Rightarrow$ 1) are trivial.

1)$\Rightarrow$ 3). Assume the contrary. Without the loss of generality we can assume that $\mathcal{M}$ is finite. Let ρ be an implicitly definable representation of α such that α is not Δ_1-definable. We can assume that ρ is compatible with a coding c for $\mathcal{M}$. We will show that $\mathcal{I}(\mathcal{M}) \subseteq \mathcal{D}(\mathbf{A} +^2 Tr_{\mathcal{M},c,\rho})$. Let ψ be a formula scheme in $\mathcal{M}$ which defines a non-empty system $\mathcal{A}$. By the Lemma 29 every $\beta < ord(\psi)$ is Δ_1-definable in $\mathcal{M}$. It follows that $\alpha \geq ord(\psi)$. By Lemma 28 there is some $A \in \mathcal{A}$ definable in $\mathbf{A} +\ ^2Tr_{\mathcal{M},c,\rho}$. Hence $\mathcal{I}(\mathcal{M}) \subseteq \mathcal{D}(\mathbf{A} +^2 Tr_{\mathcal{M},c,\rho})$. We assumed that $\mathcal{M}$ is finite and ρ is implictly definable in $\mathcal{M}$ and hence also $Tr_{\mathcal{M},c,\rho}$ is implicitly definable in $\mathcal{M}$; therefore $\mathcal{I}(\mathcal{M}) \sim \mathbf{A} +^2 Tr_{\mathcal{M},c,\rho}$. But that contradicts Corollary 2 of Proposition 13. `QED`

The statement of the proposition may be strengthened to say that every definable system of ordinals contains a definable element or even that every definable system of ordinals has a definable supremum (see [2], Chapter IV), but those modifications will not be needed here.

Proposition 31 *Let $\mathcal{M}$ be a structure. Let ψ a formula scheme in $\mathcal{M}$. Then*

1. *$ord(\psi) \leq Ord(\mathcal{M})$*
2. *If ψ is a proper implicit definition of some B then $ord(\psi) < Ord(\mathcal{M})$.*

`Proof.` 1) Let τ be the characteristic tree of ψ in L which defines ψ. Then KB_τ is definable in $\mathcal{M}$ (Proposition 24) and $ord(KB_\tau) \leq Ord(\mathcal{M})$ by Lemma 29.

2) without the loss of generality we can assume that $\mathcal{M}$ is finite and that $B \subseteq \omega$. It is sufficient to prove that $wo(KB_\tau)$ is implicitly definable in $\mathcal{M}$. Since KB_τ is definable in $\mathcal{M}$, $KB_\tau \lceil wo(KB_\tau)$ is then an implicitly definable representation of $ord(\psi)$ in $\mathcal{M}$ and therefore, by Proposition 30, $ord(\psi)$ is definable in $\mathcal{M}$. Let T be a truth predicate for $\mathcal{M} +\ ^1B$. Since T is implicitly definable in $\mathcal{M} +\ ^1B$ and B is implicitly definable in $\mathcal{M}$, T is implicitly definable in $\mathcal{M}$ and it is sufficient to prove that $wo(KB_\tau)$ is definable in $\mathcal{M} +\ ^1B +\ ^1T$.

Clearly, the two conditions are equivalent:

1. $x \in wo(KB_\tau)$
2. there is no $y \in Rng(\tau)$ such that $KB_\tau(y, x)$ and y lies on an infinite branch of τ.

On the other hand, by Lemma 25, 2) the condition 'y lies on an infinite branch of τ' is equivalent to the condition 'y lies on an infinite branch of τ definable in $\mathcal{M} +\ ^1B$'. But the later statement can be expressed using the truth predicate T, and hence the condition 2) can be expressed in $\mathcal{M} +\ ^1B +\ ^1T$. Therefore $wo(KB_\tau)$ is definable in $\mathcal{M} +\ ^1B +\ ^1T$. `QED`

[10] For the definition of R_n see page 10

Lemma 32 *Let $\mathcal{N}$ be a structure. Let $\mathcal{T}$ be a set of binary predicates such that i) every $X \in \mathcal{T}$ is of the form ${}^2Tr_{\mathcal{N},c,\rho}$, where ρ is a well-ordering definable in $\mathcal{N}$ compatible with coding c and ii) for every $\alpha < Ord(\mathcal{M})$ there exists ρ a representation of $\beta \geq \alpha$ definable in $\mathcal{N}$ and a coding c for $\mathcal{N}$ such that ${}^2Tr_{\mathcal{N},c,\rho} \in \mathcal{T}$. Then $\mathcal{I}(\mathcal{N}) \sim A + \mathcal{T}$.*

`Proof.` $\mathcal{I}(\mathcal{N}) \subseteq \mathcal{D}(A + \mathcal{T})$ follows from Lemma 28 and Proposition 31. By Proposition 15 we have also $\mathcal{T} \subseteq \mathcal{I}(\mathcal{N})$ `QED`

Proof of Theorem 21. Theorem 21 is now a direct consequence of the previous Lemma and the corollary of Theorem 20. `QED`

In order to prove Theorem 22, we shall find a linear ordering R definable in $\mathcal{N}$ such that $ord(R) = Ord(\mathcal{N})$. This will be achieved by means of a formula scheme in $\mathcal{N}$ such that $ord(\psi) = Ord(\mathcal{N})$. It must be noted that $ord(\psi) \leq Ord(\mathcal{N})$ but the condition $ord(\psi) < Ord(\mathcal{N})$ in general holds just for schemes which implicitly define a set. In $\mathcal{N}$ there may exist systems defined by a scheme ψ such that the characteristic ordinal of ψ is not definable in $\mathcal{N}$.

Observe that for a formula ψ defining a nonempty system $\mathcal{A}$ if $ord(\psi) < Ord(\mathcal{M})$ then, by Lemma 28, there is some $A \in \mathcal{A}$ implicitly definable in $\mathcal{M}$. Hence, if for every formula scheme in $\mathcal{M}$, $ord(\psi) < Ord(\mathcal{M})$ then every non-empty system definable contains an implicitly definable set in $\mathcal{M}$. This is the essence of the folowing definition.

Definition 22 *Let $\mathcal{M}$ be a structure. Then $\mathcal{M}$ is* implicitly complete *iff every non-empty system definable in $\mathcal{M}$ contains an implicitly definable set in $\mathcal{M}$.*

Lemma 33 *Let $\mathcal{M}$ be a structure. If for every R a linear ordering definable in $\mathcal{M}$, $ord(R) < Ord(\mathcal{M})$ then $\mathcal{M}$ is implicitly complete.*

`Proof.` We can assume that $\mathcal{M}$ is finite. Let ψ be a scheme in $\mathcal{M}$ in a prenex form which defines a non-empty system $\mathcal{A}$. Let τ be the characteristic tree. KB_τ is definable in $\mathcal{M}$ and by the assumption there is $ord(KB_\tau) < Ord(\mathcal{M})$. We can find a definable representation ρ in $\mathcal{M}$ of an ordinal β, $ord(KB_\tau) < \beta < Ord(\mathcal{M})$. We can assume that ρ is compatible with a coding c for $\mathcal{M}$. The set $Tr_{\mathcal{M},c,\rho}$ is implicitly definable in $\mathcal{M}$ and by Lemma 28 there is some $A \in \mathcal{A}$ definable in $\mathbf{A} + {}^2Tr_{\mathcal{M},c,\rho}$. `QED`

In definition 15 we introduced $\overline{Tr}(\mathcal{M}, c, R)$,which is a generalisation of the concept of $Tr_{\mathcal{M},c,\rho}$ if R is not a well-ordering. Similarly to Proposition 15 we may obtain:
There is a system $\mathcal{S} \subseteq P(\omega^{2,2,2})$ definable in $\mathbf{A}$ such that for every structure $\mathcal{M}$ and a coding c for $\mathcal{M}$ and a linear ordering R, $\langle D_{\mathcal{M},c}, R, U\rangle \in \mathcal{S}$ iff $U \in \overline{Tr}(\mathcal{M}, c, R)$

Lemma 34 *Let $\mathcal{N}$ be a L-finite structure, c a coding for $\mathcal{N}$. Let R be a linear ordering compatible with c such that $Ord(\mathcal{N}) \leq ord(R)$. Let $U \in \overline{Tr}(\mathcal{N}, c, R)$. If $x \in Rng(R) \setminus wo(R)$ then there is a universal predicate for $\mathcal{I}(\mathcal{N})$ definable in $A + {}^1U(x, .)$. Hence $\mathcal{I}(\mathcal{N}) \subseteq \mathcal{D}(\mathbf{A} + {}^1U(x, .)) \subseteq \mathcal{D}(\mathbf{A} + {}^2U)$.*

`Proof.` For $n \in N$ let $U_n \subseteq N^2$ be a relation such that $U_n(a, b)$ iff $U(a, b)$ and $R(a, n)$, let R_n be defined as on page 10. For $n \in wo(R)$ we have $U_n = Tr_{\mathcal{N},c,R_n}$. Since $Ord(\mathcal{N}) \leq ord(R)$ then for every ordinal α definable in $\mathcal{N}$ there is some $n \in wo(R)$ such that R_n is a representation for α. Therefore, using Lemma 32, every set implicitly definable in $\mathcal{N}$ is definable in $\mathcal{N} + \{{}^2U_n; n \in wo(\mathcal{N})\} \sim \mathcal{N} + \{{}^1U(n, .); n \in wo(\mathcal{N})\}$ (see Lemma 16). The set $U(x, .)$ is defined to be a truth predicate for the structure $\mathcal{M} := \mathbf{A} + \{{}^1U(y, .); y \prec_R x)\}$. A proper universal predicate P for $\mathcal{M}$ is therefore definable in $\mathbf{A} + {}^1U(x, .)$. But $\mathcal{I}(J) \subseteq \mathcal{D}(\mathcal{N} + \{{}^2U(n, .); n \in wo(\mathcal{N})\}) \subseteq \mathcal{M}$ and hence P is a universal predicate for $\mathcal{I}(\mathcal{N})$. `QED`

Proposition 35 *Let $\mathcal{N}$ be a L-finite structure. Then $\mathcal{N}$ is not implicitly complete.*

`Proof.` Let $\mathcal{N}$ be a L-finite structure and c a coding for $\mathcal{N}$. In $\mathcal{N}$ we can find a formula scheme in $\psi \in Fle_1^{2,2,1,3}(\mathcal{N})$ such that for every $R, U, X, Y \in P(\omega^{2,2,1,3})$, R, U, X, Y satisfies ψ iff

1. R is a linear ordering compatible with c and R has the first member 0_R,
2. $U \in \overline{Tr}(D_{\mathcal{N},c}, R)$,
3. for every $n \in N$ if n is a c-Gödel number of a formula in $\mathcal{N}$ defining a linear ordering Q then either a) $X(n,.) \subseteq Dom(Q)$ is a set without the Q-first member or b) $Y(n,.,.)$ is an isomorphism between Q and a lower segment of R. [11]

Observe that the condition 3) uses just quantifications over sets definable in $\mathcal{N}$ and hence it can be formulated using the fact that R has the smallest member and $U(R_0,.) = Tr_{\mathcal{M},c}$.

If R, U, X, Y satisfy ψ then $Ord(\mathcal{N}) \leq ord(R)$ because for every well-ordering Q definable in $\mathcal{N}$ the condition a) is not satisfied and therefore Q must be isomorphic to a lower segment of R. Moreover, for every R which is a representation of an ordinal $\alpha \geq Ord(\mathcal{N})$ compatible with c there are some U, X, Y such that R, X, Y, U satisfies ψ. Hence the system defined by ψ is non-empty.

Let us assume that $\mathcal{N}$ is implicitly complete. Then there are some R, X, Y, U satisfying ψ which are implicitly definable in $\mathcal{N}$. By Lemma 34 we have $\mathcal{I}(\mathcal{N}) \subseteq \mathcal{D}(\mathbf{A} + {}^2U)$. Since we assumed that U is implicitly definable this implies that the structure $\mathcal{I}(\mathcal{N})$ is essentially finite. But this contradicts the Corollary 2 of Proposition 13. QED

Corollary *Let $\mathcal{N}$ be finite. Then there is a linear ordering R definable in $\mathcal{N}$ such that $ord(R) = Ord(\mathcal{N})$. Hence $ord(R)$ is not definable and $wo(R)$ is not implicitly definable in $\mathcal{M}$.*
Proof. Follows from the previous Theorem and Lemma 33. QED.

Lemma 36 *Let $\mathcal{N}$ be a L-finite structure and c a coding for $\mathcal{N}$. Then there is a linear ordering R definable in $\mathcal{N}$ and some $U \subseteq \omega^2$ with the following properties:*

1. *$ord(R) = Ord(\mathcal{M})$,*
2. *R is compatible with c and $U \in \overline{Tr}(\mathcal{M}, c, R)$,*
3. *there is no $X \subseteq Rng(R)$ such that X does not have the R-first member and X is definable in $\mathcal{N} + {}^2U$.*

Proof. Assume the contrary. Let R be a definable linear ordering in $\mathcal{N}$ such that $ord(R) = Ord(\mathcal{N})$. Let c be a coding for $\mathcal{N}$. We can assume that R is compatible with c. Let $\psi \in Fle^{1,1,2}(\mathcal{N})$ be a formula scheme such that T, V, U satisfy ψ iff

1. $V \subseteq Rng(R)$ is a lower segment in R such that $Rng(R) \setminus V$ is non-empty and does not have a R-first member.
2. $U \in \overline{Tr}(\mathcal{N}, c, R \lceil V)$.
3. Let c' be the coding for the structure $\mathcal{N} + {}^2U$ such that $c \subseteq c'$ and $c'({}^2U) = min(\omega \setminus Rng(c))$. Then $T = Tr_{\mathcal{N} + {}^2U, c'}$.
4. There is no non-empty $X \subseteq Rng(R)$ definable in $\mathcal{N} + {}^2U$ such that X does not have a R-first element.

Observe that the last condition can be formulated using the truth predicate T.

Let us show that under the given assumption the formula ψ is a proper implicit definition of some T, V, U such that $V = wo(R)$.

Assume first that $V = wo(R)$. Then there is unique U which satisfies 2) because $R \lceil V$ is a well-ordering. Then there is unique T such that 3) is satisfied. The condition 4) is satisfied because $R \lceil V$ is a well-ordering.

Assume that $V \subseteq wo(R)$, $V \neq wo(R)$. Then clearly 1) is not satisfied.

Assume that $wo(R) \subseteq V$, $V \neq wo(R)$. Then 2) or 4) is not satisfied by the assumption.

Hence ψ is an implicit definition of some $T, wo(R), U$. But that is impossible. For then $R \lceil wo(R)$ is implicitly definable representation of $Ord(\mathcal{N})$ and hence $Ord(\mathcal{N})$ is implicitly definable, contrary to Proposition 30. QED

[11] I.e., for every $x \in Dom(Q)$ there is a unique $y \in Dom(R)$ such that $X(n, x, y)$ and i) if 0_Q is the Q-first member of $Dom(Q)$ then $Y(n, 0_Q, 0_R)$ and ii) for every $x \in Dom(Q)$ $Y(n, x, y)$ iff y is the R-first member of $Rng(R) \setminus \{z;$ there exists $z' \in Dom(Q),\ Y(n, z', z)$ and $z' \prec_Q x\}$.

Proof of Theorem 22. Assume that $\mathcal{I}(\mathcal{N})$ has a canonic universal predicate G_0. Let us first prove the following:

Let R be a linear ordering definable in $\mathcal{N}$ and let $U \in \overline{Tr}(\mathcal{N}, c, R)$ for a compatible coding c for $\mathcal{N}$. Assume that U is definable in $\mathcal{N} + {}^2G_0$. Then $ord(R) < Ord(\mathcal{N})$

Let R, U be as assumed and $ord(R) = Ord(\mathcal{N})$. Let $m, n \in Rng(R) \setminus wo(R)$, $m \prec_R n$. (The existence is granted since $wo(R)$ is not definable in $\mathcal{N}$ and hence $Rng(R) \setminus wo(R)$ must be infinite). Let $U_m := U(m, .)$ and $U_n := U(n, .)$. By Lemma 34 a universal predicate for $\mathcal{I}(\mathcal{N})$ is definable in $\mathcal{N} + {}^2U_m$. Therefore also G_0 is definable in $\mathcal{N} + {}^2U_m$ because G_0 is canonic. But since U is definable in $\mathcal{N} + {}^2G_0$ then U_n is definable in $\mathcal{N} + {}^2G_0$ and hence also in $\mathcal{N} + {}^2U_m$. But from the definition of $U \in \overline{Tr}(\mathcal{M}, c, R)$, U_n is a truth predicate for a structure containing $\mathcal{N} + {}^2U_m$. But that is impossible.

Let us now complete the proof. Let us take R, U as in the previous Lemma. By Lemma 34 there is a universal predicate for $\mathcal{I}(\mathcal{M})$ definable in $\mathcal{N} + {}^2U$. By the assumption, G_0 is definable in $\mathcal{N} + {}^2U$.

In $\mathcal{N} + {}^2G_0$ we can find a formula η with one free variable such that:

for every $n \in \omega$ n satisfies η iff there exists $m \in \omega$ such that m is a G_0-code of some U_n such that $U_n \in \overline{Tr}(\mathcal{N}, c, R_n)$

If $n \in wo(R)$ then R_n is a well-ordering definable in $\mathcal{N}$. Hence $Tr_{\mathcal{N},c,R_n}$ is implicitly definable in $\mathcal{N}$ and it has a G_0-code since G_0 is a universal predicate for $\mathcal{I}(\mathcal{N})$. Hence n satisfies η. If on the other hand $n \notin wo(R)$ then n does not satisfy η by the proposition. This implies that $wo(R)$ is definable in $\mathcal{N} + {}^2G_0$ and therefore also in $\mathcal{N} + {}^2U$. But this contradicts the condition 4) of the Lemma 36, since $Rng(R) \setminus wo(R)$ does not have a R-first member. `QED`

Recall the relation between Tarski and proper Tarski hierarchy as stated in Corollary 2 of Theorem 20. Hence, in order to prove Theorem 2, it is sufficient to show that the structure $\mathcal{M}_{Ord(\mathcal{N})} \sim \mathcal{M}^p_{Ord(\mathcal{N})} \sim \mathcal{I}(\mathcal{N})$ does have a canonic proper universal predicate.

Theorem 37 *Let $\mathcal{N}$ be a L-finite structure. Let $\{\mathcal{M}^p_\alpha\}_{\alpha<\lambda^p(\mathcal{N})}$ be a proper Tarski hierarchy over $\mathcal{N}$. Then $Ord(\mathcal{N}) < \lambda^p(\mathcal{N})$.*

`Proof.` Let us show that the structure $\mathcal{M}_{Ord(\mathcal{N})}$ does have a canonic proper universal predicate. By Proposition 35 and Lemma 33 there is a linear ordering R definable in $\mathcal{N}$ such that $ord(R) = Ord(\mathcal{N})$. Let $\rho = R\lceil wo(R)$. $\omega \setminus wo(R)$ is infinite and we can chose a coding c for $\mathcal{N}$ compatible with ρ. Furthermore, for every $\gamma < Ord(\mathcal{N})$ $\rho\lceil\gamma$ is a definable representation of γ in $\mathcal{N}$. Hence, from Theorem 20,

$$\mathcal{M}_{Ord(\mathcal{N})} \sim \mathcal{N} + \{Tr_{\mathcal{N},c,\rho}(\rho_\gamma, .); \gamma < Ord(\mathcal{N})\}$$

Let $\mathcal{M}_{<Ord(\mathcal{N})}$ denote the structure on the right hand side of the equivalence, and $c_{Ord(\mathcal{N})}$ be the induced coding on $\mathcal{M}_{<Ord(\mathcal{N})}$. As in the proof of Theorem 20 it is sufficient to prove that $Tr_{\mathcal{M}_{<Ord(\mathcal{N})}, c_{Ord(\mathcal{N})}}$ is definable in $\mathbf{A} + P$, for any proper universal predicate P for $\mathcal{M}_{<Ord(\mathcal{N})}$.

Let P be a proper universal predicate for $\mathcal{M}_{<Ord(J)}$. From Lemma 19 it is sufficient to prove that ρ is definable in $\mathbf{A} + P$. In $\mathcal{N} + P$ we can find a formula η with one free variable such that:

for every $n \in \omega$ n satisfies η iff there exists $m \in \omega$ such that m is a P-code of some U_n such that $U_n \in \overline{Tr}(\mathcal{N}, c, R_n)$

If $n \in wo(R)$ then R_n is a well-ordering definable in $\mathcal{N}$. Hence $Tr_{\mathcal{N},c,R_n}$ is implicitly definable in $\mathcal{N}$ and it has a P-code since P is a universal predicate for $\mathcal{I}(\mathcal{N})$. Hence n satisfies η. If on the other hand $n \notin wo(R)$ then there is no U_n implicitly definable in $\mathcal{N}$ such that $U_n \in \overline{Tr}(\mathcal{N}, c, R_n)$ (for otherwise $\mathcal{I}(\mathcal{M}) \sim \mathbf{A} + {}^2U_n$ and $\mathcal{I}(\mathcal{M})$ is essentially finite). Hence n does not satisfy η because P is a proper universal predicate for $\mathcal{I}(\mathcal{M})$. Therefore $wo(R)$ is definable in $\mathbf{A} + P$. `QED`

Corollary *There is a structure which has a canonic proper universal predicate but does not have a canonic universal predicate. Namely, if $\mathcal{N}$ is a L-finite structure then $\mathcal{I}(\mathcal{N})$ has a canonic universal but not a proper canonic universal predicate.*

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