On topologizable algebras

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Let A be a linear associative algebra. By [3] it is always possible to define a topology on A which makes of A a locally convex algebra with separately continuous multiplication (i.e. $x_\alpha, x, y \in A, x_\alpha \to x$ implies $x_\alpha y \to xy, yx_\alpha \to yx$).

On the other hand (cf. [3]) in general it is not possible to introduce a topology on A which makes of A a locally convex algebra with jointly continuous multiplication (i.e. $x_\alpha \to x, y_\beta \to y \Rightarrow x_\alpha y_\beta \to xy$). The aim of this note is to exhibit two examples which continue these investigations.

In the first example we construct a commutative algebra which admits no topology. This gives a negative answer to the question raised in [2]. In the second example we construct a topological algebra which admits no locally convex topology.

All algebras in this paper will be complex (this condition, however, is not essential).

We say that an algebra A is topologizable (topologizable as a locally convex algebra) if there exists a topology on A which makes of A a topological (locally convex) algebra with jointly continuous multiplication.

It is easy to see that an algebra A is topologizable if and only if there exists a system $\mathcal{V}$ of subset of A (zero-neighbourhoods in A) satisfying

(1) $\bigcap\limits_{V\in\mathcal{V}} V = \{0\}$

(2) $\lambda V \subset V$ for every $v \in \mathcal{V}$ and complex number $\lambda, |\lambda| \le 1$

(3) each $V \in \mathcal{V}$ is absorbent

(4) for every $V \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W + W \subset V$

(5) for every $W \in \mathcal{V}$ there exists $W \in \mathcal{V}$ such that $W \cdot W \subset V$.

For basic properties of topological algebras see e.g. [1].

Theorem 1. *There exists a commutative algebra which is not topologizable.*

PROOF. Denote by N the set of all positive integers and by $\mathcal{F}$ the set of all sequences $f = \{f_j\}_{j=1}^\infty$ of positive integers. Consider the linear space A of all formal linear combinations of elements c, x_i $(i \in N)$ and a_f $(f \in \mathcal{F})$. We define the multiplication in A by

$$\begin{aligned} cz = zc &= 0 && \text{for every } z \in A,\\ x_i x_j &= 0 && (i, j \in N),\\ a_f a_{f'} &= 0 && (f, f' \in \mathcal{F}),\\ x_n a_f = a_f x_n &= f_n \cdot c && (n \in N, f \in \mathcal{F}). \end{aligned}$$

Clearly these relations define uniquely a multiplication on A which makes of A a commutative algebra (for the associative law note that the product of any three of the basis elements is equal to zero).

We prove that A is not topologizable. Suppose on the contrary that there exists a system $\mathcal{V}$ of zero-neighbourhoods in A satisfying (1) - (5). Let $V, W \in \mathcal{V}$ satisfy $c \notin V$ and $W \cdot W \subset V$.

For $n = 1, 2, \cdots$ choose $s_n > 0$ such that $x_n \in s_n \cdot W$. Let $f = \{f_n\}_{n=1}^{\infty}$ be a sequence of positive integers f_n with $f_n > n \cdot s_n$. Then $a_f \in r \cdot W$ for some $r > 0$. We have

$$c = \frac{1}{f_n}(x_n \cdot a_f) = \frac{r \cdot s_n}{f_n}\left(\frac{x_n}{s_n} \cdot \frac{a_f}{r}\right) \in \frac{rs_n}{f_n} \cdot W \cdot W \subset \frac{rs_n}{f_n} V.$$

Since $c \notin V$ we have

$$\frac{rs_n}{f_n} > 1 \quad \text{and } r > \frac{f_n}{s_n} > n \qquad (n \in N),$$

a contradiction.

Remark. Let x be a linear space of infinite dimension and let $\mathcal{L}(X)$ be the algebra of all linear mappings acting in X. By [3], $\mathcal{L}(X)$ can not be topologized as a locally convex algebra. Using analogous method as in example 1 it is possible to show that $\mathcal{L}(X)$ is not topologizable. In fact even the algebra of all finite-dimensional operators in X is not topologizable.

THEOREM 2. *There exists a commutative topological algebra which is not topologizable as a locally convex algebra.*

PROOF. Let K be an uncountable set. Denote by $\mathcal{D}$ the set of all functions $d : N \times K \to N$. For $d \in \mathcal{D}, n \in N$ and $k \in K$ we shall write shortly d_{nk} instead of $d(n,k)$.

Clearly for every $d \in \mathcal{D}$ and $n \in N$ there exists a subset $K_{d,n} \subset K$ and a positive integer d_n such that $card K_{d,n} = d_n$ and $d_{nk} = d_n$ for every $k \in K_{dn}$. Let A be the linear space of all (finite) linear combinations of elements c, x_{nk} $(n \in N, k \in K), a_d$ $(d \in \mathcal{D})$ and y_{dnk} $(d \in \mathcal{D}, n \in N, k \in K_{dn} \subset K)$.

We define the multiplication in A by

$$\begin{aligned}
cz = zc &= 0 && (z \in A),\\
y_{dnk}z = zy_{dnk} &= 0 && (z \in A, d \in D, n \in N, k \in K_{dn}),\\
a_d a_{d'} &= 0 && (d, d' \in D),\\
x_{nk} \cdot x_{n'k'} &= 0 && (n, n' \in N, k, k' \in K),
\end{aligned}$$

$$x_{nk} \cdot a_d = a_d \cdot x_{nk} = \begin{cases} d_n y_{dnk} & (d \in D, n \in N, k \in K_{dn}) \\ 0 & (k \notin K_{dn}). \end{cases}$$

Clearly A is a commutative algebra. To define the topology on A we shall need the following notations:

Let $\mathcal{L}$ be the set of all complex valued functions $\lambda : k \mapsto \lambda_k$ defined on K with a finite support. For $\lambda \in \mathcal{L}$ and $i \in \{0, 1, 2, \ldots\}$ define

$$m_i(\lambda) = \min_{\substack{M \subset K \\ card M = i}} \max_{j \in K - M} |\lambda_j|.$$

Clearly $\max\limits_{j\in K}|\lambda_j| = m_0(\lambda) \geq m_1(\lambda) \geq \dots$ and $card\{j \in K, |\lambda_j| > m_i(\lambda)\} \leq i$.

LEMMA 3. *Let $\lambda, \mu \in \mathcal{L}$ and let $s, t \in \{0, 1, 2, \dots\}$. Then*

$$m_{s+t}(\lambda + \mu) \leq m_s(\lambda) + m_t(\mu)$$

where $\lambda + \mu \in \mathcal{L}$ is defined by $(\lambda + \mu)_k = \lambda_k + \mu_k \quad (k \in K)$.

PROOF. Suppose $j \in K, |\lambda_j + \mu_j| > m_s(\lambda) + m_t(\mu)$. Then either $|\lambda_j| > m_s(\lambda)$ or $|\mu_j| > m_t(\mu)$. Since

$$card\{j, |\lambda_j+\mu_j| > m_s(\lambda)+m_i(\mu)\} \leq card\{j, |\lambda_j| > m_s(\lambda)\}+card\{j, |\mu_j| > m_t(\lambda)\} \leq s+t, \blacksquare$$

we conclude that $m_{s+t}(\lambda + \mu) \leq m_s(\lambda) + m_t(\mu)$.

For $\lambda \in \mathcal{L}$ define $h(\lambda) = \sum\limits_{i=0}^{\infty}(i+1)m_i(\lambda)$.

LEMMA 4. *If $\lambda, \mu \in \mathcal{L}$ then*

$$h(\lambda + \mu) \leq 4\,[h(\lambda) + h(\mu)]\,.$$

PROOF. We have

$$\begin{aligned} h(\lambda+\mu) &= \sum_{r=0}^{\infty}(2r+1)m_{2r}(\lambda+\mu) + \sum_{r=0}^{\infty}(2r+2)m_{2r+1}(\lambda+\mu) \leq \\ &\leq \sum_{r=0}^{\infty}(2r+1)\,[m_r(\lambda)+m_r(\mu)] + \sum_{r=0}^{\infty}(2r+2)\,[m_r(\lambda)+m_{r+1}(\mu)] \leq \\ &\leq \sum_{r=0}^{\infty}(4r+3)\,[m_r(\lambda)+m_r(\mu)] \leq 4\,[h(\lambda)+h(\mu)]\,. \end{aligned}$$

(continuation of the proof of Theorem 2):

Let $u \in A$, i.e. u can be expressed as

$$u = \alpha c + \sum_{n\in N}\sum_{k\in K}\beta_{nk}X_{nk} + \sum_{d\in\mathcal{D}}\gamma_d a_d + \sum_{d\in\mathcal{D}}\sum_{n\in N}\sum_{k\in K_{dn}}\delta_{dnk}y_{dnk} \tag{6}$$

where $\alpha, \beta_{nk}, \gamma_d, \delta_{dnk}$ are complex numbers such that only a finite number of them is non-zero. For u of form (6) define

$$f(u) = |\alpha| + \sum_{n\in N} h\Big(\{\beta_{nk}\}_{k\in K}\Big) + \sum_{d\in\mathcal{D}}|\gamma_d| + \sum_{d\in\mathcal{D}}\sum_{n\in N}\frac{2}{d_n+1}h\Big(\{\delta_{dnk}\}_{k\in K}\Big)$$

(we put formally $\delta_{dnk} = 0$ for $k \in K - K_{dn}$).

The function $f : A \to< 0, \infty)$ has the following properties:

a) $u \in A, u \neq 0 \Rightarrow f(u) \neq 0$

b) $f(\varepsilon u) = |\varepsilon| f(u)$ for each complex number ε and $u \in A$

c) $f(u + u') \leq 4[f(u) + f(u')]$

d) $f(u, u') \leq 8 f(u) f(u')$.

The first two properties are evident, property c) follows from Lemma 4. To prove d) suppose that $u, u' \in A$ are of form (6) (i.e. $u' = \alpha' c + \sum_n \sum_k \beta'_{nk} x_{nk} + \ldots$). Then

$$\begin{aligned} f(uu') &= f\left(\sum_{d,n} \sum_{k \in K_{dn}} d_n y_{dnk} (\beta_{nk} \gamma'_d + \beta'_{nk} \gamma_d) \right) = \\ &= \sum_{d,n} \frac{2d_n}{d_{n+1}} h\Big(\{\beta_{nk}\gamma'_d + \beta'_{nk}\gamma_d\}_{k \in K_{dn}} \Big) \leq \\ &\leq 8 \sum_{d,n} \Big[|\gamma'_d| h\Big(\{\beta_{nk}\}_{k \in K_{dn}} \Big) + |\gamma_d| h\Big(\{\beta'_{nk}\}_{k \in K_{dn}} \Big) \Big] \leq 8 f(u) f(u'). \end{aligned}$$

Let $V = \{u \in A, f(u) < 1\}$ and $\mathcal{V} = \{tV, t \in (0, \infty)\}$. Then $\mathcal{V}$ satisfies conditions (1) – (5) so A with the topology given by $\mathcal{V}$ is a topological algebra.

Let $M \subset A$ be the subspace generated by the elements of form $c - \frac{1}{d_n} \sum\limits_{k \in K_{dn}} y_{dnk}$, $d \in \mathcal{D}, n \in N$. Clearly M is a two-sided ideal in A.

Let $u \in A$ be of form (6). If $\beta_{nk} \neq 0$ for some $n \in N$, $k \in K$ or $\gamma_d \neq 0$ for some $d \in \mathcal{D}$ then $(u + tV) \cap M = \phi$ for a suitable $\epsilon > 0$, so $u \notin \bar{M}$. Similarly, $u \notin \bar{M}$ if $\delta_{dnk} \neq \delta_{dnk'}$ for some d, n, k, k'. Finally, if $u = \alpha c - \sum\limits_{d,n} \sum\limits_{k \in K_{dn}} \varepsilon_{dn} y_{dnk}$ and $\alpha \neq \sum_{d,n} d_n \varepsilon_{dn}$ we have $u \notin \bar{M}$ as $f(\frac{1}{d_n} \sum\limits_{k \in K_{dn}} y_{dnk}) = 1 \quad (d \in \mathcal{D}, n \in N)$.

Hence M is a closed ideal in A and $c \notin M$. Let $B = A/M$ and let $\pi : A \longrightarrow B$ be the canonical homomorphism. Then B is a topological algebra and $\pi(c) \neq 0$.

We prove that B is not topologizable as a locally convex algebra. Suppose on the contrary that there exists a system $\mathcal{W}$ of convex zero-neighbourhoods in B satisfying (1) - (5). We shall need the following lemma:

LEMMA 5. *For every $W \in \mathcal{W}$ there exists $d \in \mathcal{D}$ and $n \in N$ such that $\pi(y_{dnk}) \in W$ for every $k \in K_{dn}$.*

PROOF. Let $W \in \mathcal{W}$. Suppose on the contrary that for every $d \in \mathcal{D}$ and $n \in N$ there exists $k \in K_{dn}$ with $\pi(y_{dnk}) \notin W$. Let $W' \in \mathcal{W}$ satisfy $W'W' \subset W$. For $n \in N$ and $k \in K$ choose $s_{nk} > 0$ such that $\pi(x_{nk}) \in s_{nk} W'$.

Choose $d = \{d_{nk}\}_{n \in N} \in \mathcal{D}$ such that $d_{nk} > n s_{nk}$ $(n \in N, k \in K)$. Then $a_d \in rW'$ for some $r > 0$.

We supposed that for every $n \in N$ there exists $k \in K_{dn}$ such that $\pi(y_{dnk}) \notin W$. On the other hand we have

$$\pi(y_{dnk}) = \frac{1}{d_n} \pi(x_{nk}) \pi(a_d) \in \frac{1}{d_n} s_{nk} W' r W' \subset \frac{s_{nk} r}{d_n} W.$$

So $s_{nk} r / d_n > 1$, $\;r > d_n / s_{nk} > n$ for every $n \in N$ which is a contradiction.

(continuation of the proof of Theorem 2):
Let $W \in \mathcal{W}$. Let $d \in \mathcal{D}$ and $n \in N$ be given by Lemma 5. Then

$$\pi(c) = \frac{1}{d_n} \sum_{k \in K_{dn}} \pi(y_{dnk})$$

and $\pi(y_{dnk}) \in W$ for every $k \in K_{dn}$. Since W is convex and $card K_{dn} = d_n$ we have $\pi(c) \in W$ for every $W \in \mathcal{W}$, a contradiction with condition (1).

Problem: Is it possible to construct separable algebras with properties of Theorem 1 (Theorem 2)?

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