# Stability of infinite ranges and kernels

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**Abstract.** Let $z \mapsto A(z)$ be a regular function defined on a connected set whose values are mutually commuting essentially Kato operators. Then the spaces $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are constant. This generalizes results of Aupetit and Zemánek.

Denote by $B(X)$ the set of all bounded linear operators on a complex Banach space $X$. For $T \in B(X)$ denote by $N(T)$ the null space and by $R(T)$ the range of $T$, respectively.

Write also $R^\infty(T) = \bigcap_{k=1}^\infty R(T^k)$ and $N^\infty(T) = \bigcup_{k=1}^\infty N(T^k)$. It is well known that the spaces $R^\infty(T - zI)$ and $\overline{N^\infty(T - zI)}$ remain constant for all $z$ in a neighbourhood of zero for various classes of operators although the ranges $R(T - zI)$ and kernels $N(T - zI)$ do change, see [GK1], [H], [MO].

As it was observed by Aupetit and Zemánek [AZ], this phenomenon is closely related to the concept of regular functions.

Denote by $\gamma(T) = \inf\{\|Tx\| : \operatorname{dist}\{x, N(T)\} = 1\}$ the reduced minimum modulus of $T$. It is well known that $\gamma(T^*) = \gamma(T)$, and $\gamma(T) > 0$ if and only if $T$ has closed range.

Let $G$ be a metric space, $w \in G$, and let $A : G \to B(X)$ be a continuous operator-valued function. We say that $A$ is regular at $w$ if $R(A(w))$ is closed and $A$ satisfies one of the following equivalent conditions:

(i) the function $z \mapsto \gamma(A(z))$ is continuous at $w$;
(ii) $\liminf_{z \to w} \gamma(A(z)) > 0$;
(iii) the function $z \mapsto R(A(z))$ is continuous at $w$ in the gap topology;
(iv) the function $z \mapsto N(A(z))$ is continuous at $w$ in the gap topology.

Recall that the gap between two subspaces $M, L \subset X$ is defined by $\hat{\delta}(M, L) = \max\{\delta(M, L), \delta(L, M)\}$ where $\delta(M, L) = \sup_{\substack{x \in M \\ \|x\| \le 1}} \operatorname{dist}\{x, L\}$.

Regular functions have been studied by a number of authors, see e.g., [Ma], [T], [J], [S], [M2]. By property (ii), the set of all regularity points is open.

The regular functions are closely connected with the important class of Kato operators (sometimes also called semiregular operators). An operator $T \in B(X)$ is called Kato if the function $z \mapsto T - z$ is regular at 0. It is well known, see e.g. [M2], p.113 that the following conditions are equivalent for an operator T with closed range:

(i) $T$ is Kato;
(ii) $N(T) \subset R^\infty(T)$;
(iii) $N^\infty(T) \subset R(T)$;
(iv) $N^\infty(T) \subset R^\infty(T)$;
(v) $N(T) \subset \bigvee_{z \ne 0} N(T - zI)$;
(vi) $R(T) \supset \bigcap_{z \ne 0} \overline{R(T - zI)}$.

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It is known that the spaces $R^\infty(T-z)$ and $\overline{N^\infty(T-z)}$ are constant on each connected subset of the set $\{\lambda \in \mathbb{C} : T-\lambda \text{ is Kato}\}$; moreover, $R^\infty(T-\lambda)$ is closed whenever $T-\lambda$ is Kato. This result was generalized in [AZ] to any regular analytic function, whose values are mutually commuting semi-Fredholm operators.

The aim of this note is to show that the assumption of analycity is not necessary. Moreover, semi-Fredholm operators can be replaced by a more general class of essentially Kato operators, see below. Thus the spaces $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are constant on each connected set for each regular function whose values are mutually commuting essentially Kato operators.

For an essential version of Kato operators we use the following notation. For subspaces $M, L \subset X$ write $M \overset{e}{\subset} L$ if $\dim M/(L \cap M) < \infty$. Equivalently, $\dim(M+L)/L < \infty$.

An operator $T \in B(X)$ is called essentially Kato if $R(T)$ is closed and $T$ satisfies any of the following equivalent conditions:

(i) $N(T) \overset{e}{\subset} R^\infty(T)$;
(ii) $N^\infty(T) \overset{e}{\subset} R(T)$;
(iii) $N^\infty(T) \overset{e}{\subset} R^\infty(T)$.
(iv) $N(T) \overset{e}{\subset} \bigvee_{z\neq 0} N(T-z)$;
(v) $\bigcap_{z\neq 0} \overline{R(T-z)} \overset{e}{\subset} R(T)$.

In particular, any semi-Fredholm operator is essentially Kato.

We summarize here the basic properties of essentially Kato operators, see [M2], p. 183–187.

**Theorem 1.**

(i) Let $T \in B(X)$ be essentially Kato. Then $R(T^k)$ is closed for all $k$. Consequently, $R^\infty(T)$ is closed;
(ii) $T \in B(X)$ is essentially Kato if and only if $T^* \in B(X^*)$ is essentially Kato;
(iii) $T \in B(X)$ is essentially Kato if and only if there exists a closed subspace $M \subset X$ such that $TM = M$ and the operator $\hat{T} : X/M \to X/M$ induced by $T$ is upper semi-Fredholm.
As the space $M$ it is possible to take $M = R^\infty(T)$;
(iv) Let $T \in B(X)$ be essentially Kato. Then the limit $\lim_{n\to\infty} \gamma(T^n)^{1/n}$ exists and is positive. Moreover,

$$\lim_{n\to\infty} \gamma(T^n)^{1/n} = \max\{r : T-z \text{ is Kato for } 0<|z|<r\}.$$

We start with the following example which shows that it is really necessary that the values of the function are mutually commuting. It is not sufficient to assume that the values $A(z)$ commute with $A(w)$ for a fixed $w$, even for analytic functions on finite dimensional spaces.

**Example 2.** Let $X = \mathbb{C}^3$. For $z \in \mathbb{C}$ let

$$A(z) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & z \\ z & 0 & z^2 \end{pmatrix}$$

Clearly $z \mapsto A(z)$ is an analytic function and $\operatorname{rank} A(z) = 1$ for all $z \in \mathbb{C}$. It is easy to see that $A$ is regular. Moreover, $A(0)A(z) = A(z)A(0)$ for all $z \in \mathbb{C}$.

We have $A(0)^2 = 0$, and so $R^\infty(A(0)) = \{0\}$. On the other hand, $A(z)\begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix} = z^2 \begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix}$, and so $\dim R^\infty(A(z)) = 1$ for all $z \neq 0$.

Similarly, $N^\infty(A(0)) = X$ and $N^\infty(A(z)) \neq X$ for $z \neq 0$. Hence $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are not constant on a neighbourhood of 0.

Moreover, $R(A(0)^k) = \{0\}$ and $R(A(z)^k) \neq \{0\}$ for all $z \neq 0$ and $k \geq 2$. So the function $z \mapsto A(z)^k$ is not regular at 0 for $k \geq 2$.

Note that $A(z)$ does not commute with $A(z')$ for $z, z' \neq 0$, $z \neq z'$.

**Remark 3.** By [FK], the limit $\lim_{k\to\infty} \gamma(A(z)^k)^{1/k}$ exists for each $z \in \mathbb{C}$ and

$$\lim_{k\to\infty} \gamma(A(z)^k)^{1/k} = \max\{r > 0 : \dim N(A(z) - u) \text{ is constant for all } u \in \mathbb{C}, 0 < |u| < r\}.$$

The previous example shows also that the function $z \mapsto \lim \gamma(A(z)^n)^{1/n}$ is not continuous although the function $z \mapsto \gamma(A(z))$ is continuous and $A(z)$ commutes with $A(0)$ for all $z$.

For our main theorem we need the following finite-dimensional lemma.

**Lemma 4.** Let $\dim X < \infty$, let $A_i \in B(X)$ $\quad (i = 0, 1, 2, \ldots)$ be a sequence of mutually commuting operators, let $\|A_i - A_0\| \to 0$ and $\operatorname{rank} A_j = \operatorname{rank} A_0$ for all $j$. Suppose that $A_0$ is nilpotent. Then $A_j$ is nilpotent for all $j$ large enough.

**Proof.** We prove the statement by induction on the dimension of $X$.

The statement is clear if $\dim X = 1$. Let $\dim X = k > 1$ and suppose that the statement is true for all spaces with dimension $< k$.

Suppose on the contrary that the statement is not true and fix a $j$ such that $A_j$ is not nilpotent. Let $M_1 = N^\infty(A_j)$ and $M_2 = R^\infty(A_j)$. Then $X = M_1 \oplus M_2$ is the spectral decomposition of $A_j$ corresponding to the sets $\{0\}$ and $\mathbb{C} \setminus \{0\}$. By the assumptions, $\dim M_1 < \dim X$ and $\dim M_2 \leq \operatorname{rank} A_j = \operatorname{rank} A_0 < \dim X$. Clearly the spaces $M_1, M_2$ are invariant with respect to all operators $A_i$.

Since the rank is a lower semicontinuous function, $\operatorname{rank} A_i|M_1 \geq \operatorname{rank} A_0|M_1$ and $\operatorname{rank} A_i|M_2 \geq \operatorname{rank} A_0|M_2$ for all $i$ large enough. Thus

$$\operatorname{rank} A_0 = \operatorname{rank} A_i = \operatorname{rank} A_i|M_1 + \operatorname{rank} A_i|M_2 \geq \operatorname{rank} A_0|M_1 + \operatorname{rank} A_0|M_2 = \operatorname{rank} A_0.$$

Hence $\operatorname{rank} A_i|M_1 = \operatorname{rank} A_0|M_1$ and $\operatorname{rank} A_i|M_2 = \operatorname{rank} A_0|M_2$. By the induction assumption, $A_i$ is nilpotent for all $i$ large enough. $\square$

The following result was proved by Livšak [L]; implicitly it is also contained in papers of Gol'dman and Kratčkovskii. However, the existing proofs of the result [L], [AZ] refer for the most difficult step of the proof to [GK2], Theorem 3, where it is stated in fact without proof. Therefore we feel that it is convenient to give a complete proof here. Moreover, we give a quantitative estimate on the norm of perturbation $S$.

Note that if $A$ is essentially Kato then the space $R^\infty(A)+N^\infty(A)$ is automatically closed. Moreover, we can write $N^\infty(A) = F + \big(R^\infty(A)\cap N^\infty(A)\big)$, where $F$ is a finite-dimensional subspace and $F \cap R^\infty(A) = \{0\}$. Since $F + \overline{R^\infty(A)\cap N^\infty(A)}$ is closed, we have $\overline{N^\infty(A)} = F + \overline{R^\infty(A)\cap N^\infty(A)}$. Since $\overline{R^\infty(A)\cap N^\infty(A)} \subset R^\infty(A)$, we have

$$R^\infty(A)\cap \overline{N^\infty(A)} = \overline{R^\infty(A)\cap N^\infty(A)}. \tag{1}$$

Similarly one can show that

$$R^\infty(A^*)\cap \overline{N^\infty(A^*)}^{w^*} = \overline{R^\infty(A^*)\cap N^\infty(A^*)}^{w^*}. \tag{2}$$

**Theorem 5.** (Livšak) Let $A \in B(X)$ be essentially Kato, let $S \in B(X)$, SA=AS and $\|S\| < \lim \gamma(A^k)^{1/k}$. Then $A+S$ is essentially Kato,

$$R^\infty(A+S)\cap\overline{N^\infty(A+S)} = R^\infty(A)\cap\overline{N^\infty(A)}$$

and

$$R^\infty(A+S)+N^\infty(A+S) = R^\infty(A)+N^\infty(A).$$

**Proof.** We prove the statement in several steps.

**(a)** $A+S$ is essentially Kato.

**Proof.** Set $M = R^\infty(A)$. Then $M$ is a closed subspace of $X$ invariant with respect to $A$ and $S$. Let $A_1 = A|M$ and $S_1 = S|M$ be the corresponding restrictions. Denote further by $\hat{A} : X/M \to X/M$ and $\hat{S} : X/M \to X/M$ the operators induced by $A$ and $S$, respectively. By Theorem 1 (iii), $A_1$ is onto and $\hat{A}$ is upper semi-Fredholm. Moreover,

$$\lim_{k\to\infty}\gamma(A^k)^{1/k} = \min\Big\{\lim_{k\to\infty}\gamma(A_1^k)^{1/k}, \lim_{k\to\infty}\gamma(\hat{A}^k)^{1/k}\Big\},$$

see [KM]. Clearly $\|S_1\| \le \|S\| < \lim_{k\to\infty}\gamma(A_1^k)^{1/k}$ and $\|\hat{S}\| \le \|S\| < \lim_{k\to\infty}\gamma(\hat{A}^k)^{1/k}$. By [Z], $A_1+S_1$ is onto and $\hat{A}+\hat{S}$ is upper semi-Fredholm. By Theorem 1, $A+S$ is essentially Kato.

**(b)** $R^\infty(A) \subset R^\infty(A+S)$.

**Proof.** Since $(A+S)M = M$, we have $R^\infty(A+S) \supset M = R^\infty(A)$.

**(c)** $\overline{N^\infty(A+S)} \subset \overline{N^\infty(A)}$.

**Proof.** We have

$$R^\infty(A) = \bigcap_{k=0}^\infty R(A^k) = \bigcap_{k=0}^\infty {}^\perp N(A^{*k}) = {}^\perp\bigcup_{k=0}^\infty N(A^{*k}) = {}^\perp N^\infty(A^*)$$

and

$$\overline{N^\infty(A)} = {}^\perp\Big(N^\infty(A)^\perp\Big) = {}^\perp\Big(\bigcap_{k=0}^\infty N(A^k)^\perp\Big) = {}^\perp\Big(\bigcap_{k=0}^\infty R(A^{*k})\Big) = {}^\perp R^\infty(A^*).$$

The analogous equalities are true also for the operator $A+S$. By duality argument we have

$$\overline{N^\infty(A)} = {}^\perp R^\infty(A^*) \supset {}^\perp R^\infty(A^*+S^*) = \overline{N^\infty(A+S)}.$$

**(d)** $R^\infty(A+S) \cap \overline{N^\infty(A+S)} \subset R^\infty(A)$.

**Proof.** By induction on $k$ we show that $R^\infty(A+S) \cap N((A+S)^k) \subset R^\infty(A)$. The statement is clear for $k=0$. Let $k \geq 1$ and suppose that $R^\infty(A+S) \cap N((A+S)^{k-1}) \subset R^\infty(A)$.

Let $x_0 \in R^\infty(A+S) \cap N((A+S)^k)$. Let $m$ be larger than

$$\begin{aligned}\dim(R^\infty(A)+N^\infty(A))/R^\infty(A) &= \dim(R^\infty(A)+\overline{N^\infty(A)})/R^\infty(A)\\ &= \dim \overline{N^\infty(A)}/(R^\infty(A) \cap \overline{N^\infty(A)}).\end{aligned}$$

Since $(A+S)R^\infty(A+S) = R^\infty(A+S)$, we can find vectors $x_1, \dots, x_m \in R^\infty(A+S)$ such that $(A+S)x_k = x_{k-1}$ $(k = 1, \dots, m)$. It is easy to see that the vectors $x_k$ are linearly independent and belong to $N^\infty(A+S) \subset \overline{N^\infty(A)}$. Therefore there is a nontrivial linear combination $y = \sum_{i=0}^m \alpha_i x_i \in R^\infty(A)$. Let $\alpha_{m'} \neq 0$ and $\alpha_i = 0$ for $i > m'$. Then $(A+S)^{m'} y \in R^\infty(A)$. But

$$(A+S)^{m'} y \in \alpha_{m'} x_0 + \big(R^\infty(A+S) \cap N((A+S)^{k-1})\big).$$

By the induction assumption, $x_0 \in R^\infty(A)$. Hence $R^\infty(A+S) \cap N((A+S)^k) \subset R^\infty(A)$ and, by (1), we have statement (d).

**(e)** Let $c$ be a positive number such that $S' = cS$ satisfies $\|S'\| < \frac{1}{2}\gamma(A|M)$. Then $R^\infty(A) \cap \overline{N^\infty(A)} \subset \overline{N^\infty(A+S')}$.

**Proof.** By (1), it is sufficient to show that $R^\infty(A) \cap N(A^n) \subset \overline{N^\infty(A+S')}$ for all $n$.

Let $n \geq 1$ and $x_0 \in N(A^n) \cap M$, where $M = R^\infty(A)$. Since $AM = M$, $SM \subset M$ and $\|S'\| < \gamma(A|M)$, we have $(A+S')M = M$ and

$$\gamma((A+S')|M) \geq \gamma(A|M) - \|S'\| \geq \frac{1}{2}\gamma(A|M).$$

Therefore we can find inductively vectors $x_1, x_2, \dots \in M$ such that $(A+S')x_k = x_{k-1}$ and $\|x_k\| < 2\gamma(A|M)^{-1}\|x_{k-1}\|$ for all $k \geq 1$.

For $k \geq n$ set $y_k = x_0 - \sum_{j=0}^{n-1} \binom{k}{j} A^j S'^{k-j} x_k$. Then $y_k \in M$ and we have

$$(A+S')^k y_k = (A+S')^k x_0 - \sum_{j=0}^{n-1} \binom{k}{j} A^j S'^{k-j} x_0 = 0.$$

Thus $y_k \in N^\infty(A+S')$ for all $k$. Moreover,

$$\begin{aligned}\|y_k - x_0\| &= \Big\|\sum_{j=0}^{n-1} \binom{k}{j} A^j S'^{k-j} x_k\Big\| \leq \sum_{j=0}^{n-1} k^j \|A^j\| \cdot \|S'\|^{k-j} \cdot \|x_k\|\\ &\leq \sum_{j=0}^{n-1} \frac{k^j \|A^j\| \cdot \|x_0\|}{\|S'\|^j} \cdot \Big(\frac{2\|S'\|}{\gamma(A|M)}\Big)^k \to 0\end{aligned}$$

as $k \to \infty$. Thus $x_0 \in \overline{N^\infty(A+S')}$, and so $\overline{N^\infty(A)} \cap R^\infty(A) \subset \overline{N^\infty(A+S')}$.

**Proof of the Theorem 5.** By statements (a) – (e), the spaces $R^\infty(A+aS) \cap \overline{N^\infty(A+aS)}$ are constant for all complex numbers $a$ with $|a|$ small enough $\big(|a| < \frac{\gamma(A|M)}{2\|S\|}\big)$. By a standard argument, these spaces are constant on each connected set for which $A+aS$ is essentially Kato. In particular,

$$R^\infty(A+S) \cap \overline{N^\infty(A+S)} = R^\infty(A) \cap \overline{N^\infty(A)}.$$

The second statement can be obtained by duality argument. As in (c), we have $N^\infty(A)^\perp = R^\infty(A^*)$ and $R^\infty(A)^\perp = \big({}^\perp N^\infty(A^*)\big)^\perp = \overline{N^\infty(A^*)}^{w^*}$.

By (2), we have

$$N^\infty(A) + R^\infty(A) = {}^\perp\Big(\big(N^\infty(A) + R^\infty(A)\big)^\perp\Big) = {}^\perp\Big(N^\infty(A)^\perp \cap R^\infty(A)^\perp\Big)$$
$$= {}^\perp\Big(R^\infty(A^*) \cap \overline{N^\infty(A^*)}^{w^*}\Big) = {}^\perp\Big(R^\infty(A^*) \cap \overline{N^\infty(A^*)}\Big)^{-w^*}.$$

Similarly,

$$N^\infty(A+S) + R^\infty(A+S) = {}^\perp\Big(R^\infty(A^*+S^*) \cap \overline{N^\infty(A^*+S^*)}\Big)^{-w^*},$$

and so

$$N^\infty(A+S) + R^\infty(A+S) = N^\infty(A) + R^\infty(A).$$

□

We need the following simple lemma.

**Lemma 6.** Let $A \in B(X)$ be a surjective operator. Then $A^{-1}\big(\overline{N^\infty(A)}\big) \subset \overline{N^\infty(A)}$.

**Proof.** Let $x \in A^{-1}\big(\overline{N^\infty(A)}\big)$. Let $0 < c < \gamma(A)$ and $\varepsilon > 0$.

Find $u \in N^\infty(A)$ such that $\|Ax - u\| < c\varepsilon$. Since $A$ is onto, we can find $v \in X$ such that $Av = u$. Thus $v \in N^\infty(A)$ and $\|A(x-v)\| < c\varepsilon$. Therefore there is a $w \in X$ with $Aw = A(x-v)$ and $\|w\| < \varepsilon$. Consequently, $x - v - w \in N(A)$ and $x - w \in N^\infty(A)$. Hence $\operatorname{dist}\{x, N^\infty(A)\} \le \|w\| < \varepsilon$. Since $\varepsilon$ was arbitrary, we have $x \in \overline{N^\infty(A)}$. □

**Theorem 7.** Let $G$ be a metric space, let $A : G \to B(X)$ be a regular function. Suppose that the values of $A$ are mutually commuting operators. Let $w \in G$ and $A(w)$ be essentially Kato. Then the spaces $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are constant on a neighbourhood of $w$.

**Proof.** Set $M = R^\infty(A(w)) + N^\infty(A(w))$. By Theorem 5, there is a neighbourhood $U$ of $w$ such that $A(z)$ is essentially Kato and $R^\infty(A(z)) + N^\infty(A(z)) = M$ for each $z \in U$.

Write $A_1(z) = A(z)|M : M \to M$. Since $M$ contains the null spaces of $A(z)$ for all $z \in U$, the function $z \mapsto A(z)|M$ is regular at $w$. Moreover, the operators $A_1(z)$ are lower semi-Fredholm with finite descent since $\operatorname{codim} R^\infty(A_1(z)) = \dim M/R^\infty(A(z)) < \infty$.

Let $M' = R^\infty(A(w)) \cap \overline{N^\infty(A(w))}$. Define $B(z) : M/M' \to M/M'$ by

$$B(z)(m + M') = A(z)m + M' \quad (z \in U).$$

Since $R(A_1(z)) \supset R^\infty(A(z)) \supset M'$, we have $R(B(z)) = R(A_1(z))/M'$. It is easy to check that $\delta(L_1/M', L_2/M') \le \delta(L_1, L_2)$ for all subspaces $L_1, L_2 \subset M$ with $L_1 \supset M'$, $L_2 \supset M'$. Therefore

$$\hat\delta\big(R(B(z)), R(B(w))\big) \le \hat\delta\big(R(A_1(z)), R(A_1(w))\big),$$

and so the function $z \mapsto R(B(z))$ is continuous at $w$ in the gap topology. Thus the function $z \mapsto B(z)$ is regular at $w$.

Clearly $\operatorname{codim} R^\infty(B(w)) \le \dim M/R^\infty(A(w)) < \infty$, and so the operator $B(w)$ has finite descent. We show that it has also finite ascent.

The Kato decomposition for the operator $A_1(w)$ implies that there exists a finite-dimensional subspace $N \subset M$ invariant with respect to $A_1(w)$ such that $M = N \oplus R^\infty(A_1(w))$ and $A_1(w)|N$ is nilpotent. By Lemma 6 for the operator $A_1(w)^n|R^\infty(A)$ we have for each $n$ that

$$A_1(w)^{-n}M' \subset N + R^\infty(A_1(w)) \cap \overline{N^\infty(A_1(w))} = N + M',$$

and so

$$\dim N(B(w)^n) = \dim\big(A_1(w)^{-n}M'\big)/M' \le \dim N.$$

Thus $\dim N^\infty(B(w)) < \infty$ and $B(w)$ has finite ascent.

Let $L_1 = R^\infty(B(w))$ and $L_2 = N^\infty(B(w))$. Then $M/M' = L_1 \oplus L_2$, $B(w)|L_1$ is invertible and $B(w)|L_2$ is a finite-dimensional nilpotent. By Lemma 4, there is a neighbourhood $U'$ of $w$ such that $B(z)|L_1$ is invertible and $B(z)|L_2$ is nilpotent for all $z \in U'$. Thus $R^\infty(A(z)) = R^\infty(A_1(z)) = \pi^{-1}\big(R^\infty(B(z))\big) = \pi^{-1}(L_1)$ and $\overline{N^\infty(A(z))} = \overline{N^\infty(A_1(z))} = \pi^{-1}\big(N^\infty(B(z))\big) = \pi^{-1}(L_2)$, where $\pi : M \to M/M'$ denotes the canonical projection. Hence the spaces $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are constant on a certain neighbourhood of $w$. □

**Remark 8.** In fact the functions $z \mapsto A^j(z)$ are regular at $w$ for all $j$ sufficiently large. Clearly this is true for all $j$ satisfying $B(w)^j|L_2 = 0$. In particular, this happens for $j \ge \dim\Big(\big(R^\infty(A(w)) + N^\infty(A(w))\big)/R^\infty(A(w))\Big)$.

**Corollary 9.** Let $G$ be a connected metric space, Let $A : G \to B(X)$ be a regular operator-valued function whose values are mutually commuting essentially Kato operators. Then the spaces $R^\infty(A(z))$ and $\overline{N(A(z))}$ are constant on $G$.

If the operator $A(w)$ is even Kato, then the function $A$ is automatically regular at $w$ and a weaker version of commutativity is sufficient.

**Theorem 10.** Let $G$ be a metric space, $w \in G$, let $A : G \to B(X)$ be a continuous function, let the operator $A(w)$ be Kato and let $A(z)A(w) = A(w)A(z)$ for all $z \in G$. Then the function $A$ is regular at $w$ and there is a neighbourhood $U$ of $w$ such that $A(z)$ is Kato and the spaces $R^\infty(A(z))$ and $\overline{N(A(z))}$ are constant for $z \in U$.

**Proof.** Set $M = R^\infty(A(w))$. Then $A(w)M = M$ and the operator $\hat{A}(w) : X/M \to X/M$ induced by $A(w)$ is bounded below. Thus $A(z)M = M$ and the operator $\hat{A}(z) : X/M \to X/M$ induced by $A(z)$ is bounded below for all $z$ close to $w$. So $A(z)$ is Kato. Let $U$ be an open connected neighbourhood of $w$ such that $A(z)$ is Kato for all $z \in U$ (by [KM], it is possible to take $U = \{z \in G : \|A(z) - A(w)\| < \lim \gamma(A^k(w))^{1/k}\}$).

Since $\overline{N^\infty(A(z))} \subset R^\infty(A(z))$ for all $z \in U$, by Theorem 5 the spaces $R^\infty(A(z))$ and $\overline{N^\infty(A(z))}$ are constant on $U$.

Let $A_1(z) \in B(M)$ be the restriction of $A(z)$ to $M$. Since $A_1(z)$ is onto for all $z \in U$, the function $z \mapsto A_1(z)$ si regular. Thus the null space $N(A_1(z))$ is changing continuously in the gap topology. Since $N(A(z)) = N(A_1(z)) \quad (z \in U)$, the function $z \mapsto A(z)$ is regular in $U$. $\square$

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