On the axiomatic theory of spectrum

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Abstract. There are a number of spectra studied in literature which do not fit into the axiomatic theory of Żelazko. This paper is an attempt to give an axiomatic theory for these spectra which, apart from the usual types of spectra, like one-sided, approximate point or essential spectra, include also the local spectra, the Browder spectrum and various versions of the Apostol spectrum (studied under various names, e.g. regular, semi-regular or essentially semi-regular).

I. Basic properties of regularities

The axiomatic theory of spectrum was introduced by W. Żelazko [21], see also Słodkowski and Żelazko [17]. He gave a classification of various types of spectra defined for commuting n-tuples of elements of a Banach algebra. The most important notion is that of subspectrum.

All algebras in this paper are complex and unital. Denote by $\mathrm{Inv}(\mathcal{A})$ the set of all invertible elements in a Banach algebra A and by $\sigma(a) = \{\lambda \in \mathbb{C}, a - \lambda \notin \mathrm{Inv}(\mathcal{A})\}$ the ordinary spectrum of an element $a \in \mathcal{A}$. The spectral radius of $a \in \mathcal{A}$ will be denoted by $r(a)$.

Definition 1.1. Let $\mathcal{A}$ be a Banach algebra. A subspectrum $\widetilde{\sigma}$ in $\mathcal{A}$ is a mapping which assigns to every n-tuple $(a_1, \dots, a_n)$ of mutually commuting elements of $\mathcal{A}$ a non-empty compact subset $\widetilde{\sigma}(a_1, \dots, a_n) \subset \mathbb{C}^n$ such that
(1) $\widetilde{\sigma}(a_1, \dots, a_n) \subset \sigma(a_1) \times \cdots \times \sigma(a_n)$,
(2) $\widetilde{\sigma}(p(a_1, \dots, a_n)) = p(\widetilde{\sigma}(a_1, \dots, a_n))$ for every commuting $a_1, \dots, a_n \in \mathcal{A}$ and every polynomial mapping $p = (p_1, \dots, p_m) : \mathbb{C}^n \to \mathbb{C}^m$.

This notion has proved to be quite useful since it includes for example the left (right) spectrum, the left (right) approximate point spectrum, the Harte (= the union of the left and right) spectrum, the Taylor spectrum and various essential spectra.

However, there are also many examples of spectrum, usually defined only for single elements of $\mathcal{A}$, which are not covered by the axiomatic theory of Żelazko. The aim of this paper is to give an axiomatic description of such spectra.

Instead of describing a spectrum, it is possible to describe equivalently the set of regular elements.

Definition 1.2. Let $\mathcal{A}$ be a Banach algebra. A non-empty subset R of $\mathcal{A}$ is called a regularity if
(1) if $a \in \mathcal{A}$ and $n \in \mathbb{N}$ then $a \in \mathcal{A} \Leftrightarrow a^n \in \mathcal{A}$,
(2) if a, b, c, d are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$.

Proposition 1.3. *Let R be a regularity in a Banach algebra $\mathcal{A}$.*

(1) *If $a, b \in \mathcal{A}, ab = ba$ and $a \in \mathrm{Inv}(\mathcal{A})$ then*

$$ab \in R \Leftrightarrow a \in R \text{ and } b \in R.$$

(2) $\mathrm{Inv}(\mathcal{A}) \subset R$.

Proof. (1) We have $a \cdot a^{-1} + b \cdot 0 = 1_{\mathcal{A}}$, so that it is possible to apply property (2) of Definition 1.2.

(2) Let $b \in R$. By (1) for $a = 1_{\mathcal{A}}$ we have $1_{\mathcal{A}} \in R$.
Let $c \in \mathrm{Inv}(\mathcal{A})$. Then $c \cdot c^{-1} = 1_{\mathcal{A}} \in R$, so that $c \in R$ by (1).

A regularity $R \subset \mathcal{A}$ defines a mapping $\widetilde{\sigma}_R$ from $\mathcal{A}$ into subsets of $\mathbb{C}$ by

$$\widetilde{\sigma}_R(a) = \{\lambda \in \mathbb{C} : a - \lambda \notin R\} \qquad (a \in \mathcal{A}).$$

This mapping will be called the spectrum corresponding to the regularity R. When no confusion can arise we will write simply $\widetilde{\sigma}(a)$.

Remarks:

(1) In general $\widetilde{\sigma}_R(a)$ is neither closed nor non-empty. Proposition 1.3, (2) implies that $\widetilde{\sigma}(a)$ is bounded, $\widetilde{\sigma}_R(a) \subset \sigma(a)$.

(2) If $ab = ba, b \in \mathrm{Inv}(\mathcal{A})$ then $a \in R \Leftrightarrow ab \in R$. In particular, if $a \in R$ and λ is a non-zero complex number then $\lambda a \in R$.

(3) Consider the following property

(P1) $\quad ab \in R \Leftrightarrow a \in R$ and $b \in R \quad$ for all commuting elements $a, b \in \mathcal{A}$.

Clearly a non-empty subset R of $\mathcal{A}$ satisfying (P1) is a regularity.

(4) Let $\widetilde{\sigma}$ be a subspectrum. It is an easy observation (see [12]) that the set R defined by $R = \{a \in \mathcal{A} : 0 \notin \widetilde{\sigma}(a)\}$ is a regularity.

(5) Let $(R_\alpha)_\alpha$ be a family of regularities. Then $R = \cap_\alpha R_\alpha$ is a regularity. The corresponding spectra satisfy

$$\widetilde{\sigma}_R(a) = \bigcup_\alpha \widetilde{\sigma}_{R_\alpha}(a).$$

Examples:

Let $\mathcal{A}$ be a Banach algebra. The following subsets of $\mathcal{A}$ are regularities:

(1) $R_1 = \mathcal{A}$; the corresponding spectrum is empty for every $a \in \mathcal{A}$.

(2) $R_2 = \mathrm{Inv}(\mathcal{A})$; this gives the ordinary spectrum $\sigma(a)$.

(3) Let $R_3(R_4)$ be the set of all left (right) invertible elements of $\mathcal{A}$. Then the corresponding spectrum is the left (right) spectrum in $\mathcal{A}$.

(4) Let $R_5(R_6)$ be the set of all elements of $\mathcal{A}$ which are not the left (right) topological divisors of zero. The corresponding spectrum is the left (right) approximate point spectrum.

In the algebra $\mathcal{L}(X)$ of all bounded operators in a Banach space X we have:

(5) R_5 is the set of all operators bounded below, R_6 is the set of all surjective operators. The corresponding spectra in this case are usually called the approximate point and the defect spectrum.

(6) Let R_7 be the set of all Fredholm operators in X. This regularity gives the essential spectrum.

(7) Let $R_8(R_9)$ be the set of all upper (lower) semi-Fredholm operators in X. The corresponding spectra are called upper (lower) semi-Fredholm or sometimes left (right) essential approximate point spectrum.

All the sets defined above satisfy (P1) so they are regularities. However, all these examples are rather trivial since it is well-known that the corresponding spectra can be extended to commuting n-tuples of elements so that they become a subspectrum.

Further, more interesting examples of regularities will be given later.

Every spectrum defined by a regularity satisfies the spectral mapping theorem:

Theorem 1.4. *Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. Then*

$$\widetilde{\sigma}(f(a)) = f(\widetilde{\sigma}(a))$$

for every $a \in \mathcal{A}$ and every function f analytic on a neighbourhood of $\sigma(a)$ which is non-constant on each component of its domain of definition.

Proof. It is sufficient to show

$$0 \notin \widetilde{\sigma}(f(a)) \Leftrightarrow 0 \notin f(\widetilde{\sigma}(a)). \tag{1}$$

Since f has only a finite number of zeros $\lambda_1, \dots, \lambda_n$ in $\sigma(a)$, it can be written as $f(z) = (z-\lambda_1)^{k_1} \cdots (z-\lambda_n)^{k_n} \cdot g(z)$, where g is a function analytic on a neighbourhood of $\sigma(a)$ and $g(z) \neq 0$ for $z \in \sigma(a)$. Then $f(a) = (a-\lambda_1)^{k_1} \cdots (a-\lambda_n)^{k_n} \cdot g(a)$ and $g(a)$ is invertible by the spectral mapping theorem for the ordinary spectrum.

Thus (1) is equivalent to

$$f(a) \in R \Leftrightarrow a - \lambda_i \in R \qquad (i = 1, \dots, n). \tag{2}$$

Since $g(a)$ is invertible then, by the second of the previous remarks and by property (1) of Definition 1.2, this is equivalent to

$$(a-\lambda_1)^{k_1} \cdots (a-\lambda_n)^{k_n} \in R \Leftrightarrow (a-\lambda_i)^{k_i} \in R \qquad (i = 1, \dots, n). \tag{3}$$

Since for all relatively prime polynomials p, q there exist polynomials p_1, q_1 such that $pp_1 + qq_1 = 1$, i.e. $p(a)p_1(a) + q(a)q_1(a) = 1_{\mathcal{A}}$, we can apply property (2) of the definition inductively to get (3). This proves the theorem.

We shall see later that the assumption that f is non-constant on each component is really necessary. However, in many cases this can be left out. We give a simple criterion (in the most interesting case of the algebra $\mathcal{L}(X)$) which is usually easy to verify.

Let R be a regularity in $\mathcal{L}(X)$ and let $X = X_1 \oplus X_2$. Denote $R_1 = \{T_1 \in \mathcal{L}(X_1) : T_1 \oplus I \in R\}$ and $R_2 = \{T_2 \in \mathcal{L}(X_2) : I \oplus T_2 \in R\}$. If $X_i \neq \{0\}$ then R_i is a regularity in $\mathcal{L}(X_i)$ $(i = 1, 2)$. Indeed, to see condition (2) of Definition 1.2 (e.g. for R_1), note that if $A_1C_1 + B_1D_1 = I_{X_1}$ for some commuting $A_1, B_1, C_1, D_1, \in \mathcal{L}(X_1)$ then

$$(A_1 \oplus I)(C_1 \oplus \frac{1}{2}I) + (B_1 \oplus I)(D_1 \oplus \frac{1}{2}I) = I_X.$$

If $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ then

$$T_1 \oplus T_2 \in R \Leftrightarrow T_1 \in R_1 \quad \text{and} \quad T_2 \in R_2.$$

Indeed, this follows from the observation that

$$(T_1 \oplus I)(0 \oplus I) + (I \oplus T_2)(I \oplus 0) = I_X.$$

Denote by $\widetilde{\sigma}_i$ the spectrum corresponding to R_i $(i = 1, 2)$.

Theorem 1.5. *Let X be a Banach space, let R be a regularity in $\mathcal{L}(X)$ and let $\widetilde{\sigma}$ be the corresponding spectrum. Suppose that for all pairs of complementary subspaces X_1, X_2, $X = X_1 \oplus X_2$ such that $R_1 = \{S_1 \in \mathcal{L}(X_1) : S_1 \oplus I \in R\} \neq \mathcal{L}(X_1)$ the corresponding specrum $\widetilde{\sigma}_1(T_1) = \{\lambda : (T_1 - \lambda) \oplus I \notin R\}$ is non-empty for every $T_1 \in \mathcal{L}(X_1)$.*

Then $\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T))$ for every $T \in \mathcal{L}(X)$ and every function f analytic on a neighbourhood of $\sigma(T)$.

Proof. It is sufficient to show that

$$0 \notin \widetilde{\sigma}(f(T)) \Leftrightarrow 0 \notin f(\widetilde{\sigma}(T)).$$

Let U_1, U_2 be open subsets of the domain of definition of f such that $U_1 \cup U_2 \supset \sigma(T)$, $f|U_1$ is identically 0 and $f_2 = f|U_2$ can be written as $f_2(z) = p(z)g(z)$ where p is a polynomial and g has no zeros in $U_2 \cap \sigma(T)$. Let X_1, X_2 be the spectral subspaces corresponding to U_1 and U_2, i.e. $X = X_1 \oplus X_2$, $T = T_1 \oplus T_2$ where $T_i = T|X_i$ and $\sigma(T_i) \subset U_i$ $(i = 1, 2)$.

Let $R_1 \subset \mathcal{L}(X_1)$ and $R_2 \subset \mathcal{L}(X_2)$ be the regularities defined above and let $\widetilde{\sigma}_1, \widetilde{\sigma}_2$ be the corresponding spectra. Clearly $\widetilde{\sigma}(T) = \widetilde{\sigma}_1(T_1) \cup \widetilde{\sigma}_2(T_2)$.

The following statements are equivalent:
$0 \notin \widetilde{\sigma}(f(T))$,
$f(T) \in R$,
$0 \in R_1$ and $f_2(T_2) \in R_2$,
$R_1 = \mathcal{L}(X_1)$ and $p(T_2) \in R_2$,
$\widetilde{\sigma}_1(T_1) = \emptyset$ and $0 \notin p(\widetilde{\sigma}_2(T_2))$,
$0 \notin f(\widetilde{\sigma}_1(T_1) \cup \widetilde{\sigma}_2(T_2))$,
$0 \notin f(\widetilde{\sigma}(T))$.

We are going to study now the continuity properties of spectra. Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. We consider the following properties of R (or $\widetilde{\sigma}$):

(P2) "Upper semi-continuity of $\widetilde{\sigma}$"
If $a_n, a \in \mathcal{A}$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$ then $\lambda \in \widetilde{\sigma}(a)$.

(P3) "Upper semi-continuity on commuting elements"
If $a_n, a \in \mathcal{A}$, $a_n \to a$, $a_n a = a a_n$ for every n, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$ then $\lambda \in \widetilde{\sigma}(a)$.

(P4) "Continuity on commuting elements"
If $a_n, a \in \mathcal{A}$, $a_n \to a$ and $a_n a = a a_n$ for every n then $\lambda \in \widetilde{\sigma}(a)$ if and only if there exists a sequence $\lambda_n \in \widetilde{\sigma}(a_n)$ such that $\lambda_n \to \lambda$.

Clearly either (P2) or (P4) implies (P3). If $\widetilde{\sigma}$ satisfies (P3) then, by considering a constant sequence $a_n = a$, we have that $\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$.

Proposition 1.6. *Let R be a regularity in a Banach algebra $\mathcal{A}$, let $\widetilde{\sigma}$ be the corresponding spectrum. The following conditions are equivalent:*

(1) *(P2),*

(2) *$\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$ and the function $a \mapsto \widetilde{\sigma}(a)$ is upper semi-continuous,*

(3) *R is an open subset of $\mathcal{A}$.*

Proof. Clearly any condition implies that $\widetilde{\sigma}(a)$ is closed for each $a \in \mathcal{A}$. The equivalence $1 \Leftrightarrow 2$ is well-known (see [2], p.25).

$3 \Rightarrow 1$; Let $a_n, a \in \mathcal{A}$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$. Then $a_n - \lambda_n \notin R$. Since $\mathcal{A} - R$ is closed, we conclude $a - \lambda \notin R$, $\lambda \in \widetilde{\sigma}(a)$.

$1 \Rightarrow 3$: We prove that $\mathcal{A} - R$ is closed. Let $a_n \in \mathcal{A} - R$, $a_n \to a$. Then $0 \in \widetilde{\sigma}(a_n)$ for each n. From (1) we conclude $0 \in \widetilde{\sigma}(a)$, $a \in \mathcal{A} - R$.

Proposition 1.7. *Let R be a regularity in a Banach algebra $\mathcal{A}$ and let $\widetilde{\sigma}$ be the corresponding spectrum. The following conditions are equivalent:*

(1) *(P3),*

(2) *$\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$, and for every $a \in \mathcal{A}$ and a neighbourhood U of $\widetilde{\sigma}(a)$, there exists $\varepsilon > 0$ such that $\widetilde{\sigma}(a+u) \subset U$ whenever $u \in \mathcal{A}$, $au = ua$ and $\|u\| < \varepsilon$.*

(3) *If $a \in R$ then there exists $\varepsilon > 0$ such that $u \in \mathcal{A}$, $ua = au$ and $\|u\| < \varepsilon$ implies $a + u \in R$.*

Proof. Analogous to Proposition 1.6.

Definition. If M, N are bounded subsets of $\mathbb{C}$, we denote by $\delta(M, N)$ the Hausdorf distance of M, N:

$$\delta(M, N) = \max\bigl\{\sup_{z \in M} \operatorname{dist}\{z, N\}, \sup_{w \in N} \operatorname{dist}\{w, M\}\bigr\}.$$

Proposition 1.8. *Let R be a regularity in a Banach algebra $\mathcal{A}$, let $\widetilde{\sigma}$ be the corresponding spectrum.*

(1) *Suppose that for all commuting $a, u \in \mathcal{A}$ with $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$ we have $a + u \in R$. Then $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \le \|a - b\|$ for all commuting $a, b \in \mathcal{A}$.*

(2) *If $\widetilde{\sigma}(a)$ is closed for every $a \in \mathcal{A}$ and $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \le \|a - b\|$ for all commuting $a, b \in \mathcal{A}$ then $\widetilde{\sigma}$ satisfies (P4).*

Proof. (1) Let $a, b \in \mathcal{A}$, $ab = ba$ and let $\lambda \in \widetilde{\sigma}(a)$. We prove $\operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\} \leq \|a - b\|$. This is clear if $\lambda \in \widetilde{\sigma}(b)$. If $\lambda \notin \widetilde{\sigma}(b)$, then

$$\|a-b\| = \|(a-\lambda)-(b-\lambda)\| \geq \inf\{|z| : z \in \widetilde{\sigma}(b-\lambda)\} = \operatorname{dist}\{0, \widetilde{\sigma}(b-\lambda)\} = \operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\}.$$

Thus

$$\sup_{\lambda \in \widetilde{\sigma}(a)} \operatorname{dist}\{\lambda, \widetilde{\sigma}(b)\} \leq \|a - b\|$$

and from the symmetry $\delta(\widetilde{\sigma}(a), \widetilde{\sigma}(b)) \leq \|a - b\|$.

(2) Let $a_n a = a a_n$, $a_n \to a$, $\lambda_n \in \widetilde{\sigma}(a_n)$ and $\lambda_n \to \lambda$. Then, for each n, there exists $\mu_n \in \widetilde{\sigma}(a)$, $|\mu_n - \lambda_n| \leq \|a_n - a\|$. Clearly $\mu_n \to \lambda$, so that $\lambda \in \widetilde{\sigma}(a)$ since $\widetilde{\sigma}(a)$ is closed. This proves the upper semi-continuity.

The lower semi-continuity is straightforward.

All regularities $R_1, \ldots, R_9$ in the examples above are open, therefore they satisfy (P2). In fact they satisfy also (P4).

Theorem 1.9. *Let $\widetilde{\sigma}$ be a subspectrum in a Banach algebra $\mathcal{A}$. If $a, u \in \mathcal{A}$, $au = ua$ and $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$ then $0 \notin \widetilde{\sigma}(a + u)$. Consequently, $\widetilde{\sigma}$ (considered for single elements of $\mathcal{A}$) satisfies (P4).*

Proof. Let $a, u \in \mathcal{A}$, $au = ua$ and $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\}$. Consider $\widetilde{\sigma}$ restricted to the commutative Banach algebra $\mathcal{A}_0$ generated by a, u and $1_{\mathcal{A}}$. By [21], Theorem 5.3 there exists a compact subset K of the maximal ideal space $\mathcal{M}(\mathcal{A}_0)$ such that $\widetilde{\sigma}(c) = \{f(c) : f \in K\}$ for every $c \in \mathcal{A}_0$. In particular $\widetilde{\sigma}(a) = \{f(a) : f \in K\}$ and $\widetilde{\sigma}(a + u) = \{f(a + u) : f \in K\}$. For $f \in K$ we have

$$|f(a + u)| = |f(a) + f(u)| \geq |f(a)| - \|u\| \geq \inf\{|z| : z \in \widetilde{\sigma}(a)\} - \|u\| > 0.$$

Thus $0 \notin \widetilde{\sigma}(a + u)$. By 1.8, $\widetilde{\sigma}$ satisfies (P4).

Remark. Frequently, a spectrum $\widetilde{\sigma}$ is defined only for single elements of $\mathcal{A}$ and we would like to extend it to commutative n-tuples of $\mathcal{A}$ so that $\widetilde{\sigma}$ becomes a subspectrum. A necessary condition for it is (P1), see [12]. Property (P4) (or more precisely, $au = ua$, $\|u\| < \inf\{|z| : z \in \widetilde{\sigma}(a)\} \Rightarrow 0 \notin \widetilde{\sigma}(a + u)$) gives another necessary condition.

Yet another necessary condition is: if $a, u \in \mathcal{A}$, $au = ua$ and $\sigma(u) = \{0\}$ then $\widetilde{\sigma}(a + u) = \widetilde{\sigma}(a)$.

It is an open problem to give some sufficient conditions.

The upper semi-continuity on commuting elements enables to weaken the axioms of regularity.

Theorem 1.10. *Let R be a non-empty subset of a Banach algebra $\mathcal{A}$ satisfying*

(1) *if $a \in R$ and $n \in \mathbb{N}$ then $a^n \in R$,*

(2) *if a,b,c,d, are mutually commuting elements of $\mathcal{A}$ and $ac + bd = 1_{\mathcal{A}}$, then $ab \in R \Leftrightarrow a \in R$ and $b \in R$,*

(3) *R satisfies (P3).*

Then R is a regularity.

Proof. It is sufficient to show $a^n \in R \Rightarrow a \in R \quad (n \geq 2)$. By (3), $a^n - \mu a = a(a^{n-1} - \mu) \in R$ for some non-zero complex number μ. Since

$$(a^{n-1} - \mu) \cdot (-\mu^{-1}) + a(\mu^{-1} a^{n-2}) = 1_{\mathcal{A}},$$

we have $a \in R$ by (2).

Theorem 1.11. *Suppose R is a regularity in a Banach algebra $\mathcal{A}$ such that the corresponding spectrum $\widetilde{\sigma}$ satisfies $\max\{|\lambda| : \lambda \in \widetilde{\sigma}(a)\} = r(a)$ for every $a \in \mathcal{A}$. Then $\partial\sigma(a) \subset \overline{\widetilde{\sigma}(a)} \qquad (a \in \mathcal{A})$.*

Proof. Suppose on the contrary $\lambda_0 \in \partial\sigma(a)$ and there exists $\varepsilon > 0$ such that $\{z : |\lambda_0 - z| < \varepsilon\} \cap \widetilde{\sigma}(a) = \emptyset$. Choose $\lambda_1 \in \mathbb{C} - \sigma(a)$, $|\lambda_1 - \lambda_0| < \varepsilon/2$. Consider the function $f(z) = (\lambda_1 - z)^{-1}$. Then

$$\begin{aligned}
\operatorname{dist}\{\lambda_1, \widetilde{\sigma}(a)\}^{-1} &= \max\{|f(z)| : z \in \widetilde{\sigma}(a)\} = \max\{|z| : z \in \widetilde{\sigma}(f(a))\} = r(f(a)) \\
&= \max\{|f(z)| : z \in \sigma(a)\} \geq \frac{1}{|\lambda_1 - \lambda_0|} > (\varepsilon/2)^{-1}.
\end{aligned}$$

Thus there exists $\lambda_2 \in \widetilde{\sigma}(a)$ with $|\lambda_2 - \lambda_1| < \varepsilon/2$, i.e. $|\lambda_2 - \lambda_0| < \varepsilon$, a contradiction.

II. Browder and Apostol spectra

Let T be an operator in a Banach space X. Denote by $R(T)$ and $N(T)$ its range and kernel, respectively. In general $N(T) \subset N(T^2) \subset \cdots$ and $R(T) \supset R(T^2) \supset \cdots$. Denote $N^\infty(T) = \bigcup_{n=0}^\infty N(T^n)$ and $R^\infty(T) = \bigcap_{n=0}^\infty R(T^n)$.

Denote by $R_0(X)$ the set of all operators $T \in \mathcal{L}(X)$ such that T is Fredholm and either T is invertible or 0 is an isolated point of $\sigma(T)$.

Theorem 2.1. *$R_0(X)$ is a regularity. Moreover, $R_0(X)$ is an open subset of $\mathcal{L}(X)$, so that the corresponding spectrum (the Browder spectrum) satisfies (P2) (upper semi-continuity).*

Proof. Clearly $T \in R_0(X)$ if and only if there exists a decomposition $X = X_1 \oplus X_2$ such that $TX_i \subset X_i \quad (i = 1, 2)$, $\dim X_1 < \infty$, $\sigma(T|X_1) \subset \{0\}$ and $T|X_2$ is invertible. It is easy to see that $X_1 = N^\infty(T)$ and $X_2 = R^\infty(T)$.

We prove that $R_0(X)$ satisfies (P1). Let $T, S \in \mathcal{L}(X)$, $TS = ST$. If $T, S \in R_0(X)$ then TS is Fredholm and the inclusion $\sigma(TS) \subset \sigma(T) \cdot \sigma(T)$ gives easily $TS \in R_0(X)$.

Conversely, suppose $TS \in R_0(X)$. Then $X = N^\infty(TS) \oplus R^\infty(TS)$, $\dim N^\infty(TS) < \infty$ and $T|R^\infty(TS)$ is invertible.

Let M be the spectral subspace corresponding to all non-zero eigenvalues of the finite dimensional operator $T|N^\infty(TS)$. Then $X = N^\infty(T) \oplus (R^\infty(TS) \oplus M)$ is the required decomposition so that $T \in R_0(X)$.

We show that $R_0(X)$ is open. Let $T \in R_0(X)$. Let $\delta > 0$ satisfy $\{z : |z| < 3\delta\} \cap \sigma(T) \subset \{0\}$. From the upper semi-continuity of the ordinary and the essential spectra there exists $\varepsilon > 0$ such that $\|S\| < \varepsilon$ implies that $T + S$ is Fredholm,

$$\sigma(T + S) \subset \{z : |z| \leq \delta\} \cup \{z : |z| \geq 2\delta\}$$

and $\sigma_e(T+S) \subset \{z : |z| \geq 2\delta\}$. It follows from the properties of the essential spectrum that either $T+S$ is invertible or 0 is an isolated eigenvalue of $T+S$ of finite multiplicity. Thus $T+S \in R_0(X)$ for every $S \in \mathcal{L}(X)$, $\|S\| < \varepsilon$.

Remark. By [3] it is possible to extend the Browder spectrum to a subspectrum defined on commuting n-tuples of operators. Thus $R_0(X)$ satisfies also (P4) by Theorem 1.9.

Let T be an operator from a Banach space X into a Banach space Y. We say that T has a generalized inverse if there exists an operator $S : Y \to X$ such that $TST = T$.

It is well-known that T has a generalized inverse if and only if T has closed range and both $N(T)$ and $R(T)$ are complemented subspaces of X and Y, respectively.

Let M, N be closed subspaces of a Banach space X. We write $M \overset{e}{\subset} N$ if there exists a finite dimensional subspace $F \subset X$ such that $M \subset N + F$. Equivalently, $\dim M/(M \cap N) < \infty$.

Notation. Let X be a Banach space. Denote by

(1) $R_1(X)$ the set of all $T \in \mathcal{L}(X)$ such that $R(T)$ is closed and $N(T) \subset R^\infty(T)$,

(2) $R_2(X)$ the set of all $T \in \mathcal{L}(X)$ such that $R(T)$ is closed and $N(T) \overset{e}{\subset} R^\infty(T)$,

(3) $R_3(X)$ the set of all $T \in \mathcal{L}(X)$ such that $N(T) \subset R^\infty(T)$ and T has a generalized inverse,

(4) $R_4(X)$ the set of all $T \in \mathcal{L}(X)$ such that $N(T) \overset{e}{\subset} R^\infty(T)$ and T has a generalized inverse.

The elements of $R_1(X)$ are called semi-regular operators (see [9]), the elements of $R_2(X)$ essentially semi-regular. Correspondingly, the elements of $R_3(X)$ and $R_4(X)$ will be called regular and essentially regular.

The semi-regular operators in Hilbert spaces were first studied by Apostol [1] (note that in Hilbert spaces semi-regular=regular) and further in [8], [10], [11], [12] and [15]. For essentially semi-regular operators see [12] and [13]. The regular operators were studied in [16] (cf. also [12] and [14]). The essentially regular operators has not been studied yet. We fill this logical gap.

We summarize now the basic properties of semi-regular and essentially semi-regular operators:

Theorem 2.2. (see [8], [10], [11]). *Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:*

(1) $N(T) \subset R^\infty(T)$,

(2) $N^\infty(T) \subset R(T)$,

(3) $N^\infty(T) \subset R^\infty(T)$,

(4) *the function $\lambda \mapsto R(T-\lambda)$ is continuous at $\lambda = 0$ in the gap topology,*

(5) *the function $\lambda \mapsto N(T-\lambda)$ is continuous at $\lambda = 0$ in the gap topology,*

(6) *the function $\lambda \mapsto c(T-\lambda)$ is continuous at $\lambda = 0$, where c is the Kato reduced minimum modulus defined by $c(S) = \inf\{\|Sx\| : \operatorname{dist}\{x, N(S)\} = 1\}$,* see [6],

(7) $\liminf_{\lambda \to 0} c(T-\lambda) > 0$,

(8) *there exist a closed subspace M of X such that $TM = M$ and the operator $\widetilde{T} : X/M \longrightarrow X/M$ induced by T is bounded below. As the subspace M it is possible to take $R^\infty(T)$.*

In fact we are going to use only conditions (1),(2),(3) and (8).

Theorem 2.3. (see [8], [12], [13]). *Let $T \in \mathcal{L}(X)$ be an operator with closed range. The following conditions are equivalent:*

(1) $N(T) \overset{e}{\subset} R^\infty(T)$,

(2) $N^\infty(T) \overset{e}{\subset} R(T)$,

(3) $N^\infty(T) \overset{e}{\subset} R^\infty(T)$,

(4) *there exist subspaces $X_0, X_1 \subset X$ such that $X = X_0 \oplus X_1$, $\dim X_0 < \infty, TX_0 \subset X_0, TX_1 \subset X_1, T|X_0$ is nilpotent and $T|X_1$ is semi-regular (the Kato decomposition),*

(5) *there exist a closed subspace M of X such that $TM = M$ and the operator $\widetilde{T} : X/M \longrightarrow X/M$ induced by T is upper semi-Fredholm $\big(R(\widetilde{T})$ is closed and $\dim N(\widetilde{T}) < \infty\big)$. As M it is possible to take $R^\infty(T)$.*

We prove that $R_i(X)$ $(i = 1, 2, 3, 4)$ are regularities. We shall need several lemmas. Most of them are known but since they are usually stated in a little bit different form and they are scattered in many papers, we give the proofs.

Lemma 2.4. (see [12], Theorem 3.5.) *If $A, B \in \mathcal{L}(X)$, $AB = BA$, $N(AB) \overset{e}{\subset} R^\infty(AB)$ and $R(AB)$ is closed then $R(A)$ and $R(B)$ are closed.*

Proof. There exists a finite dimensional subspace $F \subset X$ such that $N(AB) \subset R(AB) + F$. We prove that $R(A) + F$ is closed. Let $v_j \in X, f_j \in F$ and $Av_j + f_j \to u$. Then $BAv_j + Bf_j \to Bu$ and $Bu \in R(AB) + BF$ since $R(AB) + BF$ is closed. Thus $Bu = ABv + Bf$ for some $v \in X$ and $f \in F$ so that

$$Av + f - u \in N(B) \subset N(AB) \subset R(AB) + F \subset R(A) + F.$$

Hence $u \in R(A) + F$ and $R(A) + F$ is closed. By a lemma of Neubauer (see [12]), $R(A)$ is closed.

Lemma 2.5. (see [12], Lemma 1.7.) *If $R(A)$ is closed and $N(A) \overset{e}{\subset} R^\infty(A)$ then $R(A^n)$ is closed for every n.*

Proof. Let F be a finite dimensional subspace of $N(A)$ such that $N(A) \subset R^\infty(A)+F$. We prove by induction on n that $R(A^n)$ is closed. Suppose $n \geq 1$ and $\overline{R(A^n)} = R(A^n)$. Let $u \in \overline{R(A^{n+1})}$, i.e. $A^{n+1}v_j \to u$ $(j \to \infty)$, for some $v_j \in X$. By the induction assumption $u \in R(A^n)$, $u = A^n v$ for some $v \in X$. Thus $A(A^n v_j - A^{n-1}v) \to 0$. Consider the operator $\widetilde{A} : X/N(A) \longrightarrow X$ induced by A. Clearly $\widetilde{A}$ is bounded below and $\widetilde{A}(A^n v_j - A^{n-1}v + N(A)) \to 0$, so that $A^n v_j - A^{n-1}v + N(A) \to 0$ $(j \to \infty)$ in the quotient space $X/N(A)$. Thus there exist vectors $k_j \in N(A) \subset R(A^n) + F$ such that $A^n v_j + k_j \to A^{n-1}v$. Since $R(A^n) + F$ is closed, we have $A^{n-1}v = A^n a + f$ for some $a \in X$ and $f \in F \subset N(A)$. Hence $u = A^n v = A^{n+1}a \in R(A^{n+1})$ and $R(A^{n+1})$ is closed.

Lemma 2.6. *Let A, B, C, D be mutually commuting operators in X such that $AC + BD = I$. Then*

(1) *For every n there are $C_n, D_n \in \mathcal{L}(X)$ such that A^n, B^n, C_n, D_n are mutually commuting and $A^nC_n + B^nD_n = I$.*
(2) *For every n, $R(A^nB^n) = R(A^n) \cap R(B^n)$ and $N(A^nB^n) = N(A^n) + N(B^n)$. Further $R^\infty(AB) = R^\infty(A) \cap R^\infty(B)$ and $N^\infty(AB) = N^\infty(A) + N^\infty(B)$.*
(3) *$N^\infty(A) \subset R^\infty(B)$ and $N^\infty(B) \subset R^\infty(A)$.*

Proof. (1) We have

$$I = (AC + BD)^{2n-1} = \sum_{i=0}^{2n-1} \binom{2n-1}{i} A^iC^iB^{2n-1-i}D^{2n-1-i} = A^nC_n + B^nD_n$$

for some $C_n, D_n \in \mathcal{L}(X)$ commuting with A^n, B^n.

(2) Clearly $R(AB) \subset R(A) \cap R(B)$. If $x \in R(A) \cap R(B), x = Au = Bv$ for some $u, v \in X$, then set $w = Cv + Du$. Then

$$Bw = BCv + BDu = Cx + BDu = ACu + BDu = u,$$

so that $ABw = Au = x$. Thus $R(AB) = R(A) \cap R(B)$.

By (1) we have $R(A^nB^n) = R(A^n) \cap R(B^n)$ for every n and

$$R^\infty(AB) = \bigcap_n R(A^nB^n) = \bigcap_n (R(A^n) \cap R(B^n)) = R^\infty(A) \cap R^\infty(B).$$

Similarly $N(A) + N(B) \subset N(AB)$. If $x \in N(AB)$, then $x = ACx + BDx$, where $ACx \in N(B)$ and $BDx \in N(A)$. Thus $N(AB) = N(A) + N(B)$ and, by (1), $N(A^nB^N) = N(A^n) + N(B^n)$. Further

$$N^\infty(AB) = \bigcup_n N(A^nB^n) = \bigcup_n (N(A^n) + N(B^n)) = N^\infty(A) + N^\infty(B).$$

(3) If $x \in N(A)$ then $x = BDx \in R(B)$. Thus $N(A) \subset R(B)$ and, by 1), $N(A^n) \subset R(B^n)$ for every n. If $m \geq n$ then $N(A^n) \subset N(A^m) \subset R(B^m)$, so that $N(A^n) \subset R^\infty(B)$ and $N^\infty(A) \subset R^\infty(B)$. The inclusion $N^\infty(B) \subset R^\infty(A)$ follows from the symmetry.

Lemma 2.7. *Let $A, B \in \mathcal{L}(X), AB = BA$. If $N(AB) \subset R^\infty(AB)$ then $N(A) \subset R^\infty(A)$. If $N(AB) \overset{e}{\subset} R^\infty(AB)$ then $N(A) \overset{e}{\subset} R^\infty(A)$.*

Proof. If $N(AB) \subset R^\infty(AB)$ then

$$N(A) \subset N(AB) \subset R^\infty(AB) \subset R^\infty(A).$$

Similarly, if $N(AB) \overset{e}{\subset} R^\infty(AB)$, then

$$N(A) \subset N(AB) \overset{e}{\subset} R^\infty(AB) \subset R^\infty(A).$$

Lemma 2.8. *Let A, B, C, D be mutually commuting operators in a Banach space X, let $AC + BD = I$. Then AB has a generalized inverse if and only if both A and B have a generalized inverse.*

Proof. Suppose $ASA = A$ and $BTB = B$ for same $S, T \in \mathcal{L}(X)$. Then

$$\begin{aligned}
&ABTSAB = ABT(CA+BD)SAB = ABTCASAB + ABTBDSAB \\
=&ABTCAB + ABDSAB = ABT(I-BD)B + A(I-CA)SAB \\
=&ABTB - ABTBDB + ASAB - ACASAB \\
=&AB - ABDB + AB - ACAB = 2AB - A(BD+CA)B = AB.
\end{aligned}$$

Conversely, let $ABZAB = AB$ for some $Z \in \mathcal{L}(X)$. Then

$$\begin{aligned}
&A[C + BZ(I-AC)]A = ACA + ABZA - ABZACA = ACA + ABZA[I-CA] \\
=&ACA + ABZABD = ACA + ABD = A
\end{aligned}$$

and similarly $B[D + (I-DB)ZA]B = B$.

Lemma 2.9. *Let $A, F \in \mathcal{L}(X)$, let A have a generalized inverse and let F be a finite dimensional operator. Then $A+F$ has a generalized inverse.*

Proof. Since $R(A)$ is closed, we have $R(A)+R(F)$ is closed. Since $R(A+F)$ is of finite codimension in $R(A)+R(F)$, we conclude that $R(A+F)$ is closed.

Let M be a subspace of X such that $R(A) \oplus M = X$. Let $x_1, \dots, x_n$ be a basis in $R(F)$, $x_i = Au_i + m_i$ where $u_i \in X, m_i \in M \quad (i = 1, \dots, n)$. Set $M_0 = \vee\{m_i, i = 1, \dots, n\}$ and let M_1 be a subspace of M with $M_0 \oplus M_1 = M$. Then

$$X = R(A) \oplus (M_0 \oplus M_1) = (R(A) + R(F)) \oplus M_1$$

since $R(A) + R(F) = R(A) \oplus M_0$. Thus $R(A)+R(F)$ is complemented and $R(A+F)$ is of finite codimension in $R(A)+R(F)$. Hence $R(A+F)$ is complemented.

Similarly one can prove the complementarity of $N(A+F)$.

Lemma 2.10. *Let A be an operator with closed range such that $N(A)\overset{e}{\subset}R^\infty(A)$. Suppose that A has a generalized inverse. Then A^n has a generalized inverse for every n.*

Proof. (a) Suppose first $N(A) \subset R^\infty(A)$. Let $ASA = A$ for some $S \in \mathcal{L}(X)$. We prove by induction on n that $A^nS^nA^n = A^n$ for every n. Suppose $A^nS^nA^n = A^n$. Then

$$A^{n+1}S^{n+1}A^{n+1} = A[A^nS^n(SA-I) + A^nS^n]A^n.$$

By the induction assumption A^nS^n is a projection onto $R(A^n)$ and $SA-I$ is a projection onto $N(A) \subset R(A^n)$. Thus

$$A^{n+1}S^{n+1}A^{n+1} = A[(SA-I) + A^nS^n]A^n = A \cdot A^nS^nA^n = A^{n+1}.$$

(b) The general case $N(A)\overset{e}{\subset}R^\infty(A)$ can be reduced to (a) by the Kato decomposition (Theorem 2.3, (4)) and the previous lemma.

Lemma 2.11. *Let $A \in R_2(X)$ and let $F \in \mathcal{L}(X)$ be a finite dimensional operator. Then $A+F \in R_2(X)$.*

Proof. See [7].

Lemma 2.12. (cf. [16]) *Let $T \in \mathcal{L}(X)$ be a semi-regular operator having a generalized inverse. Then there exists $\varepsilon > 0$ such that $T - U$ has a generalized inverse for every operator $U \in \mathcal{L}(X)$ commuting with T such that $\|U\| < \varepsilon$.*

Proof. Let $TST = T$ for some $S \in \mathcal{L}(X)$. Set $\varepsilon = \|S\|^{-1}$. Let $U \in \mathcal{L}(X), UT = TU$ and $\|U\| < \varepsilon$.

We prove first by induction on n, that $U(SU)^n N(T) \subset N(T^{n+1})$ for every n. This is clear for $n = 0$. Suppose $U(SU)^{n-1}N(T) \subset N(T^n) \subset R(T)$ and let $z \in N(T)$. Then, for some $v \in X$,

$$T^{n+1}U(SU)^n z = T^n UTSTv = T^n UTv = UT^n U(SU)^{n-1}z = 0$$

by the induction assumption. Since $I - ST$ is a projection onto $N(T)$, we have

$$U(SU)^n(I - ST)X \subset N(T^{n+1}) \subset R(T) \quad (n \geq 0),$$

so that

$$(I - TS)U(SU)^n(I - ST) = 0 \quad (n \geq 0).$$

Then

$$\begin{aligned}
&(T-U)S(I-US)^{-1}(T-U) = (T-U)S\sum_{i=0}^{\infty}(US)^i(T-U)\\
=&TST - UST - TSU + TSUST\\
+&\sum_{i=0}^{\infty}(TS(US)^{i+2}T - US(US)^{i+1}T - TS(US)^{i+1}U + US(US)^iU)\\
=&T - UST - TSU + TSUST + \sum_{i=0}^{\infty}(I-TS)(US)^{i+1}U(I-ST)\\
=&T - U + (I-TS)U(I-ST) + \sum_{i=0}^{\infty}(I-TS)U(SU)^{i+1}(I-ST) = T - U.
\end{aligned}$$

Theorem 2.13. *The sets $R_i(X)$ $(i = 1, 2, 3, 4)$ are regularities satisfying (P3) (upper semi-continuity on commuting elements).*

Proof. It is easy to see that $\mathrm{Inv}(\mathcal{L}(X)) \subset R_i(X) \quad (i = 1, 2, 3, 4)$.

The implication $T \in R_i(X) \Rightarrow T^n \in R_i(X) \quad (i = 1, 2, 3, 4)$ follows from Lemmas 2.5 and 2.10 and the trivial fact that $R^\infty(T^n) = R^\infty(T)$ and $N^\infty(T^n) = N^\infty(T)$.

Suppose that A, B, C, D are commuting operators satisfying $AC + BD = I$. The implication $AB \in R_i(X) \Rightarrow A, B \in R_i(X) \quad (i = 1, 2, 3, 4)$ follows from Lemmas 2.4, 2.7 and 2.8. The opposite implication follows from Lemmas 2.6, (2) and (3) and 2.8.

By Theorem 1.10 it remains to show (P3).

Let $T \in R_1(X)$. By condition (8) of Theorem 2.2, $R^\infty(T)$ is closed, $TR^\infty(T) = R^\infty(T)$ and the induced operator $\widetilde{T} : X/R^\infty(T) \longrightarrow X/R^\infty(T)$ is bounded below. If U

is an operator commuting with T such that $\|U\|$ is small enough, then $(T+U)R^\infty(T) = R^\infty(T)$ and the induced operator $\widetilde{T+U} : X/R^\infty(T) \longrightarrow X/R^\infty(T)$ is bounded below. Thus $T+U \in R_1(X)$ by condition (8) of Theorem 2.2. Hence $R_1(X)$ satisfies (P3).

Condition (P3) for $R_3(X)$ follows from Lemma 2.8.

Let $T \in R_2(X)$ and let $X = X_1 \oplus X_2$ be the Kato decomposition: $\dim X_1 < \infty$,

$$T = \begin{pmatrix} T_1 & O \\ O & T_2 \end{pmatrix}$$

in this decomposition and $T_2 = T|X_2$ is semi-regular (i.e. $T_2 \in R_1(X_2)$). If $U \in \mathcal{L}(X), UT = TU$ and

$$U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

is the decomposition $X = X_1 \oplus X_2$ then $T_2U_{22} = U_{22}T_2$ and $\|U_{22}\| \le c \cdot \|U\|$ for some positive constant c depending only on the decomposition $X = X_1 \oplus X_2$.

If $\|U\|$ is small enough, then $T_2 + U_{22}$ is semi-regular and $T+U \in R_2(X)$ by Lemma 2.11. Hence $R_2(X)$ satisfies (P3).

Property (P3) for $R_4(X)$ can be proved analogously using Lemmas 2.9, 2.12.

Corollary 2.14. (see [10], [11], [12], [13], [15], [16]) *Let $T \in \mathcal{L}(X)$, let f be a function analytic on a neighbourhood of $\sigma(T)$. Then*

$$\widetilde{\sigma}_i(f(T)) = f(\widetilde{\sigma}_i(T)) \quad (i = 1,2,3,4),$$

where $\widetilde{\sigma}_i$ is the spectrum corresponding to the regularity $R_i(X)$, $\quad (i = 1,2,3,4)$.

Proof. If $X = X_1 \oplus X_2$ is a decomposition of $X, T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$ then

$$T_1 \oplus T_2 \in R_i(X) \Leftrightarrow T_1 \in R_i(X_1) \text{ and } T_2 \in R_i(X_2) \quad (i = 1,2,3,4).$$

Since $\widetilde{\sigma}_3(T_1) \supset \widetilde{\sigma}_1(T_1) \supset \partial\sigma(T_1)$ we have for $i = 1,3$

$$\widetilde{\sigma}_i(T_1) \neq \emptyset \Leftrightarrow X_1 \neq \{0\}.$$

Since $\widetilde{\sigma}_4(T_1) \supset \widetilde{\sigma}_2(T_1) \supset \partial\sigma_e(T_1)$, for $i = 2,4$ we have similarly

$$\widetilde{\sigma}_i(T_1) \neq \emptyset \Leftrightarrow \dim X_1 = \infty.$$

The spectral mapping theorems now follow from Theorem 1.5.

The spectra $\widetilde{\sigma}_1$ and $\widetilde{\sigma}_2$ are not only upper semi-continuous on commuting elements, they are also continuous.

Theorem 2.15. *The regularities $R_1(X)$ and $R_2(X)$ satisfy (P4).*

Proof. (a) Let $T \in R_1(X)$. Denote $\varepsilon = \inf\{|z| : T - z \notin R_1(X)\}$ and $M = R^\infty(T)$. Since $R^\infty(T-\lambda) = M$ for $|\lambda| < \varepsilon$, we have $(T-\lambda)M = M$ and the induced operator $\widetilde{T-\lambda} : X/M \longrightarrow X/M$ is bounded below.

If $UT = TU$ and $\|U\| < \varepsilon$, then $UM \subset M$ and $(T+U)M = M$ by Theorem 1.9. for the defect spectrum. Similarly the induced operator $\widetilde{T+U} : X/M \longrightarrow X/M$ is bounded below. Thus $T + U \in R_1(X)$.

(b) For $R_2(X)$ the proof can be done analogously by using condition (5) of Theorem 2.3.

Problem. *We do not know whether the regularities $R_3(X)$ and $R_4(X)$ satisfy (P4).*

Remark. The regularities $R_i(X) \quad (i = 1, 2, 3, 4)$ satisfy neither (P1) nor (P2), see [12], Examples 2.2 and 2.5.

III. Local spectra

Further examples of regularities provide the local spectra.

Notation. *Let x be a vector in a Banach space X. Denote by $R_x(X)$ the set of all operators $T \in \mathcal{L}(X)$ for which there exists a neighbourhood $U \subset \mathbb{C}$ of 0 and an analytic vector-valued function $f : U \longrightarrow X$ such that $(T - z)f(z) = x \quad (z \in U)$.*

If $f(z) = \sum_{i=0}^{\infty} x_{i+1} z^i$ is the Taylor expansion of f in a neighbourhood of 0 then $(T - z)f(z) = Tx_1 + \sum_{i=1}^{\infty} z^i (Tx_{i+1} - x_i)$ so that $Tx_{i+1} = x_i \quad (i = 1, 2, \dots)$ and $Tx_1 = x$. Thus $T \in R_x(X)$ if and only if there exist vectors $x_1, x_2, \dots \in X$ such that $Tx_i = x_{i-1} \quad (i = 1, 2, \dots)$, where $x_0 = x$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$.

We start with the following lemma.

Lemma 3.1. *Let A, B, C, D be mutually commuting operators in a Banach space X such that $AC + BD = I$, let $x_i, y_i \in X \quad (i = 0, 1, \dots)$ satisfy $Ax_i = x_{i-1}, By_i = y_{i-1} \quad (i = 1, 2, \dots), x_0 = y_0$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty, \sup_{i \geq 1} \|y_i\|^{1/i} < \infty$. Then there exist vectors $z_{ij} \in X \quad (i, j = 0, 1, \dots)$ such that $z_{i,0} = x_i, z_{0,j} = y_j \quad (i, j = 0, 1, \dots)$, $Az_{ij} = z_{i-1,j} \quad (i \geq 1)$, $Bz_{ij} = z_{i,j-1} \quad (j \geq 1)$ and $\sup_{i+j \geq 1} \|z_{ij}\|^{1/i+j} < \infty$.*

In particular $ABz_{i,i} = z_{i-1,i-1} \quad (i \geq 1)$.

Proof. Set $z_{i,0} = x_i, z_{0,j} = y_j$ and define z_{ij} inductively by $z_{ij} = Cz_{i-1,j} + Dz_{i,j-1} \quad (i, j \geq 1)$. Then

$$\begin{aligned} Az_{ij} =& ACz_{i-1,j} + ADz_{i,j-1} = z_{i-1,j} - BDz_{i-1,j} + ADz_{i,j-1} \\ =& z_{i-1,j} - Dz_{i-1,j-1} + Dz_{i-1,j-1} = z_{i-1,j} \end{aligned}$$

and

$$Bz_{ij} = BCz_{i-1,j} + BDz_{i,j-1} = z_{i,j-1} - ACz_{i,j-1} + BCz_{i-1,j} = z_{i,j-1}$$

for all $i, j \geq 1$. Further, if k is a positive constant satisfying $\|x_i\| \leq k^i$, $\|y_i\| \leq k^i \quad (i = 1, 2, \dots)$, then it is easy to show by induction that $\|z_{ij}\| \leq \max\{k, \|C\| + \|D\|\}^{i+j} \quad (i, j = 0, 1, \dots)$.

Theorem 3.2. *Let x be a vector in a Banach space X. Then $R_x(X)$ is a regularity satisfying (P3).*

Proof. If $T \in \mathcal{L}(X)$ is invertible then set $x_i = T^{-i}x \quad (i = 0, 1, \dots)$. Clearly $T \in R_x(X)$.

Suppose $T \in R_x(X)$ and let n be positive integer. Let $x_i \in X$ satisfy $Tx_i = x_{i-1} \quad (i = 1, 2, \dots)$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Set $y_i = x_{ni} \quad (i = 0, 1, \dots)$. Then $T^n y_i = T^n x_{ni} = x_{n(i-1)} = y_{i-1} \quad (i = 1, 2, \dots)$ and

$$\sup_{i \geq 1} \|y_i\|^{1/i} = [\sup_{i \geq 1} \|y_i\|^{1/ni}]^n < [\sup_{i \geq 1} \|x_i\|^{1/i}]^n < \infty.$$

Thus $T^n \in R_x(X)$.

Let A, B, C, D be mutually commuting operators with $AC + BD = I$. The implication $A, B \in R_x(X) \Rightarrow AB \in R_x(X)$ follows from the previous lemma.

Let $AB = BA \in R_x(X)$. Let $x_i \in X$ satisfy $ABx_i = x_{i-1} \quad (i = 1, 2, \dots)$ with $x_0 = x$ and let $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Set $y_i = B^i x_i$ Then $Ay_i = AB^i x_i = B^{i-1} x_{i-1} = y_{i-1} \quad (i = 1, 2, \dots)$ and $\sup_{i \geq 1} \|y_i\|^{1/i} \leq \|B\| \cdot \sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Thus $A \in R_x(X)$ and similarly $B \in R_x(X)$. In particular $T^n \in R_x(X)$ implies $T \in R_x(X)$ so that $R_x(X)$ is a regularity.

To prove property (P3), let $T \in R_x(X)$, let $x_i \in X$ satisfy $Tx_i = x_{i-1} \quad (i = 1, 2, \dots), x_0 = x$ and $\sup_{i \geq 1} \|x_i\|^{1/i} = k < \infty$. Let $U \in \mathcal{L}(X), UT = TU$ and $\|U\| < k^{-1}$. Set $g(\lambda) = \sum_{i=0}^{\infty} (U + \lambda)^i x_{i+1}$. This series is convergent for $|\lambda| < k^{-1} - \|U\|$ and we have

$$(T - U - \lambda)g(\lambda) = Tx_1 + \sum_{i=1}^{\infty} (U + \lambda)^i T x_{i+1} - \sum_{i=0}^{\infty} (U + \lambda)^{i+1} x_{i+1} = Tx_1 = x.$$

Thus $T - U \in R_x(X)$.

Denote by γ_x the spectrum corresponding to the regularity $R_x(X)$.

Remark. The standard notation is $\gamma_T(x)$ and this local spectrum has been studied intensively, see e.g. [4], [5], [18], [19], [20]. For our approach, however, the notation $\gamma_x(T)$ is much more appropriate.

Corollary 3.3. (see e.g. [18]) *Let x be a vector in a Banach space X, let $T \in \mathcal{L}(X)$. Then*

$$\gamma_x(f(T)) = f(\gamma_x(T))$$

for every function f analytic on a neighbourhood of $\sigma(T)$ which is non-constant on every component of its domain of definition.

Remarks.

(1) The assumption that f is non-constant on each component is really necessary, since $\gamma_x(T)$ might be empty, cf. [19].

(2) $R_x(X)$ does not satisfy (P2). To see this consider a 2-dimensional space X with a basis e_1, e_2, $x = e_1$,

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then $T \in R_x(X)$ and

$$\begin{pmatrix} 1 & 0 \\ \varepsilon & 0 \end{pmatrix} \notin R_x(X)$$

for every $\varepsilon > 0$.

(3) We do not know whether $R_x(X)$ satisfies (P4).

Consider now the subset $R(X) \subset \mathcal{L}(X)$ defined by: $T \notin R(X)$ if and only if there exists a function $f : U \longrightarrow X$ analytic in a neighbourhood U of 0 such that f is not identically equal to 0 and $(T - z)f(z) = 0 \quad (z \in U)$.

As before it is easy to see that $T \notin R(X)$ if and only if there exist vectors $x_i \in X \quad (i = 1, 2, \ldots)$ not all of them equal to 0 such that $Tx_i = x_{i-1} \quad (i = 1, 2, \ldots)$, where $x_0 = 0$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. We can assume that $x_1 \neq 0$.

Theorem 3.4. *$R(X)$ is a regularity.*

Proof. If $T \in \mathcal{L}(X)$ is an invertible operator and $x_i \in X \quad (i = 1, 2, \ldots)$ satisfy $Tx_i = x_{i-1} \quad (i = 1, 2, \ldots)$ where $x_0 = 0$, then $T^i x_i = 0$, so that $x_i = 0$ for every i. Hence $T \in R(X)$ and $R(X)$ is non-empty.

Let $A, B \in \mathcal{L}(X), AB = BA \notin R(X)$. We prove that either $A \notin R(X)$ or $B \notin R(X)$. Let $x_i \in X$ satisfy $ABx_i = x_{i-1} \quad (i = 1, 2, \ldots)$, where $x_0 = 0, x_1 \neq 0$ and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Set $u_i = B^i x_i \quad (i = 0, 1, \ldots)$. Then $u_0 = 0, Au_i = u_{i-1} \quad (i = 1, 2, \ldots)$ and $\sup_{i \geq 1} \|u_i\|^{1/i} < \infty$. If $u_1 \neq 0$ then $A \notin R(X)$.

Suppose on the contrary $u_1 = Bx_1 = 0$. Set $v_0 = 0, v_i = A^{i-1} x_i \quad (i = 1, 2, \ldots)$. Then $Bv_i = v_{i-1} \quad (i = 1, 2, \ldots), \sup_{i \geq 1} \|v_i\|^{1/i} < \infty$ and $v_1 = x_1 \neq 0$. Thus $B \notin R(X)$. Hence $A, B \in R(X), AB = BA$ implies $AB \in R(X)$.

In particular $A \in R(X) \Rightarrow A^n \in R(X) \quad (n = 1, 2, \ldots)$.

Let $A \notin R(X)$ and let $x_i \in X$ satisfy the required conditions. Then $y_i = x_{ni}$ satisfy all the required conditions for A^n, so that $A^n \notin R(X)$. Hence $A \in R(X) \Leftrightarrow A^n \in R(X)$.

Suppose that A, B, C, D are mutually commuting operators satisfying $AC + BD = I$ and $A \notin R(X)$. Let $x_i \in X$ satisfy $Ax_i = x_{i-1} \quad (i = 1, 2, \ldots), x_0 = 0$, not all of x_i's are equal to 0 and $\sup_{i \geq 1} \|x_i\|^{1/i} < \infty$. Set $y_i = 0 \quad (i = 0, 1, \ldots)$. By Lemma 3.1 there are $z_i \in X$ not all of them equal to 0 such that $ABz_i = z_{i-1}, z_0 = 0$ and $\sup_{i \geq 1} \|z_i\|^{1/i} < \infty$. Thus $AB \notin R(X)$, so that $AB \in R(X) \Rightarrow A, B \in R(X)$.

Denote by $\widetilde{\sigma}$ the spectrum corresponding to the regularity $R(X)$. In general $\widetilde{\sigma}(T)$ is not closed (on contrary, it is always open) , so that $R(X)$ can not satisfy (P2), (P3) or (P4). Neither $R(X)$ satisfies (P1). To see this, let X be a separable Hilbert space, $A = 0$ and let B be a backward shift. It is easy to see that $0 = AB \in R(X)$ and $B \notin R(X)$.

The closure of $\widetilde{\sigma}(T)$ is usually denoted by S_T and called the analytic residuum of T.

Corollary 3.5. (see [18]) *Let $T \in \mathcal{L}(X)$ and let f be a function analytic on a neighbourhood of $\sigma(T)$ which is non-constant on each component of its domain of definition. Then*

$$\widetilde{\sigma}(f(T)) = f(\widetilde{\sigma}(T)) \quad \text{and} \quad S_{f(T)} = f(S_T).$$

Proposition 3.6. *Let $T \in \mathcal{L}(X), x \in X, x \neq 0$. Then $\widetilde{\sigma}(T) \cup \gamma_x(T) \neq \emptyset$.*

Proof. Suppose on the contrary that $\widetilde{\sigma}(T) \cup \gamma_x(T) = \emptyset$. Then for every $z \in \mathbb{C}$ there exists a neighbourhood U_z of z and an analytic function $f_z : U_z \longrightarrow X$ such that $(T - \lambda)f_z(\lambda) = x \quad (\lambda \in U_z)$. Since $\widetilde{\sigma}(T) = \emptyset$, functions f_z and f_w coincide on $U_z \cap U_w \quad (z, w \in \mathbb{C})$, so that in fact we have an entire function $f : \mathbb{C} \longrightarrow X$ such that $(T - \lambda)f(\lambda) = x \quad (\lambda \in \mathbb{C})$. For $|\lambda| > r(T)$ we have $f(\lambda) = (T - \lambda)^{-1}x$, so that $\lim_{\lambda\to\infty} |f(\lambda)| = 0$. By the Liouville theorem $f = 0$, so that $x = 0$, a contradiction.

The closure of $\widetilde{\sigma}(T) \cup \gamma_x(T)$ will be denoted by $\sigma_x(T)$. (the standard notation is again rather $\sigma_T(x)$ instead of $\sigma_x(T)$; this set is also called the local spectrum).

Theorem 3.7. *Let $T \in \mathcal{L}(X), x \in X, x \neq 0$ and let f be a function analytic on a neighbourhood of $\sigma(T)$. Then*

$$\widetilde{\sigma}(f(T)) \cup \gamma_x(f(T)) = f(\widetilde{\sigma}(T)) \cup f(\gamma_x(T)) \quad \text{and} \quad \sigma_x(f(T)) = f(\sigma_x(T)).$$

Proof. Let $X = X_1 \oplus X_2$ be a decomposition of X, let $x = x_1 \oplus x_2$ be the corresponding decomposition of x, let $T_1 \in \mathcal{L}(X_1)$ and $T_2 \in \mathcal{L}(X_2)$. It is easy to see that

$$T_1 \oplus T_2 \in R_x(X) \Leftrightarrow T_1 \in R_{x_1}(X_1) \text{ and } T_2 \in R_{x_2}(X_2)$$

and

$$T_1 \oplus T_2 \in R(X) \Leftrightarrow T_1 \in R(X_1) \text{ and } T_2 \in R(X_2).$$

The previous theorem together with Theorem 1.5 completes the proof.

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