# Rotations of hypercyclic and supercyclic operators

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**Abstract**

Let $T \in B(X)$ be a hypercyclic operator and $\lambda$ a complex number of modulus 1. Then $\lambda T$ is hypercyclic and has the same set of hypercyclic vectors as $T$. A version of this results gives for a wide class of supercyclic operators that $x \in X$ is supercyclic for $T$ if and only if the set $\{tT^n x : t > 0, n = 0, 1, \dots\}$ is dense in $X$. This gives answers to several questions studied in literature.

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Let $T$ be a bounded linear operator acting on a separable complex Banach space $X$. A vector $x \in X$ is called hypercyclic for $T$ if the set $\{T^n x : n = 0, 1, \dots\}$ is dense in $X$. The operator $T$ is called hypercyclic if there exists a vector hypercyclic for $T$.

By [A], if $T$ is hypercyclic then $T^n$ is hypercyclic for each $n$. Moreover, $T$ and $T^n$ have the same sets of hypercyclic vectors. Consequently, if $\lambda = e^{2\pi i r}$ where $r$ is a rational number, then $T$ and $\lambda T$ have the same sets of hypercyclic vectors.

Let $T \in B(X)$ be a hypercyclic operator. It is easy to show that the set of all $\lambda \in \mathbb{C}$, $|\lambda| = 1$ such that $\lambda T$ is hypercyclic is a $G_\delta$ dense subset of the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Indeed, it is dense by the above mentioned result of [A], and $G_\delta$ by the following observation:

**Observation.** The set of all hypercyclic operators is a $G_\delta$ subset of $B(X)$.

**Proof.** Let $(U_j)$ be a countable base of open subsets of $X$. By [GS], $T \in B(X)$ is hypercyclic if and only if for all $j, k$ there is an $n$ such that $T^n U_j \cap U_k \neq \emptyset$. Clearly, the set $M_{j,k}$ of all operators $T \in B(X)$ such that $T^n U_j \cap U_k \neq \emptyset$ for some $n$ is an open subset of $B(X)$. Thus the set of all hypercyclic operators is $G_\delta$, since it is equal to the intersection $\bigcap_{j,k} M_{j,k}$.

The aim of this paper is to show that in fact $\lambda T$ is hypercyclic for all $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. This gives a positive answer to a question that has been posed and studied simultaneously by several mathematicians. Moreover, $T$ and $\lambda T$ have the same sets of hypercyclic vectors. This gives also a positive answer to a question of Salas posed in [S], 6.5.

Our main result has important consequences also in the supercyclicity setting. An operator $T \in B(X)$ is called supercyclic if there is a vector $x \in X$ such that the set $\{\nu T^n x : \nu \in \mathbb{C}, n = 0, 1, \ldots\}$ is dense in $X$; the vector $x$ with this property is called supercyclic for $T$. The concept of supercyclic vectors was introduced by Hilden and Wallen in [HW]. They stand between hypercyclic and cyclic vectors (a vector $x$ is called cyclic for $T \in B(X)$ if the powers $T^n x$ span a dense subspace of $X$).

In a similar way it is possible to define the (apparently different) concept of $\mathbb{R}^+$-supercyclicity. A vector $x$ is called $\mathbb{R}^+$-supercyclic for $T$ if the set $\{tT^n x : t > 0, n = 0, 1, \ldots\}$ is dense, see [BBP].

In strong contrast with the class of hypercyclic operators, D. A. Herrero discovered in [H] that there are two types of supercyclic operators $T$:

(i) operators satisfying $\sigma_p(T^*) = \emptyset$ where $\sigma_p$ denotes the point spectrum;

(ii) operators with $\sigma_p(T^*) = \{\alpha\}$ for some non-zero $\alpha \in \mathbb{C}$; in this case we have $\dim \ker(T^* - \alpha) = 1 = \dim \ker(T^* - \alpha)^k$ for all $k \geq 1$.

Note that operators of class (ii) have a nontrivial invariant subspace, and so they are less interesting (at least from the point of view of invariant subspace problem and related questions).

We show that for all operators in class (i) the concept of supercyclicity is equivalent to the concept of $\mathbb{R}^+$-supercyclicity. This gives a positive answer to a problem raised in [BBP]. Moreover, the $\mathbb{R}^+$ -supercyclicity is easier to handle in practical applications than the supercyclicity, see e.g., [M].

As it was pointed out in [BBP], this equivalence is not true in general for operators of type (ii).

**Theorem 1.** Let $\mathcal{M} \subset B(X)$ be a semigroup of operators and let $x \in X$ satisfy that the set $\{\mu Sx : S \in \mathcal{M}, \mu \in \mathbb{C}, |\mu| = 1\}$ is dense in $X$. Suppose that there is an operator $T \in B(X)$ with $\sigma_p(T^*) = \emptyset$ satisfying $TS = ST$ for each $S \in \mathcal{M}$. Then the set $\{Sx : S \in \mathcal{M}\}$ is dense.

**Proof.** For each $u \in X$ set $M_u = \{Su : S \in \mathcal{M}\}^-$. For $u, v \in X$ set

$$F_{u,v} = \{\mu \in \mathbb{C} : |\mu| = 1, \mu v \in M_u\}.$$

Clearly $F_{u,v}$ is a closed subset of the unit circle $\mathbb{T} = \{\mu \in \mathbb{C} : |\mu| = 1\}$. Let $X_0$ be the set of all vectors $u \in X$ such that $\{\mu Su : S \in \mathcal{M}, \mu \in \mathbb{T}\}^- = X$.

The proof will be done in several steps:

**(a)** Let $u \in X_0$. Then $F_{u,v} \neq \emptyset$ for all $v \in X$.

**Proof.** Since the set $\{\mu Su : S \in \mathcal{M}, \mu \in \mathbb{T}\}$ is dense in $X$, there are sequences $(S_n) \subset \mathcal{M}$ and $(\mu_n) \subset \mathbb{T}$ such that $\mu_n S_n u \to v$. Passing to a subsequence if necessary, we can assume that $(\mu_n)$ is convergent, $\mu_n \to \mu$ for some $\mu \in \mathbb{T}$. Then

$$\|S_n u - \mu^{-1} v\| \leq \|S_n u - \mu_n^{-1} v\| + \|(\mu_n^{-1} - \mu^{-1}) v\| \to 0.$$

Thus $\mu^{-1} \in F_{u,v}$.

**(b)** Let $u, v, w \in X$, $\mu_1 \in F_{u,v}$ and $\mu_2 \in F_{v,w}$. Then $\mu_1\mu_2 \in F_{u,w}$.

**Proof.** Let $\varepsilon > 0$. There exists $S_1 \in \mathcal{M}$ such that $\|S_1 v - \mu_2 w\| < \varepsilon/2$ and $S_2 \in \mathcal{M}$ such that $\|S_2 u - \mu_1 v\| < \frac{\varepsilon}{2\|S_1\|}$. Then

$$\big\|S_1 S_2 u - \mu_1\mu_2 w\big\| \le \big\|S_1(S_2 u - \mu_1 v)\big\| + \big\|\mu_1(S_1 v - \mu_2 w)\big\| < \varepsilon.$$

Hence $\mu_1\mu_2 \in F_{u,w}$.

Fix now $x \in X_0$. By (a) and (b), $F_{x,x}$ is a non-empty closed subsemigroup of the unit circle $\mathbb{T}$.

Suppose first that $F_{x,x} = \mathbb{T}$. Then (a) and (b) imply that $F_{x,y} = \mathbb{T}$ for each $y \in X$. Thus $M_x = X$, and so the set $\{Sx : S \in \mathcal{M}\}$ is dense in $X$.

In the following we shall assume that $F_{x,x} \ne \mathbb{T}$. We show that this assumption leads to a contradiction.

**(c)** There exists $k \in \mathbb{N}$ such that $F_{x,x} = \{e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}$.

**Proof.** Let $s = \inf\{t > 0 : e^{2\pi i t} \in F_{x,x}\}$. Clearly $s > 0$ since otherwise $F_{x,x}$ would be dense in $\mathbb{T}$. Thus $e^{2\pi i s} \in F_{x,x}$. Let $k = \min\{n \in \mathbb{N} : ns \ge 1\}$. If $ks > 1$ then $e^{2\pi i(ks-1)} \in F_{x,x}$ and $0 < ks - 1 < s$, a contradiction with the definition of $s$. Hence $ks = 1$ and

$$F_{x,x} \supset \{e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}.$$

If there is an $\mu \in F_{x,x} \setminus \{e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}$ then $\mu = e^{2\pi i t}$ and $j_0/k < t < (j_0+1)/k$ for some $j_0$, $0 \le j_0 \le k-1$. Then $\mu \cdot e^{-2\pi i j_0/k} = e^{2\pi i(t - j_0/k)} \in F_{x,x}$ where $0 < t - j_0/k < 1/k = s$, which is again a contradiction with the definition of $s$.

Thus $F_{x,x} = \{e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}$.

**(d)** Let $y \in X_0$. Then there exists $\mu_y \in \mathbb{T}$ such that $F_{x,y} = \{\mu_y e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}$.

**Proof.** By (a), there are $\mu_y \in F_{x,y}$ and $\alpha \in F_{y,x}$. By (b), we have $\mu_y F_{x,x} \subset F_{x,y}$ and $\alpha F_{x,y} \subset F_{x,x}$. In particular, $\operatorname{card} F_{x,y} = \operatorname{card} F_{x,x}$ and $F_{x,y} = \mu_y F_{x,x} = \{\mu_y e^{2\pi i j/k} : j = 0, 1, \ldots, k-1\}$.

**(e)** $(T - z)x \in X_0$ for all $z \in \mathbb{C}$.

**Proof.** Since $\sigma_p(T^*) = \emptyset$, we have $\overline{(T-z)X} = X$. Thus

$$\{\mu S(T-z)x : S \in \mathcal{M}, \mu \in \mathbb{T}\}^- \supset (T-z)\{\mu Sx : S \in \mathcal{M}, \mu \in \mathbb{T}\}^- = (T-z)X,$$

which is a dense subset of $X$.

For each non-zero vector $y$ in the subspace generated by $x$ and $Tx$ define $f(y) = \mu^k$ where $\mu$ is any element of $F_{x,y}$. Clearly the function $f$ is well-defined by (d).

**(f)** $f$ is a continuous function.

**Proof.** Suppose on the contrary that there exist non-zero vectors $u_n, u \in \bigvee\{x, Tx\}$ such that $u_n \to u$ and $f(u_n) \not\to f(u)$. Without loss of generality we can assume that the sequence $(f(u_n))$ converges to some $\alpha \in \mathbb{T}$, $\alpha \neq f(u)$. Let $\mu_n \in F_{x,u_n}$. Passing to a subsequence if necessary we can assume that $\mu_n \to \mu$ for some $\mu \in \mathbb{T}$. Then $\mu_n u_n \in M_x$ and $\mu_n u_n \to \mu u$. Since $M_x$ is closed, we have $\mu u \in M_x$ and $\mu \in F_{x,u}$. Hence $\alpha = \lim f(u_n) = \lim \mu_n^k = \mu^k = f(u)$, a contradiction. Hence $f$ is continuous on the set $\bigvee\{x, Tx\} \setminus \{0\}$.

**Proof of Theorem 1.** We first show that the vectors $x$ and $Tx$ are linearly independent. Suppose on the contrary that $Tx = \alpha x$ for some $\alpha \in \mathbb{C}$. Then $S \ker(T - \alpha) \subset \ker(T - \alpha)$ for all $S \in \mathcal{M}$ and $X = \{\mu S x : S \in \mathcal{M}, \mu \in \mathbb{T}\}^- \subset \ker(T - \alpha)$. Thus $T$ is a scalar multiple of the identity, which contradicts to the assumption that $\sigma_p(T^*) = \emptyset$.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$ denote the closed unit disc. Let $g : \mathbb{D} \to \mathbb{T}$ be the function defined by $g(z) = f\big(zx + (1-|z|)Tx\big)$. Clearly $g$ is continuous. For all $z$ satisfying $|z| = 1$ we have $F_{x,zx} = z^{-1}F_{x,x}$ and $g(z) = f(zx) = z^{-k}f(x) = z^{-k}$. It is well-known that such a function $g$ cannot exist, see e.g. [R], Theorem 10.40. Indeed, the function $g$ would provide a homotopy between the constant path $\gamma_1 : \langle 0, 2\pi\rangle \to \mathbb{T}$ defined by $\gamma_1(t) = g(0)$ and the path $\gamma_2 : \langle 0, 2\pi\rangle \to \mathbb{T}$ given by $\gamma_2(t) = g(e^{it}) = e^{-kit}$, which has the winding number $-k$.

Thus $F_{x,x} = \mathbb{T}$ and the set $\{Sx : S \in \mathcal{M}\}$ is dense in $X$. □

**Corollary 2.** Let $T \in B(X)$. Then $x \in X$ is hypercyclic for $T$ if and only if the set $\{\mu T^n x : \mu \in \mathbb{T}, n = 0, 1 \dots\}$ is dense in $X$.

**Proof.** One implication is trivial. To show the second implication, let $x \in X$ satisfy $\{\mu T^n x : \mu \in \mathbb{T}, n = 0, 1, \dots\}^- = X$. Let $\mathcal{M} = \{T^n : n = 0, 1, \dots\}$. It is sufficient to show that $\sigma_p(T^*) = \emptyset$.

Suppose on the contrary that $\alpha \in \sigma_p(T^*)$. Let $x^* \in X^*$ be the corresponding eigenvector, $T^* x^* = \alpha x^*$. We have

$$\begin{aligned} \mathbb{C} &= \{\langle \mu T^n x, x^*\rangle : \mu \in \mathbb{T}, n = 0, 1, \dots\}^- = \{\langle \mu x, \alpha^n x^*\rangle : \mu \in \mathbb{T}, n = 0, 1, \dots\}^- \\ &= \langle x, x^*\rangle \cdot \{\mu \alpha^n : \mu \in \mathbb{T}, n = 0, 1, \dots\}^-. \end{aligned}$$

If $|\alpha| \leq 1$ or $\langle x, x^*\rangle = 0$ then the last set is bounded and therefore non-dense in $\mathbb{C}$. If $|\alpha| > 1$ and $\langle x, x^*\rangle \neq 0$ then the last set is bounded below, and therefore non-dense in $\mathbb{C}$, either. Hence $\sigma_p(T^*) = \emptyset$.

The statement now follows from Theorem 1. □

**Corollary 3.** Let $T \in B(X)$ be hypercyclic and $\lambda \in \mathbb{T}$. Then the operator $\lambda T$ is hypercyclic and has the same set of hypercyclic vectors as $T$.

The last corollary has a reformulation for supercyclic operators.

**Corollary 4.** Let $T \in B(X)$ satisfy $\sigma_p(T^*) = \emptyset$ and let $x \in X$. Then $x$ is supercyclic for $T$ if and only if the set $\{tT^n x : t > 0, n = 0, 1, \dots\}$ is dense in $X$.

**Proof.** Let $\mathcal{M} = \{t \cdot T^n : t > 0 : n = 0, 1, \dots\}$. The statement follows now from Theorem 1. □

**Concluding remarks**

**(i)** The following problem is open (cf. [BP]).

**Problem 5.** Let $T \in B(X)$ be a hypercyclic operator. Let $(n_k)$ be an increasing sequence of positive integers such that $\sup_k(n_{k+1} - n_k) < \infty$. Is the sequence $(T^{n_k})$ hypercyclic, i.e., is there a vector $x \in X$ such that the set $\{T^{n_k}x : k \in \mathbb{N}\}$ is dense in $X$?

In fact, Problem 5 is a generalization of Corollary 3. Indeed, suppose that Problem 5 has a positive answer. Let $T \in B(X)$ be hypercyclic and let $\lambda \in \mathbb{C}$, $|\lambda| = 1$. For each $s \in \mathbb{N}$ consider the set $\{n \in \mathbb{N} : |\lambda^n - 1| < 1/s\}$. It is easy to see that this set forms an increasing sequence $(n_k)$ satisfying $\sup_k(n_{k+1} - n_k) < \infty$. By the assumption, the sequence $(T^{n_k})$ is hypercyclic. Moreover, it is easy to show that the set $M_s$ of all vectors hypercyclic for the sequence $(T^{n_k})$ is $G_\delta$ dense.

By the Baire category theorem, the intersection $\bigcap_s M_s$ is also a $G_\delta$ dense subset of $X$. Let $x \in \bigcap_s M_s$. We show that $x$ is hypercyclic for $\lambda T$. Let $y \in X$ and $\varepsilon > 0$. Let $s \in \mathbb{N}$, $1/s < \varepsilon$. Then there is an $n$ such that $\|T^n x - y\| < \varepsilon$ and $|\lambda^n - 1| < s^{-1}$. Thus

$$\|(\lambda T)^n x - y\| \le \|\lambda^n (T^n x - y)\| + \|(\lambda^n - 1)y\| < \varepsilon + \varepsilon\|y\|.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $x$ is hypercyclic for $\lambda T$.

Although Problem 5 is open, it is known that in general the set of all vectors hypercyclic for a sequence $(T^{n_k})$ where $\sup_k(n_{k+1} - n_k) < \infty$ cannot be equal to the set of all vectors hypercyclic for $T$. Moreover, for each $x \in X$ hypercyclic for $T$ there exists a sequence $(n_k)$ satisfying $\sup(n_{k+1} - n_k) < \infty$ such that $x$ is not hypercyclic for the sequence $(T^{n_k})$, see [MS], Proposition 2.5.

**(ii)** It is easy to construct a hypercyclic operator $T$ such that $\lambda T$ is not hypercyclic whenever $|\lambda| \neq 1$. Let $H$ be a separable Hilbert space with an orthonormal basis $\{e_i\}$ $(i \in \mathbb{Z})$. Let $T \in B(H)$ be the weighted shift defined by $Te_i = w_i e_{i+1}$ $(i \in \mathbb{Z})$ where $w_i = \frac{i+1}{i+2}$ $(i \ge 0)$ and $w_i = \frac{i-1}{i}$ $(i < 0)$. Then $\|T^n\| = \sup_i(w_i \cdots w_{i+n-1}) = n+1$, and the spectral radius $r(T) = \lim \|T^n\|^{1/n}$ is equal to 1. Moreover, $T$ is invertible and $T^{-1}$ is the weighted shift given by $T^{-1}e_i = w_{i-1}^{-1} e_{i-1}$; in fact $T^{-1}$ is unitarily equivalent to $T$.

Let $H_0$ be the dense subset of $H$ formed by all finite linear combinations of the vectors $e_i$ $(i \in \mathbb{Z})$. It is easy to see that $T^n x \to 0$ and $T^{-n}x \to 0$ for all $x \in H_0$. By the hypercyclicity criterion, see e.g. [K], [GS], $T$ is hypercyclic.

If $\lambda \in \mathbb{C}$, $|\lambda| < 1$ then $r(\lambda T) < 1$ and $\lambda T$ cannot be hypercyclic. Similarly, if $|\lambda| > 1$ then $r((\lambda T)^{-1}) < 1$ and $(\lambda T)^{-1}$ is not hypercyclic. Hence $\lambda T$ is not hypercyclic either, see [K], [GS].

**(iii)** An example of a supercyclic operator $T \in B(X)$ with $\sigma_p(T^*) \neq \emptyset$ that is not $\mathbb{R}^+$-supercyclic was given in [BBP]. In fact, it is easy to see that if $\sigma_p(T^*) =$

$\{s \cdot e^{2\pi i r}\}$ where $s > 0$ and $r$ is a rational number, then $T$ cannot be $\mathbb{R}^+$-supercyclic.

This leads to the following question:

**Problem 6.** Let $T \in B(X)$ be a supercyclic operator such that $\sigma_p(T^*) = \{s \cdot e^{2\pi i m}\}$ where $s > 0$ and $m$ is irrational. Is then the operator $T$ $\mathbb{R}^+$-supercyclic?

Clearly, the $\mathbb{R}^+$-supercyclicity does not depend on the positive number $s$, so the question can be formulated only for operators $T$ with $\sigma_p(T^*) = \{e^{2\pi i m}\}$, where $m$ is irrational.

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