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That ℓ_p cannot sit in ℓ_q shown via Stegall's variational principle

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Motivated by an argument from [1], we use Stegall's variational principle to prove a well known fact:

Theorem. *Given two distinct numbers $p, q \in [1, +\infty)$, then ℓ_q does not contain an isomorphic copy of ℓ_p.*

Proof. Assume that there is a linear isomorphism $T : \ell_p \to \ell_q$ into. Thus there are $a, b > 0$ so that

$$a\|x\|_p \le \|Tx\|_q \le b\|x\|_p \quad \text{for every} \quad x \in \ell_p.$$

Here $\|\cdot\|_p,\ \|\cdot\|_q$ mean the canonical norms in ℓ_p and ℓ_q, respectively.

Let $p < q$. Consider the function $\varphi : \ell_p \to \mathbb{R}$ defined by

$$\varphi(x) = \|Tx\|_q{}^q - \|x\|_p{}^p, \quad x \in \ell_p.$$

Then $\varphi(x) \ge a^q \|\|x\|_p{}^q - \|x\|_p{}^p$ for all $x \in \ell_p$, and hence $\varphi(x)/\|x\|_p > 1$ whenever $x \in \ell_p$ and $\|x\|_p$ is large enough. Moreover the space ℓ_p is reflexive and hence dentable. Thus, Stegall's variational principle, say [2, Corollary 5.22], aplies and so we get a point $x \in \ell_p$ and a functional $\xi \in \ell_p{}^*$ such that

$$\varphi(x+h) - \varphi(x) - \langle \xi, h \rangle \ge 0 \quad \text{for every} \quad h \in \ell_p.$$

Hence

$$\varphi(x+h) + \varphi(x-h) - 2\varphi(x) \ge 0 \quad \text{for every} \quad h \in \ell_p.$$

Taking here $h = te_i$, where $t > 0$ and e_i is the $i-$th element of the canonical basis in ℓ_p, and then reorganizing the above inequality, we get

$$\|Tx + tTe_i\|_q{}^q + \|Tx - tTe_i\|_q{}^q - 2\|Tx\|_q{}^q \ge \|x + te_i\|_p{}^p + \|x - te_i\|_p{}^p - 2\|x\|_p{}^p.$$

Letting here $i \to \infty$, we get $e_i \to 0$ weakly, $Te_i \to 0$ weakly, and hence the above inequality with some extra effort yield

$$\big(2t^q b^q \ge \big)\ 2t^q \liminf_{i\to\infty} \|Te_i\|_q{}^q \ge 2t^p \quad \text{for every} \quad t > 0,$$

which is impossible for $t > 0$ small enough.

Let $p > q$. Consider the function $\psi : \ell_p \to \mathbb{R}$ defined by

$$\psi(x) = \|x\|_p{}^p - \|Tx\|_q{}^q, \quad x \in \ell_p.$$

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Then $\psi(x) \geq \|x\|_p{}^p - b^q \|x\|_p{}^q$ for all $x \in \ell_p$, and hence $\psi(x)/\|x\|_p > 1$ whenever $x \in \ell_p$ and $\|x\|_p$ is large enough. By Stegall's variational principle, there is a point $x \in \ell_p$ such that

$$\psi(x+h) + \psi(x-h) - 2\psi(x) \geq 0 \quad \text{for every} \quad h \in \ell_p.$$

Taking here $h = te_i$, where $t > 0$ and e_i is the i–th element of the canonical basis in ℓ_p, and reorganizing, we have

$$\|x + te_i\|_p{}^p + \|x - te_i\|_p{}^p - 2\|x\|_p{}^p \geq \|Tx + tTe_i\|_q{}^q + \|Tx - tTe_i\|_q{}^q - 2\|Tx\|_q{}^q.$$

Letting here $i \to \infty$, we get

$$2t^p \geq 2t^q \limsup_{i\to\infty} \|Te_i\|_q{}^q \ \big(\geq 2t^q a^q\big) \quad \text{for every} \quad t > 0,$$

which is impossible for $t > 0$ small enough. ∎

Remarks. 1. The second case, when $p > q$, also follows from Pitt's theorem, which can also be derived from Stegall's variational principle. Indeed, $T\big(B_{\ell_p}\big)$ is then a relatively compact set and hence $\|Te_i\| \to 0$ as $i \to \infty$. But $\|Te_i\|_q \geq a$ (> 0), a contradiction.

2. The above theorem can be extended by replacing "copy of ℓ_p" by "copy of an infinite-dimensional subspace of ℓ_p". In fact. Let $Y \subset \ell_p$ be such. Since the origin belongs to the weak closure of the unit sphere of Y, and the weak topology on Y is metrizable, there is a sequence (y_i) of norm-one elements of Y such that $y_i \to 0$ weakly as $i \to \infty$. Then a slight adaptation of the above argument, where the e_i's are replaced by y_i's, works.

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