Parallel strategies

Pavel Pudlák*
Mathematical Institute, AV ČR, and
Institute for Theoretical Computer Science,
Prague, Czech Republic

April 11, 2003

Abstract

We consider combinatorial principles based on playing several two person games simultaneously. We call strategies for playing two or more games simultaneously *parallel*. The principles are easy consequences of the determinacy of games, in particular they are true for all finite games. We shall show that the principles fail for infinite games. The statements of these principles are of lower logical complexity than the sentence expressing the determinacy of games, therefore, they can be studied in weak axiomatic systems for arithmetic (Bounded Arithmetic). We pose several open problems about the provability of these statements in Bounded Arithmetic and related computational problems.

1 Introduction

Every finite 2-player game is determined, which means that one of the players has a winning strategy. Our main motivation is to study games from the point of view of computational complexity and Bounded Arithmetic. As the statement that a game is determined is of too high complexity, we are looking for lower complexity consequences of determinacy. Such statements are likely to provide us with new principles independent of bounded arithmetic. Because proving independence results of this kind is very hard, instead we shall consider a simpler problem of constructing infinite models in which such principles fail. To this end we shall use countably infinite games, a concept introduced in [2], which plays an important role in contemporary set theory.

We shall define two principles parametrized by a natural number $k \geq 1$. Both are based on playing k games simultaneously in a synchronized manner. To avoid confusion, since some of the k games may be equal, we shall say that the games are played on k boards. We call

*A main part of this work was done while visiting Institute for Advanced Study, Princeton, USA, and supported by a grant of the state of New Jersey. Supported by grants No. A1019901 of the Academy of Sciencec of the Czech Republic and No. 201/01/1195 of the Grant Agency of the Czech Republic.

a strategy for playing several games simultaneously a *parallel strategy.* The first principle says, roughly speaking, that when playing the same game simultaneously on k boards with the same player being the first to move on all boards, it is not possible that there exists a parallel strategy for the first player to win on at least one board and at the same time there exists a strategy for the second player to win on at least one board. This is, clearly, a consequence of the game being determined. We shall show that this principle fails for infinite games already for $k = 2$. (Though it looks paradoxical, there is no immediate contradiction involved: when playing the two strategies against each other, the first player wins on one board and the second player wins on the other board.)

The second principle speaks about playing the same game twice with interchanging the roles of being the player who starts. Trivially, it is not possible that one of the players has a strategy that guarantees winning in both rounds. We shall show that the same is true for two and three games played simultaneously on two, respectively three boards, but when playing four such games simultaneously, there are infinite games for which this principle fails.

The second principle was discovered when analyzing the meaning of the Herbrand consistency of Bounded Arithmetic (see [3] for the definition). Our goal is to understand the combinatorial substance of Gödel sentences. Furthermore, since concrete independence results are so difficult to come by in Bounded Arithmetic, finding a suitable principle may considerably help us to obtain such a result. The principles proposed in this paper seem to be essentially different from all principles studied in Bounded Arithmetic before. We shall discuss these prospects in the last section and pose some open problems on provability and computational complexity.

2 Infinite games

A game is a subset $G \subseteq \{0,1\}^\omega$, where $\omega = \{0,1,2,\ldots\}$. For an $X \in \{0,1\}^\omega$, the even elements are the moves of the first player, the odd elements are the moves of the second player and X is viewed as the final position. If $X \in G$ then we say that the first player won, otherwise the second one won. A strategy for the first (second resp.) player is a function $f : \{0,1\}^* \to \{0,1\}$ ($f : \{0,1\}^+ \to \{0,1\}$ resp.). We denote by A^* (A^+ resp.) the set of all (nonempty resp.) strings of elements of A; Λ denotes the empty string. If $X \in \{0,1\}^\omega$ is a sequence $(x_0, x_1, x_2, \ldots)$ and $a \in \{0,1\}$, then aX denotes $(a, x_0, x_1, x_2, \ldots)$

Playing several games $G_1, \ldots, G_k$ simultaneously means that the first player makes the first moves on all k games, then the second player makes the second moves on all k games etc. Thus a strategy for the first (second resp.) player is a function $f : (\{0,1\}^k)^* \to \{0,1\}^k$ ($f : (\{0,1\}^k)^+ \to \{0,1\}^k$ resp.) which determines the next move of the first (second resp.) player from the previous moves of the second (first resp.) player.

Playing several games simultaneously is like playing one game with a larger number of possible moves. Here we consider games with only two moves $\{0,1\}$, so k games correspond to a game with moves $\{0,1\}^k$. The combination of games, however, enables us to ask more questions than one can ask for a single game.

Theorem 1 *There exists a game G and strategies f and g for playing G simultaneously on two boards such the following holds. f is a strategy for the first player that ensures that she* *always wins on at least one board and g is a strategy for the second player that ensures that he always wins on at least one board.*

Proof. First we define the strategies and then we show how to choose G. We shall use the following notation. $X^I = \{x_0^I, x_1^I, \ldots\}$ are the moves of the first player on the first board, $Y^I = \{y_0^I, y_1^I, \ldots\}$ are the moves of the first player on the second board, $X^{II} = \{x_0^{II}, x_1^{II}, \ldots\}$ are the moves of the second player on the first board, and $Y^{II} = \{y_0^{II}, y_1^{II}, \ldots\}$ are the moves of the second player on the second board. Furthermore, it will be more convenient to represent a final position by a pair (X^I, X^{II}) or (Y^I, Y^{II}), thus a game is a subset $G \subseteq \{0,1\}^\omega \times \{0,1\}^\omega$.

The strategies are very simple. The first player starts with 0 on the first board and 1 on the other. In her next moves she simply copies second player's moves on the second board onto the first board and copies the second player's moves on the first board onto the second board. The strategy for the second player is essentially the same, except that he plays as the second one, and thus he will be one step behind when copying the moves of the first player. Here are formal definitions of the strategies f of the first player and g of the second player.

$$f(\Lambda) = (0,1) \text{ and for } n \geq 0,\ f((x_0^{II}, y_0^{II}) \ldots (x_n^{II}, y_n^{II})) = (y_n^{II}, x_n^{II});$$

$$g(x_1^I, y_1^I) = (0,1) \text{ and for } n \geq 1,\ g((x_0^I, y_0^I) \ldots (x_n^I, y_n^I)) = (y_{n-1}^I, x_{n-1}^I).$$

Now suppose that the first player uses strategy f and the moves of the second player on the first (second resp.) board are X^{II} (Y^{II} resp.). Then the final position on the first (second resp.) board will be $(0Y^{II}, X^{II})$ ($((1X^{II}, Y^{II})$ resp.). Next suppose that the second player uses strategy g and the moves of the first player on the first (second resp.) board are X^I (Y^I resp.). Then the final position on the first (second resp.) board will be $(X^I, 0Y^I)$ ($((Y^I, 1X^I)$ resp.). Thus to prove the theorem it suffices to show that there exists a G such that for every pair of the form $((0Y, X), (1X, Y))$ either $(0Y, X) \in G$ or $(1X, Y) \in G$, and for every pair of the form $((X, 0Y), (Y, 1X))$ either $(X, 0Y) \notin G$ or $(Y, 1X) \notin G$. (In fact, in our construction always exactly one of the two possibilities occurs.)

Let E^I and E^{II} be the symmetric graphs on $\{0,1\}^\omega \times \{0,1\}^\omega$ the edges of which are such pairs, thus

$$E^I = \{((aY, X), (\bar{a}X, Y));\ X, Y \in \{0,1\}^\omega, a \in \{0,1\}\},$$
$$E^{II} = \{((X, aY), (Y, \bar{a}X));\ X, Y \in \{0,1\}^\omega, a \in \{0,1\}\},$$

where $\bar{a}$ denotes $1 - a$. Clearly, there are no loops in these graphs (this is ensured by having $a \neq \bar{a}$ at the same position in the two vertices). It is easily seen that both graphs are, in fact, perfect matchings. (Consider, eg., E^I. Clearly every vertex is of the form (aY, X) for some a, X, Y and this vertex uniquely determines the only other vertex connected to it $(\bar{a}X, Y)$.) Consequently, $E^I \cup E^{II}$ does not contain odd cycles, hence it is bipartite. (In fact, the graph is a disjoint union of even cycles and edges.) Take G to be one of the two blocks of a bipartition of $E^I \cup E^{II}$ and we are done. $\square$

Since the game in the theorem above is necessarily non-determined, we get another proof of the existence of non-determined games.

Let us consider another type of a combination of several games. Two players, called A (Alice) and B (Bob), will play k games $G_1, \dots, G_k$ as follows. First they play simultaneously these games on k boards and A starts on each board. This is the same as above and it will be called the first round. Then they exchange the roles and play the same k games; this is the second round. Exchanging roles means that in the second round B starts on all boards. Parallel strategies are defined in the same natural way as above. In particular, in the second round the moves of the strategy depend on everything that happened in the first round. Formally, a strategy for A is a pair of mappings (h, H) such that

$$h : (\{0,1\}^k)^* \to \{0,1\}^k, \text{ and } H : (\{0,1\}^\omega)^k \times (\{0,1\}^k)^+ \to \{0,1\}^k,$$

(h is for the first round, H is for the second round).

If the games are determined, then none of the players has a strategy that would ensure them winning twice on the same board. We shall show that this consequence of determinacy may fail.

Theorem 2

1. There are games $G_1, \dots, G_4$ and a strategy (h, H) for A that ensures her to win twice on at least one of the four boards.

2. There are no such games and strategies for $k = 1, 2, 3$.

Proof. 1. Let $G_1 = \ldots = G_4 = G$, where G is the game from Theorem 1. Let f, g be the strategies from that theorem. A's strategy is as follows. In the first round she uses f for the first two boards and, independently, she uses it also for the last two boards. This ensures her winning on at least two boards in the first round. In the second round she applies g to the two boards on which she won in the first round (ignoring the other two). This ensures her winning on at least on one of them also in the second round.

2. Let $k = 1$ and suppose (h, H) is such a strategy. Then, in particular, h is a winning strategy for the first player in G_1. Hence G_1 is determined and (h, H) cannot work.

Let $k = 2$ and suppose (h, H) is such a strategy. First observe that h cannot ensure A always winning on both boards in the first round. If this were so, then B could use h to defeat A on both boards in the second round. Suppose w.l.o.g. that A playing h can lose on the first board. In the first round, if she loses on the first board she has to win on the second board. Let X (Y resp.) be moves of B in the first round such that A playing h loses on the first board and wins on the second one. Let $H_{X,Y}$ denote H with the moves in the first round fixed to X and Y. Thus $H_{X,Y}$ is a parallel strategy for playing G_1 and G_2 simultaneously which ensures the second player to win on the second board. To obtain a winning strategy for the second player in G_2 we can substitute arbitrary values for the moves of the first player on the first board, say all zeros. Having a winning strategy for the second player in G_2, the problem easily reduces to the case $k = 1$. To this end it suffices to use this strategy for the moves of B on the second board in the first round and use arbitrary fixed moves for him on the second board in the second round.

Finally, let $k = 3$ and, again, suppose (h, H) is such a strategy. The same argument as above shows that B can play against h so that he wins at least on one board in the first round. Suppose it is the first board. Let X, Y, Z be the moves by which he can achieve it.

Then player A must win on the second or on the third board in the second round. Hence, by fixing the moves in the first round to X, Y, Z and the moves of the first player on the first board in the second round arbitrarily (say, to all zeros) in H, we get a parallel strategy for the pair of games G_2, G_3 that ensures the second player winning at least on one board. Let this strategy be denoted by h'.

Suppose that in the first round A plays h and B uses h' on the second and the third boards and a strategy of his choice for the first board. Suppose that in such a case B will never win on the first board (we are still considering only the first round). It means that this combination of strategies h and h' is a winning strategy for the first player in game G_1. As B can use this strategy to win on the first board in the second round, we have eliminated the first game. For two games we have shown the theorem already above.

It remains to consider the case when B playing h' on the last two boards can also win on the first board, assuming A is playing h. Then we know that B playing against A that uses h can win on at least two boards. Suppose w.l.o.g. he wins on the first two boards. Then A has to win on the third board in the second round. Then, in the same way as above, we get a winning strategy for the second player in G_3. Once we have a winning strategy for G_3, the problem is reduced to two games again. □

In the proof of Theorem 2, part 1, we have constructed the games and the strategy from a game and strategies of Theorem 1. Now we shall prove that the converse is also possible.

Theorem 3 *Suppose that games $G_1, \ldots, G_4$ and a strategy (h, H) satisfy the condition of Theorem 2. Then for every $i = 1, \ldots, 4$ it is possible to define explicitly strategies f_i and g_i such that G_i, f_i and g_i satisfy the condition of Theorem 1.*

Proof. Let games $G_1, \ldots, G_4$ and a strategy (h, H) satisfying the condition of Theorem 2 be given.

First we observe that none of the games is determined, because otherwise we could reduce four games to three which is impossible by Theorem 2, part 2.

We shall consider which subset of the four games player A can lose when playing as the first player and using strategy h. We shall prove several claims.

Claim 1. A can lose any of the games.

Suppose she cannot lose G_i. Then we could use h as a winning strategy for G_i by playing arbitrary (eg. all zeros) moves for the second player on the other boards.

Claim 2. For every i there exists j, $j \neq i$, such that A can lose both G_i and G_j.

Suppose it is not true for i. By Claim 1, B can play so that A loses G_i in the first round. Then H with the information about the first round is essentially a strategy for winning one of the games G_j, $j \neq i$ as the second player. Let us denote by s such a parallel strategy for the three games G_j, $j \neq i$. Combine h with s, thus obtaining a strategy for the first player in G_i. This must be a winning strategy, because s ensures that the first player loses at least one of the games G_j, $j \neq i$. But this is impossible, hence Claim 2 is proven.

If A wins only one game G_i in the first round, then she must win G_i in the second round. Thus we would get a winning strategy for the second player for G_i. Hence h ensures that

A wins always at least two games in the first round. Let us denote by R the set of pairs $\{i, j\}$, $i \neq j$, such that A can lose G_i and G_j when playing strategy h. Think of R as a graph on the set of vertices $V := \{1, 2, 3, 4\}$. Let $S_1(i, j)$ (respectively $S_2(i, j)$) denote the set of parallel strategies for playing games G_i and G_j simultaneously as the first (respectively second) player in both games and such that they ensure winning at least in one of the two games.

Claim 3. For every $\{k, l\} \in R^* := \{V \setminus \{i, j\};\ \{i, j\} \in R\}$, there exists an $s \in S_2(k, l)$.

Indeed, if A loses G_i and G_j in the first round, then A must win one of the two games G_m, $m \neq i, j$ in the second round. Thus we can construct an $s \in S_2(k, l)$ from H (and the moves in the first round).

Claim 4. The degree of every vertex of R is at least 2.

Suppose i is only in the edge $\{i, j\} \in R$, let $\{k, l\} = V \setminus \{i, j\}$. Take an $s \in S_2(k, l)$ and suppose B uses s for G_k and G_l in the first round. Then A always wins G_i, because she loses G_k or G_l and neither $\{i, k\}$ nor $\{i, l\}$ is in R. But that would give a winning strategy for G_i (for the first player), which is impossible. Hence the claim is true.

Claim 5. For every $\{i, j\} \in R$ (which is equal to $(R^*)^*$) there exists an $s' \in S_1(i, j)$.

Let $\{k, l\} = V \setminus \{i, j\}$ and $s \in S_2(k, l)$. Combine h with s. Since A loses one of the games G_k and G_l, she must win at least one from G_i and G_j. Thus we get $s' \in S_1(i, j)$.

Claim 6. For every $\{i, j\}$, $i \neq j$ there exists an $s' \in S_1(i, j)$.

We only need to construct an $s' \in S_1(i, j)$ for $\{i, j\} \notin R$. But for such a pair h must always win G_i or G_j.

Now we are ready to finish the proof of the theorem. By Claim 4, R contains a 4-cycle. Suppose this 4-cycle is 1–2–3–4–1. By the definition of R^* the same 4-cycle is in R^*.

First we construct a strategy f_1 which for G_1 played on two boards simultaneously ensures the first player to win at least on one board. By Claims 3 and 6 we have strategies $s_{12} \in S_1(1, 2), s_{23} \in S_2(2, 3), s_{13} \in S_1(1, 3)$. The strategy f_1 will be constructed by combining together s_{12}, s_{23}, s_{13} as follows. Denote by 1 and $1'$ the two boards on which game G_1 should be played. To play f_1 the player will use two more auxiliary boards numbered 2 and 3. The player will use s_{12} on boards 1 and 2 playing as the first player; she will use s_{23} on boards 2 and 3 playing as the second player; she will use s_{13} on boards $1'$ and 3 playing as the first player. Notice that the moves on the auxiliary boards are completely determined, so this gives a strategy for the first player to play on boards 1 and $1'$. Suppose the first player lost on board 1; then she won on board 2; hence the second (simulated) player lost on board 2, hence he won on board 3; hence the first player lost on board 3, hence she won on board $1'$.

To construct g_1 we use $r_{12} \in S_2(1, 2), r_{24} \in S_1(2, 4), r_{14} \in S_2(1, 4)$ and argue in the same way as above. $\square$

3 Games in bounded arithmetic

In this section we briefly explain the relevance of these concepts to the research in bounded arithmetic and proof complexity. By Bounded Arithmetic we mean the first order theory that results from Peano Arithmetic when the induction schema is restricted to bounded formulas. In order to be able to formalize polynomial time computations, one has also to accept an axiom saying that $x^{\lceil \log(x+1) \rceil}$ is a total function. Functions that are not bounded by a finite number of iterations of this function (ie., by $2^{(\log x)^k}$ for k a constant) are not provably total. There are various equivalent axiomatizations of this theory denoted by $I\Delta_0 + \Omega_1$, S_2, T_2 etc., see [1, 4]. In Bounded Arithmetic it is possible to formalize not only deterministic polynomial time computations (for which a smaller fragment S_2^1 suffices), but also computations with any finite number of alternations, put otherwise, computations with an oracle in **PH** (= *Polynomial Hierarchy)*. However, it is not possible to formalize computations that need time larger than polynomial.

Finite two player games can be naturally formalized in Bounded Arithmetic as follows. Suppose that we are working in Bounded Arithmetic and n is a number such that 2^n exists; thus we can speak about sequences of 0's and 1's of length n. We shall assume that G, the set of strings of length n that are winning positions for the first player, is definable and the definition gives a polynomial time decision algorithm. In general the set G cannot be coded by a number as it may be too large. It is well-known that it is **PSPACE**-complete to determine which of the players has a winning strategy. Therefore we do not expect that one can describe all strategies by computations in the **PH**. Thus, not only we cannot prove the determinacy of all finite games, but we even cannot state it, unless **PSPACE** collapses to the **PH**. Instead of considering general strategies we can consider polynomial time computable ones. We cannot state the determinacy of finite games using polynomially computable strategies, but we can state some consequences of the determinacy.

We shall consider principles corresponding to Theorems 1 and 2. To this end we have to explain how we formalize finite games and strategies. Recall that polynomial time computations can be formalized by Σ_1^b formulas (a class of formulas corresponding to **NP**). We shall assume that we have a Σ_1^b formula $\gamma(p,u)$ which for each parameter p defines a game (ie., a set of strings) of length equal to the length of p, and similarly, we have a Σ_1^b formula $\sigma_k(q,v)$ which for every parameter q defines a parallel strategy for playing k games of length equal to the length of q simultaneously on k boards. For a parallel strategy f (say for the first player), *'to beat f on ℓ boards'* means that there exist k sequences (moves of the second player) such that when we apply f to them we obtain k sequences (finial positions on the two boards) such that at least ℓ are not in G. Hence this can also be formalized by a Σ_1^b sentence. In this way the principles below can be formalized as universal Σ_1^b sentences.

The k-Game Principle I. *Suppose G is a finite polynomial time computable game and f and g are polynomial time computable strategies for playing G simultaneously on k boards as the first player in the case of f and as the second player in the case of g. Then either it is possible to beat f on all boards, or to beat g on all boards.*

The k-Game Principle II. *Suppose $G_1, \ldots, G_k$ are finite polynomial time computable games and (h, H) is a polynomial time computable strategy for A in the sense of Theorem 2. Then (h, H) does not ensure A to win twice on the same board.*

In an earlier version of this paper we conjectured that 2-Game Principle I and 4-Game Principle II are unprovable in Bounded Arithmetic. The supporting evidence was the fact that they fail for infinite games. However, recent investigation shows that they are most likely provable. We have a sketch of a proof, but some details must be verified yet. The main idea is essentially a reduction to coloring a graph by two colors. Since this problem is computationally easy (it is in **P**), also the principles are easy. For larger k such a reduction only leads to coloring *hypergraphs* by two colors, which is an **NP**-complete problem. Thus we still conjecture that for larger values of the parameter k these principles are unprovable in Bounded Arithmetic. In a forthcoming paper we are going to show connections to consistency statements for subtheories of Bounded Arithmetic and subsystems of the Quantified Sequent Calculus, which is another supporting evidence for unprovability.

Problem 1 *For which k are these principles provable in Bounded Arithmetic?*

It seems unlikely that such an independence result will be proved with currently available techniques. Thus it is more promising to try the oracle version, where the games and the strategy are given as a new uninterpreted predicate and new uninterpreted functions. For the fragment S^1_2 one can also use the following observation.

Proposition 1 *Suppose the k-Game Principle I is provable in S^1_2. Then for every game G and polynomial time strategies f and g for playing G simultaneously on k boards, it is possible to find in polynomial time either moves that beat f on all boards or moves that beat g on all boards.*

Proof. By a theorem of Buss [1], if a sentence of the form $\forall x \exists y \phi(x, y)$, with $\phi \in \Sigma^b_1$, is provable in S^1_2, then there exists a polynomial time algorithm that constructs y from x so that $\phi(x, y)$. Since the k-Game Principle I is formalized in this way, we can apply Buss's theorem to obtain the conclusion of the proposition. $\square$

This suggests a natural problem about the complexity of computations. We will state a more explicit version of this problem in terms of circuits.

Problem 2 *Suppose a circuit that defines winning positions of a game G of length n and k circuits that define strategies f and g for playing G simultaneously on k boards as the first player in case of f and as the second player in case of g are given. Is it then possible to find in time polynomial in the sizes of the circuits moves that beat f on all boards or moves that beat g on all boards? Is it possible at least to decide in polynomial time which of the two strategies can be beaten on all boards?*

Again, we conjecture that the answer to this problem is negative for $k \geq 3$; that would imply the unprovability of the k-Game Principle in S^1_2.

Let us note that the reductions proved for infinite games in the previous section can be adapted to the setting of Bounded Arithmetic. This gives a proof that k-Game Principles II are provable in Bounded Arithmetic for $k = 1, 2, 3$. Furthermore, the 2-Game Principle I and the 4-Game Principle II are equivalent in Bounded Arithmetic. The later fact is a positive message, since the study of the Herbrand consistency of Bounded Arithmetic leads to a very complicated principle in which many games are played in a non-synchronized way.

So the above equivalence gives some hope to extract a simpler combinatorial principle from this complicated statement. To this end it would be interesting to show more reductions or the impossibility of such reductions. In particular we do not know if the strength of the principles increases with increasing k. Let us note that such problems about reductions can also be studied in the context of infinite games if we interpret reductions as explicit definitions.

Acknowledgment. I would like to thank to Johan Håstad, Jan Krajíček, Alexander Razborov and Jiří Sgall for several useful discussions and comments to this paper.

References

[1] S. Buss, Bounded arithmetic, Bibliopolis, 1986.

[2] D. Gale, F. Steward, Infinite games with perfect information, Ann. Math. Studies 28 (1953), 245-266

[3] P. Hájek, P. Pudlák, Metamathematics of First-Order Arithmetic, Springer, 1993.

[4] J. Krajíček, Bounded Arithmetic, Propositional Logic and Complexity Theory, Cambridge Univ. Press, 1995.