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A variational principle in reflexive spaces with Kadec-Klee norm

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Abstract. We prove a variational principle in reflexive Banach spaces X with Kadec-Klee norm, which asserts that any Lipschitz (or any proper lower semicontinuous bounded from below extended real-valued) function in X can be perturbed with a parabola in such a way that the perturbed function attains its infimum (even more can be said — the infimum is well-posed). In addition, we have genericity of the points determining the parabolas. We prove also that the validity of such a principle actually characterizes the reflexive spaces with Kadec-Klee norm. This principle turns out to be an analytic counterpart of a result of K.-S. Lau on nearest points.

1. Introduction and main result

In 1961, Bishop and Phelps proved a lemma on supporting closed sets by cones [BP]. An analytic counterpart of it, on supporting functions, is the Ekeland variational principle [E], stating that any (extended real-valued) lower semicontinuous bounded from below function in a complete metric space can be perturbed by a distance-like function in such a way that the perturbed function attains its minimum. In 1974, Phelps [Ph] proved that in a Banach space which has a geometric property, called Radon-Nikodým property, every nonempty bounded closed convex set is the closed convex hull of its strongly exposed points. An analytic result related to this geometric fact is the Stegall's variational principle [S1,S2] which shows that in such a space one can perturb a lower semicontinuous bounded form below function (with domain in a closed bounded convex set) by a linear continuous functional, in order to obtain a perturbation which attains its minimum.

The aim of this note is to go on in this vein and to show one more example of such a correspondence between a geometrical and an analytical statement. In 1978, Lau [L] proved that in a reflexive Banach space $(X, \|\cdot\|)$, with Kadec-Klee norm, for every nonempty closed set $C \subset X$ there are residually many points in $X \backslash C$ possessing a nearest point in C. An analytic counterpart of this result is the following theorem. And, likewise, as in the previous cases, we can formulate it in the form of equivalence(s).

Theorem 1. *For a Banach space $(X, \|\cdot\|)$ the following statements are equivalent:*

(i) The space X is reflexive and its norm $\|\cdot\|$ is Kadec-Klee.

(ii) For every Lipschitzian function $f : X \to (-\infty, +\infty)$ there exists a `residual` *set $\Omega \subset X$ such that for every $u \in \Omega$ there is a residual set $R_u \subset (0, +\infty)$ such that for each $c \in R_u$ the problem of minimizing $f + c\|\cdot - u\|^2$ is well-posed.*

(ii') For every proper lower semicontinuous function $f : X \to (-\infty, +\infty]$, with $\inf f > -\infty$, there exists a `residual` *set $\Omega \subset X$ such that for every $u \in \Omega$ there are $a_u \in (0, +\infty]$*

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and a residual set $R_u \subset (0, a_u)$ such that for each $c \in R_u$ the problem of minimizing the function $f + c\|\cdot - u\|^2$ is well-posed.

(iii) For every f as in (ii) there exists a `dense` *set $D \subset X$ such that for every $u \in D$ the function $f + \|\cdot - u\|^2$ attains its infimum.*

(iii') For every f as in (ii') there exists a `dense` *set $D \subset X$ such that for every $u \in D$ there is $c > 0$ so that the function $f + c\|\cdot - u\|^2$ attains its infimum.*

We follow the usual notation in Banach space theory, see, e.g. [F∼]. Let us recall that the norm $\|\cdot\|$ on X is called *Kadec-Klee* if $\lim_{n\to\infty} \|x_n - x_0\| = 0$ whenever a sequence $x_0, x_1, x_2, \ldots \in X$ satisfies $\|x_n\| \to \|x_0\|$ and $x_n \to x_0$ weakly. In Russian literature, a reflexive Banach space, with Kadec-Klee norm, is called a *Efimov-Stečkin space.* A set in a Baire space, in particular, in a Banach space, is called *residual* if its complement is the union of countably many nowhere dense sets. An extended real-valued function $g : X \to (-\infty, +\infty]$ is called *proper* if its effective domain $\operatorname{dom} g = \{x \in X; g(x) < +\infty\}$ is nonempty. Given such a function g, with $\inf g > -\infty$, we say that *the problem of minimizing g is well-posed*, if every minimizing sequence $(x_n)_{n\in\mathbb{N}}$ in X for g (i.e. $g(x_n) \downarrow \inf g$) has a norm-cluster point $x \in X$; hence $g(x) = \inf g$ whenever g is at least lower semicontinuous. Obviously, if the problem to minimize g is well-posed, then the set of minimizers for g is compact.

In the proof, we shall need the following topological fact going back to Ulam and Kuratowski.

Lemma. *Let X be a Banach space and let $O \subset X \times \mathbb{R}$ be an open subset such that its canonical projection, along $\mathbb{R}$, is all of X. Let $E \subset O$ be a residual subset in O. Then there exists a residual subset $\Omega \subset X$ such that for every $x \in \Omega$ the set $\{t \in \mathbb{R};\ (x,t) \in E\}$ is residual in $\{t \in \mathbb{R};\ (x,t) \in O\}$.*

Proof. Put $F = E \cup \big[(X \times \mathbb{R})\backslash O\big]$. It is easy to check that F is residual in $X \times \mathbb{R}$. Indeed,

$$(X \times \mathbb{R})\backslash F = \big[(X \times \mathbb{R})\backslash E\big] \cap O = O\backslash E,$$

and the latter set is of the first Biare category in O, and hence in $X \times \mathbb{R}$. Now, by applying [Ku, §22.V, Corollary 1a], we find a residual set $\Omega \subset X$ such that for every $x \in \Omega$ the set $\{t \in \mathbb{R};\ (x,t) \in F\}$ is residual in $\mathbb{R}$. Hence, the set $\{t \in \mathbb{R};\ (x,t) \in E\}$ is residual in $\{t \in \mathbb{R};\ (x,t) \in O\}$. ∎

Proof of Theorem 1. (i)⇒(ii). We will split the proof of this implication into several steps.

`Step 1`. Let f be as in (ii). Consider the (reflexive) space $X \times \mathbb{R}$ and the (closed) set $\operatorname{epi} f = \big\{(x,t) \in X \times \mathbb{R};\ f(x) \le t\big\}$. Let $L > 0$ be a Lipschitzian constant of f and put

$$B = \big\{(x,t) \in X \times \mathbb{R};\ |t| \le -\|x\|^2 + L^2\big\}. \tag{1}$$

It is elementary to check that the set B is closed, convex, symmetric, bounded, and that it contains the origin $(0,0)$ in its interior. Let $|||\cdot|||$ denote its Minkowski functional; this will be an equivalent norm on $X \times \mathbb{R}$.

For sure, this norm will have also the Kadec-Klee property. Indeed, for $n = 0, 1, 2, \dots$ consider $(x_n, t_n) \in X \times \mathbb{R}$ such that $|||(x_n, t_n)||| = 1$, and $(x_n, t_n) \to (x_0, t_0)$ weakly as $n \to \infty$. Then, of course, $x_n \to x_0$ weakly and $t_n \to t_0$ as $n \to \infty$. Now, for all $n = 0, 1, \dots$, from the very definition of $|||\cdot|||$, we must have $|t_n| + \|x_n\|^2 = L^2$ and hence

$$\|x_0\| \le \liminf_{n\to\infty} \|x_n\| \le \limsup_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \sqrt{L^2 - |t_n|} = \sqrt{L^2 - |t_0|} = \|x_0\|.$$

Thus, the Kadec-Klee property of $\|\cdot\|$ yields that $\|x_n - x_0\| \to 0$ and so $|||(x_n, t_n) - (x_0, t_0)||| \to 0$ as $n \to \infty$.

`Step 2.` Put $O = (X \times \mathbb{R}) \setminus \operatorname{epi} f$ and consider the distance function for $\operatorname{epi} f$

$$\operatorname{dist}\big((x,t), \operatorname{epi} f\big) := \inf\big\{|||(x,t) - (u,s)|||;\ (u,s) \in \operatorname{epi} f\big\}, \quad (x,t) \in O.$$

We can immediately see that O is an open set and that its projection, along $\mathbb{R}$, is equal to all of X. By the mentioned result of Lau, [L, Theorem 4] (see also [BF, Corollary 5.8]), we find a residual subset E of O, such that for every $(x_0, t_0) \in E$, if $(y_1, s_1), (y_2, s_2), \ldots \in \operatorname{epi} f$, and

$$|||(y_n, s_n) - (x_0, t_0)||| \longrightarrow \operatorname{dist}\big((x_0, t_0), \operatorname{epi} f\big) \quad \text{as} \quad n \to \infty,$$

then the sequence $((y_n, s_n))_{n\in\mathbb{N}}$ has a norm-cluster point (such a cluster point is evidently a nearest point to (x_0, t_0) in $\operatorname{epi} f$). In other words, the best approximation problem of any $(x_0, t_0) \in E$ to $\operatorname{epi} f$ is well-posed. By the above Lemma applied for our E and O, we find the corresponding residual set $\Omega \subset X$. Fix, for a longer while, any $x_0 \in \Omega$ and then find a residual set $S \subset (-\infty, f(x_0))$ such that $\{x_0\} \times S \subset E$. Further, fix, for a longer while, any $t_0 \in S$. Let $(x_1, t_1) \in \operatorname{epi} f$ be a nearest point to (x_0, t_0) in $\operatorname{epi} f$, i.e.,

$$(2) \qquad (0 <)\ \rho := |||(x_1, t_1) - (x_0, t_0)||| \le |||(x, t) - (x_0, t_0)||| \quad \text{for all} \quad (x, t) \in \operatorname{epi} f.$$

From the very definition of the norm $|||\cdot|||$, we have that

$$(3) \qquad \rho|t_1 - t_0| + \|x_1 - x_0\|^2 = \rho^2 L^2.$$

Consider any $\varepsilon \in (0, 1)$. Then

$$|||((x_1 - \varepsilon(x_1 - x_0), 2t_0 - t_1 - \varepsilon(t_0 - t_1)) - (x_0, t_0)||| = (1-\varepsilon)|||(x_1 - x_0, t_0 - t_1)||| = (1-\varepsilon)\rho < \rho,$$

and so $(x_1 - \varepsilon(x_1 - x_0), 2t_0 - t_1 - \varepsilon(t_0 - t_1)) \notin \operatorname{epi} f$, that is,

$$2t_0 - t_1 - \varepsilon(t_0 - t_1) < f(x_1 - \varepsilon(x_1 - x_0)).$$

Letting $\varepsilon \downarrow 0$ in the last inequality we obtain $2t_0 - t_1 \le f(x_1)$. However, $(x_1, t_1) \in \operatorname{epi} f$, and so $f(x_1) \le t_1$. Therefore $2t_0 - t_1 \le t_1$, that is, $t_0 \le t_1$. Thus (3) has the form

$$(4) \qquad \rho(t_1 - t_0) + \|x_1 - x_0\|^2 = \rho^2 L^2.$$

We actually have that $t_1 = f(x_1)$. Indeed, assume that $t_1 = f(x_1) + 2\delta$ for some $\delta > 0$. Then the above obtained inequality $2t_0 - t_1 \le f(x_1)$ entails that $t_1 - \delta \ge t_0$. Thus

$$\rho|t_1 - \delta - t_0| + \|x_1 - x_0\|^2 = \rho(t_1 - \delta - t_0) + \|x_1 - x_0\|^2 = -\rho\delta + \rho^2 L^2 < \rho^2 L^2,$$

and hence $|||(x_1, t_1 - \delta) - (x_0, t_0)||| < \rho$, which is a contradiction since $(x_1, t_1 - \delta) \in \operatorname{epi} f$. Thus (4) reads as

$$\rho(f(x_1) - t_0) + \|x_1 - x_0\|^2 = \rho^2 L^2. \tag{5}$$

`Step 3.` In this step we show that if x_1 is such that $(x_1, f(x_1))$ is a nearest point in $\operatorname{epi} f$ to (x_0, t_0) with distance ρ, then x_1 is a minimum point for the function $f + \frac{1}{\rho}\|\cdot - x_0\|^2$. To this end, first fix any $x \in X$ satisfying $f(x) \ge t_0$. We then have $|||(x, f(x)) - (x_0, t_0)||| \ge \rho$, and (1) with (5) yield

$$\rho(f(x) - t_0) + \|x - x_0\|^2 \ge \rho^2 L^2 = \rho(f(x_1) - t_0) + \|x_1 - x_0\|^2.$$

Therefore

$$f(x) + \tfrac{1}{\rho}\|x - x_0\|^2 \ge f(x_1) + \tfrac{1}{\rho}\|x_1 - x_0\|^2 \quad \text{whenever} \quad x \in X \quad \text{and} \quad f(x) \ge t_0. \tag{6}$$

Now, fix for a while any $y \in X$, with $f(y) < t_0$. Then $(y, t_0) \in \operatorname{epi} f$, and so $|||(y, t_0) - (x_0, t_0)||| \ge \rho$, that is, $\rho|t_0 - t_0| + \|y - x_0\|^2 \ge \rho^2 L^2$, which yields $\|y - x_0\| \ge \rho L$. Put $x_2 = x_0 + \frac{\rho L}{\|y - x_0\|}(y - x_0)$; then $\|x_2 - x_0\| = \rho L$, and we claim that $f(x_2) \ge t_0$. Indeed, assume $f(x_2) < t_0$. From the continuity of f there is x_3 in the open linear segment, with the end points x_2 and x_0, such that $f(x_3) < t_0$. Thus $(x_3, t_0) \in \operatorname{epi} f$ and $\|x_3 - x_0\| < \|x_2 - x_0\| = \rho L$. It then follows $|||(x_3, t_0) - (x_0, t_0)||| \ge \rho$, and so $\|x_3 - x_0\|^2 = \rho|t_0 - t_0| + \|x_3 - x_0\|^2 \ge \rho^2 L^2$, a contradiction. Therefore $f(x_2) \ge t_0$.

Now, the Lipschitz property of f and (6), used for $x := x_2$, yield

$$\begin{aligned} f(y) &\ge f(x_2) - L\|y - x_2\| \ge f(x_1) - \tfrac{1}{\rho}\|x_2 - x_0\|^2 + \tfrac{1}{\rho}\|x_1 - x_0\|^2 - L\|y - x_2\| \\ &= f(x_1) - L\|y - x_0\| + \tfrac{1}{\rho}\|x_1 - x_0\|^2 \ge f(x_1) - \tfrac{1}{\rho}\|y - x_0\|^2 + \tfrac{1}{\rho}\|x_1 - x_0\|^2 \end{aligned}$$

for every $y \in X$ with $f(y) < t_0$. Here we first used that

$$\|y - x_2\| = \|y - x_0\| - \|x_2 - x_0\| = \|y - x_0\| - \rho L$$

and then that $L \le \frac{1}{\rho}\|y - x_0\|$. We have thus proved that the function $f + c\|\cdot - x_0\|^2$, with $c := \frac{1}{\rho}$, attains its infimum at the point x_1.

`Step 4.` Next, we shall show that the problem of minimizing the function $f + c\|\cdot - x_0\|^2$ is well-posed. To this end, assume that a sequence $(y_n)_{n\in\mathbb{N}}$ in X is such that

$$f(y_n) + \tfrac{1}{\rho}\|y_n - x_0\|^2 \longrightarrow \inf\big(f + \tfrac{1}{\rho}\|\cdot - x_0\|^2\big) \ \big(= f(x_1) + \tfrac{1}{\rho}\|x_1 - x_0\|^2\big) \quad \text{as} \quad n \to \infty. \tag{7}$$

From (7) we can easily deduce that $(y_n)_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ are bounded sequences. We claim that $\liminf_{n\to\infty} f(y_n) \geq t_0$. Indeed, assume, by contradiction, that there are $\delta > 0$ and an infinite set $N \subset \mathbb{N}$ such that $f(y_n) < t_0 - \delta$ for every $n \in N$. Find $\Delta > \frac{\delta}{L}$ so big that $\|y_n - x_0\| < \Delta$ for every $n \in \mathbb{N}$. Put

$$z_n = \tfrac{\delta}{L\Delta} x_0 + \big(1 - \tfrac{\delta}{L\Delta}\big) y_n, \quad n \in N.$$

Fix any $n \in N$. Then

$$\begin{aligned} f(z_n) &\leq f(y_n) + L\|z_n - y_n\| < t_0 - \delta + L\|z_n - y_n\| \\ &= t_0 - \delta + L \cdot \tfrac{\delta}{L\Delta}\|x_0 - y_n\| \leq t_0, \end{aligned}$$

and hence $(z_n, t_0) \in \operatorname{epi} f$. Thus $|||(z_n, t_0) - (x_0, t_0)||| \geq \rho$, and so $\rho|t_0 - t_0| + \|z_n - x_0\|^2 \geq \rho^2 L^2$, that is, $\|z_n - x_0\| \geq \rho L$. But then

$$\|y_n - x_0\| = \frac{1}{1 - \frac{\delta}{L\Delta}}\|z_n - x_0\| \geq \tfrac{\rho L^2\Delta}{L\Delta - \delta} > 0 \quad \text{for every} \quad n \in N.$$

Now, the Lipschitz property of f and the latter inequality used twice yield

$$\begin{aligned} f(y_n) + \tfrac{1}{\rho}\|y_n - x_0\|^2 &\geq f(x_0) - L\|y_n - x_0\| + \tfrac{1}{\rho}\|y_n - x_0\|^2 \\ &\geq f(x_0) - L\|y_n - x_0\| + \tfrac{L^2\Delta}{L\Delta - \delta}\|y_n - x_0\| \\ &= f(x_0) + \tfrac{\delta L}{L\Delta - \delta}\|y_n - x_0\| \geq f(x_0) + \tfrac{\delta L}{L\Delta - \delta} \cdot \tfrac{\rho L^2\Delta}{L\Delta - \delta}. \end{aligned}$$

Therefore,

$$\begin{aligned} (0 <)\ \tfrac{\rho\delta L^3\Delta}{(L\Delta - \delta)^2} &\leq -f(x_0) + \lim_{n\in N}\big(f(y_n) + \tfrac{1}{\rho}\|y_n - x_0\|^2\big) \\ &= -f(x_0) + \inf\big(f + \tfrac{1}{\rho}\|\cdot - x_0\|^2\big)\ (\leq 0), \end{aligned}$$

a contradiction. We thus have proved that $\liminf_{n\to\infty} f(y_n) \geq t_0$.

`Step 5.` We shall show that

$$\rho_n := |||(y_n, f(y_n)) - (x_0, t_0)||| \longrightarrow \rho \quad \text{as} \quad n \to \infty. \tag{8}$$

Since, as we noticed above, the sequences $(y_n)_{n\in\mathbb{N}}$ and $(f(y_n))_{n\in\mathbb{N}}$ are bounded, so is $(\rho_n)_{n\in\mathbb{N}}$. In order to prove (8), it is enough to show that $\lim_{\mathcal{U}} \rho_n = \rho$ for every free ultrafilter $\mathcal{U}$ over $\mathbb{N}$. So fix one such $\mathcal{U}$. From (7) and (5) we have

$$\begin{aligned} \rho\big(\lim_{\mathcal{U}} f(y_n) - t_0\big) + \lim_{\mathcal{U}}\|y_n - x_0\|^2 &= \rho\lim_{\mathcal{U}}\big(f(y_n) + \tfrac{1}{\rho}\|y_n - x_0\|^2\big) - \rho t_0 \\ &= \rho\big(f(x_1) + \tfrac{1}{\rho}\|x_1 - x_0\|^2\big) - \rho t_0 \\ &= \rho\big(f(x_1) - t_0\big) + \|x_1 - x_0\|^2 = \rho^2 L^2. \end{aligned} \tag{9}$$

From the definition of the norm $|||\cdot|||$ we have

$$\rho_n\big|f(y_n) - t_0\big| + \|y_n - x_0\|^2 = \rho_n^2 L^2 \quad \text{for all} \quad n = 1, 2, 3, \ldots,$$

and hence, as $\lim_{\mathcal{U}} f(y_n) \geq t_0$,

$$\lim_{\mathcal{U}} \rho_n \big(\lim_{\mathcal{U}} f(y_n) - t_0 \big) + \lim_{\mathcal{U}} \|y_n - x_0\|^2 = \lim_{\mathcal{U}} \rho_n^2 L^2. \tag{10}$$

And, subtracting (10) from (9), we get

$$\big(\rho - \lim_{\mathcal{U}} \rho_n\big)\big(\lim_{\mathcal{U}} f(y_n) - t_0\big) = \big(\rho^2 - \lim_{\mathcal{U}} \rho_n^2\big) L^2.$$

If $\lim_{\mathcal{U}} \rho_n \neq \rho$, then we get $\lim_{\mathcal{U}} f(y_n) - t_0 = \big(\rho + \lim_{\mathcal{U}} \rho_n\big) L^2$, and regarding (10), we have

$$\lim_{\mathcal{U}} \rho_n \big(\rho + \lim_{\mathcal{U}} \rho_n\big) L^2 + \lim_{\mathcal{U}} \|y_n - x_0\|^2 = \lim_{\mathcal{U}} \rho_n^2 L^2,$$

$$\lim_{\mathcal{U}} \rho_n \rho L^2 + \lim_{\mathcal{U}} \|y_n - x_0\|^2 = 0.$$

And this is impossible since $\rho > 0$ and $\rho_n \geq \rho$ for all $n = 1, 2, 3, \ldots$ Therefore, $\lim_{\mathcal{U}} \rho_n = \rho$. Here $\mathcal{U}$ was any free ultrafilter on $\mathbb{N}$. Therefore $\lim_{n\to\infty} \rho_n = \rho$. We thus have proved (8).

Now, since (8) means that $(y_n, f(y_n))_{n\in\mathbb{N}}$ is a minimizing sequence for the best approximation problem of (x_0, t_0) to epi f, using Step 2, we get that the sequence $((y_n, f(y_n))_{n\in\mathbb{N}}$ has a norm-cluster point. Hence so does the minimizing sequence $(y_n)_{n\in\mathbb{N}}$. (Since the norm $\|\cdot\|$ is not assumed to be strictly convex, there is no guarantee that a cluster point of $(y_n)_{n\in\mathbb{N}}$ must be equal to x_1; see also Remark 2 below). We have thus proved that the problem of minimizing the function $f + \frac{1}{\rho}\|\cdot - x_0\|^2$ is well-posed.

`Step 6.` Keeping still the same fixed x_0, it remains to show the residuality of the c's in question (so far we have found only one c). Consider the function $\varphi : (-\infty, f(x_0)) \to (0, +\infty)$ defined by

$$\varphi(\tau) = \big[\mathrm{dist}((x_0, \tau), \mathrm{epi}\, f)\big]^{-1}, \quad \tau \in (-\infty, f(x_0)).$$

It is easy to verify that φ is locally Lipschitzian, and that $\varphi(\tau) \to +\infty$ as $\tau \uparrow f(x_0)$. Also, the Lipschitz property of f easily yields that $\varphi(\tau) \to 0$ as $\tau \downarrow -\infty$. We shall show that φ is strictly increasing. So fix any $\tau_1 < \tau_2 < f(x_0)$. Find $\tau \in (\tau_1, \tau_2)$ so that $\tau \in S$ (S is the residual subset of $(-\infty, f(x_0))$ from Step 2). Then there is $(u, t) \in \mathrm{epi}\, f$ so that $\varphi(\tau) = |||(x_0, \tau) - (u, t)|||^{-1}$. We have $(u, t + \tau_2 - \tau) \in \mathrm{epi}\, f$, and this point actually lies in the interior of epi f (as f is continuous). Thus

$$\mathrm{dist}\big((x_0, \tau_2), \mathrm{epi}\, f\big) < |||(x_0, \tau_2) - (u, t + \tau_2 - \tau)||| = |||(x_0, \tau) - (u, t)|||.$$

Hence $\varphi(\tau_2) > \varphi(\tau)$. Now, consider any $\varepsilon > 0$ and find $(v, \sigma) \in \mathrm{epi}\, f$ so that

$$\mathrm{dist}\big((x_0, \tau_1), \mathrm{epi}\, f\big) + \varepsilon > |||(x_0, \tau_1) - (v, \sigma)|||.$$

Then $(v, \sigma + \tau - \tau_1) \in \mathrm{epi}\, f$, and so

$$\begin{aligned} \varphi(\tau)^{-1} &= \mathrm{dist}\big((x_0, \tau), \mathrm{epi}\, f\big) \leq |||(x_0, \tau) - (v, \sigma + \tau - \tau_1)||| \\ &= |||(x_0, \tau_1) - (v, \sigma)||| < \varphi(\tau_1)^{-1} + \varepsilon. \end{aligned}$$

Hence, letting $\varepsilon \downarrow 0$, we get $\varphi(\tau) \geq \varphi(\tau_1)$. Therefore $\varphi(\tau_2) > \varphi(\tau_1)$.

Putting together the above facta we get that φ is a homeomorphism from $(-\infty, f(x_0))$ onto $(0, +\infty)$. Hence, as S is a residual subset of $(-\infty, f(x_0))$, the set $R_{x_0} := \varphi(S)$ is residual in $(0, +\infty)$. And from Step 5 we already know that for every $c \in R_{x_0}$ the problem of minimizing the function $f + c\|\cdot - x_0\|^2$ is well-posed. We have thus fully proved (ii). ∎

(ii)⇒(iii). Assume (ii) holds. Fix any $z \in X$ and any $\varepsilon > 0$. Let $L > 0$ be a Lipschitzian constant of f. Let $|||\cdot|||$ be the Minkowski functional of the body B defined by (1) in Step 1 of the proof of the implication (i)⇒(ii); we already know that it is an equivalent norm on $X \times \mathbb{R}$. According to (ii) there is x_0 with $\|x_0 - z\| < \varepsilon/2$ and $c \in \left(\frac{2L}{2L+\varepsilon}, 1\right)$ so that $f + c\|\cdot - x_0\|^2$ attains its infimum at, say, $x_1 \in X$. Thus

$$f(x) + c\|x - x_0\|^2 \geq f(x_1) + c\|x_1 - x_0\|^2 \quad \text{for every} \;\; x \in X. \tag{11}$$

The choice $x := x_0$ and the Lipschitz property of f yield that $\|x_1 - x_0\| \leq \frac{L}{c}$. Put $t_0 = f(x_1) + c\|x_1 - x_0\|^2 - \frac{L^2}{c}$. Then $t_0 \leq f(x_1)$ and

$$\tfrac{1}{c}\big(f(x_1) - t_0\big) + \|x_1 - x_0\|^2 = \tfrac{1}{c^2}L^2,$$

which means that $|||(x_1, f(x_1)\big) - (x_0, t_0)||| = \frac{1}{c}$. Moreover, for $(x, t) \in \operatorname{epi} f$, with $t \geq t_0$, we have

$$\begin{aligned}\tfrac{1}{c}(t - t_0) + \|x - x_0\|^2 &\geq \tfrac{1}{c}\big(f(x) + c\|x - x_0\|^2 - t_0\big)\\ &\geq \tfrac{1}{c}\big(f(x_1) + c\|x_1 - x_0\|^2 - t_0\big) = \tfrac{1}{c^2}L^2,\end{aligned}$$

and so $|||(x, t) - (x_0, t_0)||| \geq \frac{1}{c}$.

On the other hand, take $(x, t) \in \operatorname{epi} f$, with $t < t_0$. Then $f(x) \leq t < t_0$ and, using the definition of t_0 and (11), we obtain

$$t_0 + c\|x - x_0\|^2 > f(x_1) + c\|x_1 - x_0\|^2 = t_0 + \tfrac{1}{c^2}L^2.$$

Thus $\|x - x_0\|^2 > \frac{L^2}{c^2}$, and consequently

$$\frac{1}{c}|t - t_0| + \|x - x_0\|^2 > \tfrac{L^2}{c^2},$$

showing again that $|||(x, t) - (x_0, t_0)||| > \frac{1}{c}$. Therefore, we have shown that the point $(x_1, f(x_1))$ is a nearest point to (x_0, t_0) in $\operatorname{epi} f$ with distance $\frac{1}{c}$ (> 1).

Now, put

$$u = x_1 + c(x_0 - x_1), \quad \tau = f(x_1) + c\big(t_0 - f(x_1)\big).$$

We have $|||(x_1, f(x_1)) - (u, \tau)||| = c|||(x_1, f(x_1)) - (x_0, t_0)||| = 1$ and the point (u, τ) lies in the interior of the line segment joining (x_0, t_0) and $(x_1, f(x_1))$. Since $(x_1, f(x_1))$ is a nearest point to (x_0, t_0) in $\operatorname{epi} f$ this means that $(x_1, f(x_1))$ is a nearest point also to (u, τ) with $\rho = \operatorname{dist}((u, \tau), \operatorname{epi} f) = 1$. But we already showed in Step 3 that in such a case the function $f + \frac{1}{\rho}\|\cdot - u\|^2 = f + \|\cdot - u\|^2$ attains its infimum at x_1.

Finally, note that

$$\begin{aligned}\|u-z\| &\le \|u-x_0\|+\|x_0-z\| = (1-c)\|x_0-x_1\|+\|x_0-z\| \\ &< (1-c)\cdot\tfrac{L}{c}+\tfrac{\varepsilon}{2} = \big(\tfrac{1}{c}-1\big)L+\tfrac{\varepsilon}{2} < \big(\tfrac{2L+\varepsilon}{2L}-1\big)L+\tfrac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

Thus we have completely proved (iii). ∎

(iii)⇒(i). Assume that (iii) holds. As regards the reflexivity of X, fix any $\xi \in X^*$. From (iii) find $x_0 \in X$, and $x_1 \in X$ so that $\xi(x)+\|x-x_0\|^2 \ge \xi(x_1)+\|x_1-x_0\|^2$ for all $x \in X$. Rearranging this inequality we get

$$\xi(x-x_1) \ge \|x_1-x_0\|^2-\|x-x_0\|^2 \quad \text{for all} \quad x\in X. \tag{12}$$

Assume that $x_1 = x_0$. Then $\xi(x-x_0)\ge 0$ for every $x \in X$ and hence $\xi = 0$. Further assume $x_1 \ne x_0$. Consider any $t \in (0,1)$. Taking $x := x_0+t(x_1-x_0)$ in (12), we have

$$\begin{aligned}(1-t)\xi(x_0-x_1) &= \xi(x_0+t(x_1-x_0)-x_1) \\ &\ge \|x_1-x_0\|^2-\|x_0+t(x_1-x_0)-x_0\|^2 = (1-t^2)\|x_1-x_0\|^2.\end{aligned}$$

Dividing both sides here by $(1-t)$, we get $\xi(x_0-x_1)\ge(1+t)\|x_1-x_0\|^2$. And letting $t\uparrow 1$, we obtain

$$\xi(x_0-x_1)\ge 2\|x_1-x_0\|^2. \tag{13}$$

On the other hand, fixing any $h \in X$, with $\|h\| = 1$, and any $t > 0$, when putting $x := x_1-th$ in (12), we get

$$\begin{aligned}-t\xi(h) &= \xi(x_1-th-x_1) \ge \|x_1-x_0\|^2-\|x_1-th-x_0\|^2 \\ &\ge \|x_1-x_0\|^2-\|x_1-x_0\|^2-2t\|x_1-x_0\|-t^2 = -t(2\|x_1-x_0\|+t).\end{aligned}$$

Here, keeping still h fixed, then dividing by $(-t)$, and finally letting $t\downarrow 0$, we get $\xi(h)\le 2\|x_1-x_0\|$. Thus $\|\xi\|\le 2\|x_1-x_0\|$. Therefore, comparing this inequality with (13), we get

$$2\|x_1-x_0\|^2 \le \xi(x_0-x_1)\le\|\xi\|\|x_1-x_0\|\le 2\|x_1-x_0\|^2,$$

and so, since $x_1\ne x_0$, we get

$$\xi\Big(\frac{x_1-x_0}{\|x_1-x_0\|}\Big)=2\|x_1-x_0\|=\|\xi\|.$$

Thus, in both cases, our ξ attains its norm. Then James' theorem guarantees the reflexivity of X.

Now we shall check that the norm $\|\cdot\|$ is Kadec-Klee. Assume (iii) holds and that $\|\cdot\|$ is not Kadec-Klee. We shall profit from a counterexample due to Konjagin [K], [BF]. Using [BF, Lemma 5.9], we find $\delta\in(0,\frac{1}{4})$, $\xi\in X^*$, and $y_n\in 2B_X\cap\xi^{-1}(1)$, $n\in\mathbb{N}$, such that $\lim_{n\to\infty}\|y_n\| = 1 = \|\xi\|$ and $\|y_n - y_m\| > 9\delta$ whenever $n, m \in \mathbb{N}$ are distinct. Find $k \in \mathbb{N}$ such that $|\|y_k\| - 1| < \frac{\delta}{4}$ and $2^{-k} < \frac{\delta}{4}$. Put

$$M = \bigcup_{n=k}^{\infty} M_n \quad \text{where} \quad M_n = (1 + 2^{-n})y_n + \big(3\delta B_X \cap \xi^{-1}(0)\big).$$

Here, each M_n is closed, and for all distinct $n, m \in \{k, k+1, \ldots\}$ the distance between M_n and M_m is at least δ. Therefore the whole set M is closed. Take $\alpha > \max\{4, \frac{4}{\delta}(1 + \delta)^2\}$. Define then the function $f : X \to [0, +\infty)$ by $f = \alpha \inf\{\| \cdot - u\|;\ u \in M\}$; note that f is Lipschitzian. We shall show that for any $x_0 \in \frac{\delta}{4}B_X$ the function $f + \| \cdot - x_0\|^2$ does not attain its infimum. So fix such an x_0 and assume, by contradiction, that there is $x_1 \in X$ so that

$$f(x) + \|x - x_0\|^2 \geq f(x_1) + \|x_1 - x_0\|^2 \quad \text{for all} \quad x \in X. \tag{14}$$

We shall show that x_1 must lie in M. Put $u = y_k(1 + 2^{-k})$. Then $u \in M_k \subset M$, and so

$$\begin{aligned} f(u) + \|u - x_0\|^2 = \|y_k + 2^{-k}y_k - x_0\|^2 &\leq (\|y_k\| + \tfrac{\delta}{4}\|y_k\| + \|x_0\|)^2 \\ &< (1 + \tfrac{\delta}{4} + \tfrac{\delta}{4}(1 + \tfrac{\delta}{4}) + \tfrac{\delta}{4})^2 < (1 + \delta)^2. \end{aligned}$$

Now, we claim that $\frac{1}{\alpha}f(x_1) < \frac{\delta}{4}$. Indeed, otherwise, using the latter inequality and the choice of α, we would have

$$f(x_1) + \|x_1 - x_0\|^2 \geq \tfrac{\alpha\delta}{4} > (1 + \delta)^2 > f(u) + \|u - x_0\|^2,$$

which contradicts with (14). Thus, $\frac{1}{\alpha}f(x_1) < \frac{\delta}{4}$, and since the distance between the sets M_n and M_m for distinct $n, m > k$ is greater than δ, this entails that $\frac{1}{\alpha}f(x_1) = \inf\{\|x_1 - y\|;\ y \in M_{n_1}\}$ for some (unique) $n_1 \geq k$.

Since each M_n is closed convex and bounded, and since X is already shown to be reflexive, there must exist $y_1 \in M_{n_1}$ such that $\|x_1 - y_1\| = \frac{1}{\alpha}f(x_1)$. Assume that $x_1 \notin M_{n_1}$. Then, $\|x_1 - y_1\| > 0$, and using the fact that $\alpha > 2\|y_1 - x_0\|$, we can estimate

$$\begin{aligned} f(x_1) + \|x_1 - x_0\|^2 &= \alpha\|x_1 - y_1\| + \|x_1 - x_0\|^2 \geq \alpha\|x_1 - y_1\| + \big(\|y_1 - x_0\| - \|x_1 - y_1\|\big)^2 \\ &= \alpha\|x_1 - y_1\| + \Big(\|y_1 - x_0\|^2 - 2\|y_1 - x_0\|\|x_1 - y_1\| + \|x_1 - y_1\|^2\Big) \\ &> (\alpha - 2\|y_1 - x_0\|)\|x_1 - y_1\| + \|y_1 - x_0\|^2 \\ &> \|y_1 - x_0\|^2 = f(y_1) + \|y_1 - x_0\|^2, \end{aligned}$$

which contradicts (14). Therefore $x_1 \in M_{n_1} \subset M$.

Now, for every $n \geq k$ and every $x \in M_n$ we thus have from (14) that $\|x - x_0\|^2 \geq \|x_1 - x_0\|^2$ and hence $\|x - x_0\| \geq \|x_1 - x_0\|$. Fix any $n \geq k$. Put $w_n = (1 + 2^{-n})y_n + x_0 - \xi(x_0)y_n$. Then $\|w_n - (1 + 2^{-n})y_n\| \leq \|x_0\| + 2\|x_0\| \leq 3\delta$ and $\xi(w_n - (1 + 2^{-n})y_n) = 0$. Hence $w_n \in M_n$ and we thus have $\|w_n - x_0\| \geq \|x_1 - x_0\|$. Moreover,

$$\|w_n - x_0\| = (1 + 2^{-n} - \xi(x_0))\|y_n\| \to 1 - \xi(x_0) \quad \text{as} \quad n \to \infty.$$

Therefore

$$1-\xi(x_0)=\lim_{n\to\infty}\|w_n-x_0\|\geq\|x_1-x_0\|\geq\xi(x_1-x_0)=1+2^{-n_1}-\xi(x_0),$$

a contradiction. We have completely proved (i). ∎

(i)⇒(ii'). Assume that the function $f: X\to(-\infty,+\infty]$ is proper, lower semicontinuous and bounded below. We shall imitate the argument from the proof of the implication (i)⇒(ii). Take $B=\big\{(x,t)\in X\times\mathbb{R};\ |t|\leq-\|x\|^2+1\big\}$ and let $|||\cdot|||$ be the Minkowski functional of B. Let the sets $O,\ E,\ \Omega$, and $x_0\in\Omega$ be found for this norm $|||\cdot|||$ as before. We find a number $t_0<\inf f$ so that $(x_0,t_0)\in E$. Let $(x_1,t_1)\in\operatorname{epi}f$ and ρ be found for our (x_0,t_0) so that (2) is satisfied. Then for sure $t_1>t_0$ and the formula (3) has the form

$$\rho(t_1-t_0)+\|x_1-x_0\|^2=\rho^2.$$

And, as in the proof of the implication (i)⇒(ii), we can see that actually $t_1=f(x_1)$. Thus we have formula (5) with $L=1$.

Now, for any $x\in X$ we have $f(x)>t_0$. Hence the reasoning from the proof of (i)⇒(ii) leading to formula (6) in Step 3 yields

$$f(x)+\tfrac{1}{\rho}\|x-x_0\|^2\geq f(x_1)+\tfrac{1}{\rho}\|x_1-x_0\|^2\quad\text{whenever}\quad x\in X.$$

Further we proceed as in the proof of (i)⇒(ii). The claim in Step 4 there is satisfied automatically. Step 5 works without any change. Then the problem of minimizing the function $f+\frac{1}{\rho}\|\cdot-x_0\|^2$ is well-posed.

It remains to prove Step 6. For the still fixed x_0 we put $a_{x_0}=\varphi(\inf f)$ (the latter could be ∞), where the function φ is as in Step 6. One easily sees that $\varphi(\tau)\to a_{x_0}$ as $\tau\uparrow\inf f$. The fact that f is bounded below yields that $\varphi(\tau)\to 0$ as $\tau\downarrow-\infty$. We need to show that φ is strictly increasing. As in Step 6 above, fix $\tau_1<\tau_2<\inf f$ and take $\tau\in(\tau_1,\tau_2)$ so that $\tau\in S$ (S is the residual subset of $(-\infty,f(x_0))$ from Step 2 corresponding to x_0). Then there is $(u,t)\in\operatorname{epi}f$ so that $\varphi(\tau)=|||(x_0,\tau)-(u,t)|||^{-1}=:\frac{1}{\rho_0}$. Thus, according to the definition of the norm $|||\cdot|||$, we have (since $t\geq\inf f$)

$$\rho_0(t-\tau)+\|x_0-u\|^2=\rho_0^2.$$

On the other hand, putting $\rho=|||(x_0,\tau_2)-(u,t)|||$, then

$$\rho(t-\tau_2)+\|x_0-u\|^2=\rho^2.$$

Consequently

$$\rho_0^2-\rho_0(t-\tau)=\rho^2-\rho(t-\tau_2)=\|x_0-u\|^2\ (\geq 0).$$

And, since $t-\tau>t-\tau_2>0$, we conclude that $\rho<\rho_0$. Therefore

$$\operatorname{dist}\big((x_0,\tau_2),\operatorname{epi}f\big)\leq\rho<\rho_0,$$

which gives $\varphi(\tau_2) > \varphi(\tau)$. The fact that $\varphi(\tau) \geq \varphi(\tau_1)$ is proved as in Step 6.

Then clearly there is a residual set $R_{x_0} \subset (0, a_{x_0})$ satisfying (ii') for x_0. We have thus verified (ii'). ∎

(ii')⇒(iii') is trivial.

(iii')⇒(i). Suppose (iii') is true. In order to prove the reflexivity of X let us fix any $\xi \in X^*$, with $\|\xi\| = 1$. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \xi^{-1}(0), \\ +\infty & \text{if } x \in X \backslash \xi^{-1}(0). \end{cases}$$

Clearly, f is lower semicontinuous and bounded below. To this f, let $D \subset X$ be the dense set given by (iii'). Pick then $x_0 \in D \cap (X \setminus \xi^{-1}(0))$ and let c be such that $f + c\| \cdot - x_0\|^2$ attains its minimum at some x_1, i.e. $f + c\| \cdot - x_0\|^2 \geq f(x_1) + c\|x_1 - x_0\|^2$. We necessarily have that $\xi(x_1) = 0$. Thus, for every $x \in \xi^{-1}(0)$ we have $c\|x - x_0\|^2 \geq c\|x_1 - x_0\|^2$, that is, $\|x - x_0\| \geq \|x_1 - x_0\|$ (> 0). We shall show that $|\xi(x_1 - x_0)| = \|x_1 - x_0\|$. Assume not; then $|\xi(x_1 - x_0)| < \|x_1 - x_0\|$. Find $z \in X$ so that $\|z - x_0\| = \|x_1 - x_0\|$ and $\xi(z - x_0) > \frac{1}{2}\big(\|x_1 - x_0\| + |\xi(x_1 - x_0)|\big)$. Then, putting

$$w = x_0 + \frac{\xi(x_1 - x_0)}{\xi(z - x_0)}(z - x_0),$$

we have $w \in \xi^{-1}(0)$ and

$$\begin{aligned} \|x_1 - x_0\| \leq \|w - x_0\| &= \frac{|\xi(x_1 - x_0)|}{\xi(z - x_0)}\|z - x_0\| \\ &< \frac{\|x_1 - x_0\|}{\xi(z - x_0)}\big(2\xi(z - x_0) - \|x_1 - x_0\|\big) \leq \|x_1 - x_0\|; \end{aligned}$$

a contradiction. Therefore $|\xi(x_1 - x_0)| = \|x_1 - x_0\|$, and James' theorem guarantees that X must be reflexive.

If the norm in X is not Kadec-Klee, then (iii') is violated. Indeed, assume that the norm in X is not Kadec-Klee and take the closed set M from the proof of the implication (iii)⇒(i). Let the function f be defined as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \in M, \\ +\infty & \text{if } x \in X \backslash M. \end{cases}$$

Evidently, f is lower semicontinuous and bounded below. Let $\delta > 0$ be the number associated with the set M. We shall show that for any $x_0 \in \frac{\delta}{4}B_X$ and any $c > 0$ the function $f + c\| \cdot - x_0\|^2$ does not attain its infimum. Suppose this is not the case. Then there are $x_0 \in \frac{\delta}{4}B_X$, $c > 0$, and $x_1 \in X$ so that $f(x) + c\|x - x_0\|^2 \geq f(x_1) + c\|x_1 - x_0\|^2$ for every $x \in X$. Clearly $x_1 \in M$, and so we get $\|x - x_0\| \geq \|x_1 - x_0\|$ for every $x \in M$. Now we proceed exactly as at the end of the the proof of the implication (iii)⇒(i) to get a contradiction. We have thus completely verified (i), which ends the proof of Theorem 1. ∎

The genericity of the set of perturbations yields the following

Corollary 1. *Let $(X, \|\cdot\|)$ be a reflexive Banach space, with Kadec-Klee norm, and let a function $f : X \to \mathbb{R}$ be either Lipschitzian or be continuous and bounded. Then there exists a residual set $\Omega \subset X$ such that for every $u \in \Omega$ there are $a_u \in (0, +\infty]$ and a residual set $R_u \subset (0, a_u)$ so that for each $c \in R_u$* $\inf\big(f + c\|\cdot - u\|^2\big)$ *and* $\sup\big(f - c\|\cdot - u\|^2\big)$ *are both attained. (In the Lipschitzian case $a_u = +\infty$.)*

2. Concluding remarks and results

This section contains several remarks strengthening some of the conclusions in our principle as well as giving some corollaries. We also give some relations to other similar variational principles.

Remark 1. To prove our Theorem 1 we used the result of K.-S. Lau [L, Theorem 4], asserting that *given a closed nonempty set C in a reflexive Banach space X with Kadec-Klee norm, there is a residual subset Ω of X such that for each $x \in \Omega$ the best approximation problem of x to C is well-posed.* Actually, Theorem 1 contains this result as a special case. This can be readily seen by taking such a $C \subset X$ and applying (ii') to the function $f : X \to (-\infty, +\infty]$ given by $f(x) = 0$ if $x \in C$ and $f(x) = +\infty$ otherwise. We believe that a direct proof of our theorem using Baire category argument exists without using the Lau's result. A forthcoming paper [F1] contains a more direct approach to the proof of (i)$\Longrightarrow$(ii) in Theorem 1. It is based on a technique of K.-S. Lau [L].

Remark 2. (*Uniqueness of the minimum point for the perturbations*). If the problem to minimize a given proper bounded from below function $g : X \to (-\infty, +\infty]$ is well-posed and with unique minimizer, then actually every minimizing sequence for g converges to this unique minimizer. This property is known in Optimization as *the minimization problem for g is Tykhonov well-posed*, and in Analysis as *g admits a strong minimum.* Lau's result [L, Theorem 4] can be used (see [BF, Theorem 6.6]) to show that, if the norm in the reflexive space X is both Kadec-Klee and strictly convex, then for every nonempty closed subset C of X the problem of best approximation in C is generically (in X) well-posed with unique solution. Thus, there exists a variant of Theorem 1: if in (i) we put *The space X is reflexive and the norm is strictly convex and Kadec-Klee*, this will be equivalent to all of the other assertions simply adding in them that the minimizer for the corresponding perturbations is unique. We omit rather routine proofs.

Remark 3. The norm $\|\cdot\|$ in a reflexive space X can be taken, both Kadec-Klee and Fréchet smooth according to well known renormings. Thus we get from Theorem 1 a "Fréchet smooth" variational principle. If X is even superreflexive, then it admits an equivalent norm $\|\cdot\|$ which is Kadec-Klee, and moreover has a modulus of smoothness of power type, that is, there are $1 < p \leq 2$ and $c > 0$ such that $\|x + th\| + \|x - th\| - 2 \leq ct^p$ whenever x, h lie in the unit sphere of X, and $t > 0$; then our principle provides supporting by an even smoother function.

Remark 4. Given a Lipschitz function f on X, the set of points of supporting by a parabola may not be residual. Indeed, if so, then, when applying our principle also for $-f$, we would get that f is Fréchet differentiable at the points of a residual set, contradictory to a known counterexample in Hilbert space due to Preiss [P].

Remark 5. If X is reflexive, with Kadec-Klee norm $\|\cdot\|$, and $f : X \to \mathbb{R}$ is lower semicontinuous with no further properties, then it may happen that $f + c\|\cdot - u\|^2$ attains its infimum for no $c \geq 0$ and for no $u \in X$. Indeed, define $f : \mathbb{R} \to \mathbb{R}$ by $f(t) = t^3,\ t \in \mathbb{R}$. Then $\inf_{t\in\mathbb{R}} \big(f(t) + c(t-u)^2\big) = -\infty$ for every $c \geq 0$ and every $u \in \mathbb{R}$.

Remark 6. Our argument does not guarantee if, in (ii), we can have $c = 1$ for some $u \in \Omega$. This is done in (iii) for the price of eventual loss of the residuality of the set of u's and eventual loss of the well-posedness of the problems of minimizing $f + \|\cdot - u\|^2$. Note that in (iii) we can replace $f + \|\cdot - u\|^2$ by $f + c\|\cdot - u\|^2$ with an arbitrary a priori fixed $c > 0$, just by considering the norm $\sqrt{c}\|\cdot\|$.

Remark 7. (*Relations with Borwein-Preiss smooth variational principle*) This principle is usually understood as that with a perturbing function in form of the sum of countably many parabolas, see [BP, Theorem 2.6], [Ph1, Theorems 4.20]. Yet in [BP, Theorem 5.2], dealing with reflexive spaces with Kadec norm, a perturbing function is just one parabola. This is very close to (iii') in our Theorem 1. In (ii) and (ii') we have in addition well-posedness of the perturbations as well as *genericity* of the points u which generate perturbing parabolas, while Borwein-Preiss principles do not have it. Moreover, our Theorem 1 covers also the case of Lipschitz, not necessarily bounded below, functions. Let us mention that genericity of the perturbations occurs also in the Stegall variational principle [S1,S2] (see also below) and Deville-Godefroy-Zizler variational principle [DGZ] (cf. also a strengthening of the latter in [DR]).

On the other hand, unlike of Borwein-Preiss principle we do not have the same kind of control over the location of the minimum point. What we have as a direct consequence from formula (5) is that the minimum point x_1 of $f + c\|\cdot - x_0\|^2$ found in the proofs of (i)$\Rightarrow$(ii) (and (i)$\Rightarrow$(ii')) satisfies $\|x_1 - x_0\| < \frac{L}{c}$ (with $L = 1$ in the non-Lipschitzian case). This means that in the Lipschitzian case we can have a minimum point of the perturbation as close to a given point as we want, at the price of having the coefficient c of the perturbing parabola big. In particular, this shows that in such a case we have density of the points realizing the minima of the perturbations.

The same kind of control could be obtained also in the case of a proper lower semicontinuous bounded from below function f which is continuous at the points of its effective domain $\operatorname{dom} f$ (the latter implies that $\operatorname{dom} f$ is open). To see this, let $\varepsilon > 0$. Then, by the proof of (i)$\Rightarrow$(ii') there is a residual set Ω' in the nonempty open set $\{x \in \operatorname{dom} f;\ f(x) < \inf f + \varepsilon\}$ so that for each $x_0 \in \Omega'$ there is $a_{x_0} > 0$ such that $f + c\|\cdot - x_0\|^2$ attains its minimum for residually many $c \in (0, a_{x_0})$ at some corresponding x_1. Formula (5), with $L = 1$, gives $\|x_1 - x_0\| \leq \frac{1}{c}$. On the other hand, we know from Step 6 that $a_{x_0} = \lim_{\tau\uparrow\inf f} [\operatorname{dist}((x_0, \tau), \operatorname{epi} f)]^{-1}$. Since for $\tau < \inf f$ we have

$$\operatorname{dist}((x_0, \tau), \operatorname{epi} f) \leq |||(x_0, \tau) - (x_0, f(x_0))||| = f(x_0) - \tau,$$

we obtain that $a_{x_0} > \frac{1}{\varepsilon}$. Thus for residually many $c \in (\frac{1}{\varepsilon}, a_{x_0})$ the minimum point x_1 will satisfy $\|x_1 - x_0\| \leq \frac{1}{c} < \varepsilon$. Observe that in this case, by formula (5) with $L = 1$, we have also $f(x_1) - t_0 \leq \frac{1}{c}$ which gives $f(x_1) < \inf f + \frac{1}{c} < \inf f + \varepsilon$ for c as above. That is, we have also control over the value of f at x_1: The minimum point x_1 of $f + c\|\cdot - x_0\|^2$ is an ε-minimum of f.

Remark 8. (*Relations with Stegall variational principle*) Let us recall some notions: Given a nonempty set A of a Banach space X a *slice* in A is every set of the form

$$\{x \in A : x^*(x) > \sup_{y \in A} x^*(y) - \alpha\}$$

for some $x^* \in X^*$ and $\alpha > 0$. The set A is called *dentable* if it admits slices of arbitrary small diameter. The Banach space X has the *Radon-Nikodým property* (in short RNP) if every nonempty bounded subset of X is dentable.

The Stegall variational principle [S1,S2] (see also [Ph1]) states the following: *Let $f : C \to \mathbb{R} \cup \{+\infty\}$ be a proper bounded from below lower semicontinuous function defined on a closed convex and bounded subset C of a Banach space X with RNP. Then the set $\{x^* \in X^* : f + x^*$ attains its strong minimum on $C\}$ is a residual subset of X^*.*

This principle can be used to obtain a perturbing function in the form $\|\cdot\|^2 + x^*$ where $x^* \in X^*$. More precisely, we have the following

Theorem 2. *Let $(X, \|\cdot\|)$ be a Banach space with RNP. Let $f : X \to (-\infty, +\infty]$ be a proper lower semicontinuous function bounded below by $a - b\|\cdot\|^p$ where $a \in \mathbb{R}$, $b > 0$, and $1 \leq p < 2$. Then there exists a residual set $\Omega \subset X^*$ such that for every $x^* \in \Omega$ the function $f + \|\cdot\|^2 + x^*$ attains strongly its infimum. If $(X, \|\cdot\|)$ is a Hilbert space, then there even exists a residual set $\Omega \subset X$ such that for every $x^* \in \Omega$ the function $f + \|\cdot - x^*\|^2$ attains strongly its infimum.*

Proof: We can see that

$$\tfrac{1}{\|x\|}(f(x) + \|x\|^2) \geq \tfrac{1}{\|x\|}(a - b\|x\|^p + \|x\|^2) \to +\infty \quad \text{as} \quad x \in X \ \text{ and } \ \|x\| \to +\infty.$$

Hence, for every $n \in \mathbb{N}$ there is $a_n \in \mathbb{R}$ such that $f + \|\cdot\|^2 \geq a_n + 2n\|\cdot\|$. Fix $n > 1$. We will show that there is a residual subset $\Omega_n \subset nB_{X^*}$ (where B_{X^*} is the unit ball in X^*) such that for any $x^* \in \Omega_n$ the function $f + \|\cdot\|^2 + x^*$ attains strong minimum on X.

To this end, we follow the lines of the proof of Corollary 5.22 from [Ph1]. Since we can replace $f + \|\cdot\|^2$ by $f + \|\cdot\|^2 - a_n$, we may suppose that $a_n = 0$. Let us note that in such a case for any $x^* \in nB_{X^*}$ and any $x \in X$ we have

$$f(x) + \|x\|^2 + x^*(x) \geq 2n\|x\| - n\|x\| = n\|x\|. \tag{15}$$

Let $r := (1/n)(f(0)+1)$ and set $C := rB_X$. Applying now the Stegall variational principle for the restriction $(f + \|\cdot\|^2)|_C$ of $f + \|\cdot\|^2$ on C we obtain a residual subset G of X^* such that for any $x^* \in G$ the function $(f + \|\cdot\|^2)|_C + x^*$ attains its strong minimum on C. Put $\Omega_n := nB_{X^*} \cap G$. The latter set is residual in nB_{X^*}. We will show that, in fact, for any $x^* \in \Omega_n$ the function $f + \|\cdot\|^2 + x^*$ attains its strong minimum also in X, not only in C.

To prove this, let $x^* \in \Omega_n$ and let $x_0 \in C$ be the strong minimum of $f + \|\cdot\|^2 + x^*$ on C. Take $x \in X$ and suppose that $f(x) + \|x\|^2 + x^*(x) \leq f(x_0) + \|x_0\|^2 + x^*(x_0) = \inf_C(f + \|\cdot\|^2 + x^*) \leq f(0)$. Thus, by (15) we have $\|x\| \leq (1/n)f(0)$ and hence $x \in C$ yielding $x = x_0$. Therefore, x_0 is the unique minimum of $f + \|\cdot\|^2 + x^*$ on X. Further, let $(x_k)_{k \in \mathbb{N}}$ be a minimizing sequence for $f + \|\cdot\|^2 + x^*$, that is $f(x_k) + \|x_k\|^2 + x^*(x_k) \to f(x_0) + \|x_0\|^2 + x^*(x_0)$. Then for k large enough we will have $f(x_k) + \|x_k\|^2 + x^*(x_k) < f(0) + 1$, and therefore, again from (15) we see that x_k must be in C and hence $x_k \to x_0$. Thus, $f + \|\cdot\|^2 + x^*$ attains also its strong minimum on X and our assertion is proved.

It suffices now to set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. This is a residual set in X^* and satisfies the conclusion of the theorem. The case of a Hilbert space can be obtained immediately from the geometry of the Hilbertian norm. Indeed, we have $\|x\|^2 - x^*(x) = \|x - \frac{x^*}{2}\|^2 - \frac{1}{4}\|x^*\|^2$ for every $x \in X$. ∎

For Hilbert spaces, the above result gives a stronger conclusion than our statement (iii) because we do not have the residuality of u's in (iii). Further, it should be noted that Stegall principle does not need the Kadec-Klee property of the norm neither the reflexivity of the space. Only the the fact that X possesses RNP is assumed (every reflexive space has RNP, see for instance [Ph1, Chapter 5]). Also, the boundedness below by the function $a - b\|\cdot\|^p$ is guaranteed by both the Lipschitz property of f as well as by $\inf f > -\infty$.

Remark 9. We note that in a reflexive space any Kadec-Klee norm is already Kadec, which means that the weak and the norm topologies on the sphere coincide. This is a consequence of a more general fact that *in any Asplund space, bounded weakly sequentially closed sets are weakly closed.* Indeed, consider an Asplund space X and a bounded weakly sequentially closed set $M \subset X$, and let x be an element of the weak closure of M. By Kaplanski's theorem [F$\sim$, p. 129], there is a countable set $C \subset M$ such that x belongs to the weak closure of C. Let Y be the closed linear span of C. Since X is Asplund, Y^* must be separable. Let S be a countable norm-dense subset in B_{Y^*}. Find then a sequence $(x_n)_{n\in\mathbb{N}}$ in C such that $y^*(x_n) \to y^*(x)$ as $n \to \infty$ for every $y^* \in S$. Then also $x^*(x_n) \to x^*(x)$ as $n \to \infty$ for every $x^* \in X^*$. And since M was weakly sequentially closed, we conclude that $x \in M$.

Finally, let us mention that it might be possible to use some ideas from [BF] in order to extend our principle for non-reflexive spaces.

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