Local behaviour of operators

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Introduction

Let T be a bounded operator in a Banach space X and let $x \in X$. Denote by $\mathcal{P}$ the set of all complex polynomials. We are going to study the following problem:

A. What can we say about the set $\{p(T)x : p \in \mathcal{P}\}$?

A weaker version of this problem is:

B. What can we say about the set $\{T^k x : k = 0, 1, \dots\}$?

The sets $\{T^k x : k = 0, 1, \dots\}$ for operators in Hilbert spaces were called "orbits" by Rolewicz [15] and intensively studied by Beauzamy [2].

Questions of type A or B appear naturally in many problems of operator theory.

Examples:

1) Local spectral radius $r(T, x)$ of an operator T at a point $x \in X$ can be defined by $r(T, x) = \limsup_{k\to\infty} \|T^k x\|^{1/k}$, i.e. it is a quantity defined in terms of B. The local spectral radius plays an important role in the local spectral theory.

2) As an analogy to the local spectral radius for the set of all polynomials can be considered the local capacity (see later).

3) The invariant subspace problem can be also easily reformulated by using the sets $\{p(T)x : p \in \mathcal{P}\}$: An operator T in X has no non-trivial invariant subspace if and only if $\{p(T)x : p \in \mathcal{P}\}$ is dense for all $x \in X$. Many of the positive results (e.g. results based on the Scott Brown technique) consist in finding $x \in X$ such that $\|p(T)x\| \geq 1$ for all polynomials p with $p(0) = 1$.

The present paper is a survey of results obtained in [8]–[14]. The results show that for every Banach space X and every bounded linear operator T on X there exists $x \in X$ such that $\|p(T)x\|$ is big enough for all polynomials p.

I. Essential approximate point spectrum

Denote by $B(X)$ the algebra of all bounded operators in a Banach space X. Denote by **C** and **N** the set of all complex numbers and positive integers, respectively.

Let T be a bounded operator in a Banach space X. If x is an eigenvalue of T, $Tx = \lambda x$ for some complex λ, then $p(T)x = p(\lambda)x$ for every polynomial p so that we have a complete information about the set $\{p(T)x : p \in \mathcal{P}\}$. Unfortunately, operators in infinite dimensional Banach spaces have usually no eigenvalues. The proper tool appears to be the notion of the essential approximate point spectrum of T.

Denote by $\sigma_e(T)$ the essential spectrum of $T \in B(X)$, i.e. the spectrum of $\rho(T)$ in the Calkin algebra $B(X)|K(X)$, where $K(X)$ is the ideal of all compact operators in X and $\rho : B(X) \longrightarrow B(X)|K(X)$ is the canonical projection. Denote further by $\sigma_{\pi e}(T)$ the essential approximate point spectrum of T, i.e. $\sigma_{\pi e}(T)$ is the set of all complex λ such that

$$\inf\{\|(T - \lambda)x\| : x \in M, \|x\| = 1\} = 0$$

for every subspace $M \subset X$ with $\operatorname{codim} \mathrm{M} < \infty$.

It is easy to see that $\lambda \notin \sigma_{\pi e}(T)$ if and only if $\dim \operatorname{Ker}(\mathrm{T} - \lambda) < \infty$ and $T - \lambda$ has closed range, i.e. if $T - \lambda$ is upper semi-Fredholm.

The terminology is not unified, the essential approximate point spectrum was studied under various names (see e.g. [1], [3], [7]).

By [7], $\sigma_{\pi e}(T)$ contains the topological boundary of the essential spectrum, in particular it is always a non-empty compact subset of $\sigma_e(T)$.

We start with studying the elements $\lambda \in \sigma_{\pi e}(T)$ for operators in Hilbert spaces. We show that, for given $k \in \mathbf{N}$, there always exists x such that the powers $T^i x \quad (0 \le i \le k)$ are smaller and smaller and almost orthogonal to each other.

Proposition 1. Let T be an operator in a Hilbert space H such that $0 \in \sigma_{\pi e}(T)$ and $\operatorname{codim} \overline{\mathrm{TH}} < \infty$. Let $k \in \mathbf{N}$ and $\varepsilon > 0$. Then there exists $x \in X$ with $\|x\| = 1$ such that
(1) $\|T^{i+1}x\| \le \varepsilon \|T^i x\| \qquad (i = 0, 1, \dots, k-1)$,
(2) $\|p(T)x\| \ge (1 - \varepsilon)|p(0)| \qquad (p \in \mathcal{P}, \deg p \le k)$.
(3) $|\langle T^i x, T^j x\rangle| \le \varepsilon \|T^i x\| \cdot \|T^j x\| \qquad (0 \le i, j \le k, i \ne j)$.

Remark. The condition $\operatorname{codim} \overline{TH} < \infty$ is only technical and rather weak. If this condition is not satisfied then 0 is an eigenvalue of T^*. In particular T has a non-trivial invariant subspace, so that this case is not interesting (at least from the point of view of the invariant subspace problem).

The following lemma is an important tool for various constructions in Banach spaces. It enables to generalize constructions in Hilbert spaces which use the orthogonal complement of a finite-dimensional subspace to general Banach spaces.

Lemma 2. Let E be a finite-dimensional subspace of a Banach space X and let $\varepsilon > 0$. Then there exists a subspace $Y \subset X$ with $\operatorname{codim} Y < \infty$ such that

$$\|e + y\| \ge (1 - \varepsilon) \max\Big\{\|e\|, \frac{1}{2}\|y\|\Big\}$$

for every $e \in E$ and $y \in Y$.

By using Lemma 2 we can get an analogue to Proposition 1 for operators in Banach spaces.

Proposition 3. Let T be an operator in a Banach space X such that $0 \in \sigma_{\pi e}(T)$ and $\operatorname{codim} \overline{\mathrm{TX}} < \infty$. Let $k \in \mathbf{N}$ and $\varepsilon > 0$. Then there exists $x \in X$ with $\|x\| = 1$ such that
(1) $\|T^{i+1}x\| \le \varepsilon \|T^i x\| \qquad (i = 0, 1, \dots, k-1)$,
(2) $\|p(T)x\| \ge \frac{1-\varepsilon}{2}|p(0)| \qquad (p \in \mathcal{P}, \deg p \le k)$.

By an inductive costruction which uses the previous proposition we can construct a point $x \in X$ (actually, a dense subset of X) such that $\|p(T)x\|$ is big enough for all polynomials p (see [11]).

Theorem 4. Let $T \in B(X)$, $\lambda \in \sigma_{\pi e}(T)$. Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of positive numbers with $\lim_{k\to\infty} a_k = 0$. Then there exists $x \in X$ such that

$$\|p(T)x\| \ge a_{\deg p} \cdot |p(\lambda)|$$

for every polynomial p.

Theorem 5. Let $T \in B(X)$, $\lambda \in \sigma_{\pi e}(T)$, $x \in X$, $\varepsilon > 0$. Let $\{a_k\}_{k=0}^\infty$ be a sequence of positive numbers with $\lim_{k\to\infty} a_k = 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \le \varepsilon$ and

$$\|p(T)y\| \ge C a_{\deg p} \cdot |p(\lambda)|$$

for every polynomial p.

Remark. Let $T \in B(X)$, $\lambda \in \sigma_{\pi e}(T)$ and suppose that there exist $y \in X$ and a constant $c > 0$ such that $\|p(T)y\| \ge c \cdot |p(\lambda)|$ for every polynomial p. Then either $(T-\lambda)y = 0$ or $M = \overline{\{(T-\lambda)p(T)y : p \in \mathcal{P}\}}$ is a non-trivial invariant subspace. Indeed, $y \notin M$ as $\|y - (T-\lambda)p(T)y\| \ge c$ for every polynomial p.

As there are examples of operators in Banach spaces without non-trivial invariant subspaces, in general it is not possible to replace the sequence $\{a_k\}$ by a constant $c > 0$. Thus Theorems 4 and 5 are the best possible, at least for Banach spaces.

Denote by $r(T)$ and $r_e(T)$ the spectral radius and the essential spectral radius of an operator $T \in B(X)$, respectively.

From Theorems 4 and 5 follow easily corresponding results for powers $T^i x$, cf. [9] or [2].

Corollary 6. Let $T \in B(X)$ and let $\{a_k\}_{k=0}^\infty$ be a sequence of positive numbers with $\lim_{k\to\infty} a_k = 0$. Then there exists $x \in X$ such that

$$\|T^k x\| \ge a_k \cdot r(T)^k \qquad (k = 0, 1, \dots).$$

Proof. Let $\lambda \in \sigma_{\pi e}(T)$ with $|\lambda| = \max\{|z| : z \in \sigma(T)\} = r(T)$. Then either λ is an eigenvalue of T and $\|T^k x\| = r(T)^k$ for the corresponding eigenvector x or $\lambda \in \sigma_{\pi e}(T)$ and we can apply Theorem 4.

Corollary 7. Let $T \in B(X)$, $x \in X$, $\varepsilon > 0$ and let $\{a_k\}_{k=0}^\infty$ be a sequence of positive numbers with $\lim_{k\to\infty} a_k = 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \le \varepsilon$ and

$$\|T^k y\| \ge C a_k \cdot r(T)^k \qquad (k = 0, 1, \dots).$$

Corollary 8 (see [17]) **.** Let $T \in B(X)$. Then the set $\{x \in X : r(T, x) = r(T)\}$ is dense in X.

As another corollary we get that the infimum and the supremum in the spectral radius formula

$$r(T) = \inf_{k\in\mathbf{N}} \|T^k\|^{1/k} = \inf_{k\in\mathbf{N}} \sup_{\|x\|=1} \|T^k x\|^{1/k}$$

can be exchanged.

Corollary 9. Let $T \in B(X)$. Then

$$r(T) = \inf_{k \in \mathbf{N}} \sup_{\|x\|=1} \|T^k x\|^{1/k} = \sup_{\|x\|=1} \inf_{k \in \mathbf{N}} \|T^k x\|^{1/k}.$$

II. Capacity

In the previous section we expressed the estimate of $\|p(T)x\|$ by means of $|p(\lambda)|$ where λ was a fixed element of $\sigma_{\pi e}(T)$. In this section we are looking for an estimate in terms of $\max\{|p(\lambda)| : \lambda \in \sigma_{\pi e}(T)\}$. As $\delta\sigma_e(T) \supset \sigma_{\pi e}(T)$ and by the spectral mapping theorem for σ_e we have

$$\max_{\lambda \in \sigma_{\pi e}(T)} |p(\lambda)| = \max_{\lambda \in \sigma_e(T)} |p(\lambda)| = \max\{|z| : z \in \sigma_e(p(T))\} = r_e(p(T)).$$

An important tool for the results in this section is the following classical lemma of Fekete [4]:

Lemma 10. Let K be a non-empty compact subset of the complex plane and let $k \geq 1$. Then there exist points $u_0, u_1, \ldots, u_k \in K$ such that

$$\max\{|p(z)| : z \in K\} \leq (k+1) \cdot \max_{0 \leq i \leq k} |p(u_i)|$$

for every polynomial p with $\deg p \leq k$.

By using the previous lemma and results of the previous section we can get (see [10])

Proposition 11. Let $T \in B(X)$, $\varepsilon \geq 0$ and $k \geq 1$. Then there exists $x \in X$ with $\|x\| = 1$ and

$$\|p(T)x\| \geq \frac{1-\varepsilon}{2(k+1)^2} \, r_e(p(T))$$

for every polynomial p with $\deg p \leq k$.

Theorem 12. Let $T \in B(X)$, $x \in X$ and $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \leq \varepsilon$ and

$$\|p(T)y\| \geq C \cdot (1 + \deg p)^{-(2+\varepsilon)} \, r_e(p(T))$$

for every polynomial p.

Remark. In case of a Hilbert space operator one can get a better estimate

$$\|p(T)y\| \geq C \cdot (1 + \deg p)^{-(1+\varepsilon)} \, r_e(p(T)).$$

The notion of capacity of an operator was defined by Halmos in [5]. If $T \in B(X)$ then

$$\operatorname{cap} T = \lim_{k\to\infty} (\operatorname{cap}_k T)^{1/k} = \inf_k (\operatorname{cap}_k T)^{1/k}$$

where

$$\operatorname{cap}_k T = \inf\left\{\|p(T)\| : p \in \mathcal{P}_k^1\right\}$$

and $\mathcal{P}_k^1$ is the set of all monic (i.e. with leading coefficient equal to 1) polynomials of degree k.

This is a generalization of the classical notion of capacity of a compact subset K of the complex plane:

$$\operatorname{cap} K = \lim_{k\to\infty} (\operatorname{cap}_k K)^{1/k} = \inf_k (\operatorname{cap}_k K)^{1/k}$$

where

$$\operatorname{cap}_k K = \inf\left\{\|p\|_K : p \in \mathcal{P}_k^1\right\} \qquad \text{and} \qquad \|p\|_K = \sup\{|p(z)| : z \in K\}.$$

By the main result of [5], $\operatorname{cap} T = \operatorname{cap} \sigma(T)$.

The local capacity of T at x can be defined analogously:

$$\operatorname{cap}_k(T,x) = \inf\left\{\|p(T)x\| : p \in \mathcal{P}_k^1\right\}$$

and

$$\operatorname{cap}\,(T,x) = \limsup_{k\to\infty} \operatorname{cap}_k(T,x)^{1/k}$$

(in general the limit does not exist).

It is easy to see that $\operatorname{cap}(T,x) \le \operatorname{cap} T$ for every $x \in X$.

Corollary 13. Let $T \in B(X)$. Then the set $\{x \in X : \operatorname{cap}(T,x) = \operatorname{cap} T\}$ is dense in X.

Proof. By Theorem 12 there exists a dense subset $Y \subset X$ such that

$$\|p(T)y\| \ge \frac{C}{(\deg p + 1)^3}\; r_e(p(T))$$

for every polynomial p. Then

$$\operatorname{cap}_k(T,y) = \inf\{\|p(T)y\| : p \in \mathcal{P}_k^1\} \ge \inf_{p\in\mathcal{P}_k^1} \frac{C}{(k+1)^3}\; r_e(p(T))$$

$$= \inf_{p\in\mathcal{P}_k^1} \frac{C}{(n+1)^3}\; \sup\{|p(\lambda)| : \lambda \in \sigma_e(T)\} = \frac{C}{(n+1)^3}\; \operatorname{cap}_k \sigma_e(T).$$

Thus

$$\operatorname{cap}\,(T,y) = \limsup_{k\to\infty} \operatorname{cap}_k(T,y)^{1/k} \ge \limsup_{k\to\infty} \left(\frac{C}{(k+1)^3}\right)^{1/k} (\operatorname{cap}_k \sigma_e(T))^{1/k} = \operatorname{cap}\sigma_e(T). \blacksquare$$

Further $\operatorname{cap} \sigma_e(T) = \operatorname{cap} \sigma(T)$ as $\sigma(T) - \sigma_e(T)$ contains only countably many isolated points in the unbounded component of the complement of $\sigma_e(T)$ and $\operatorname{cap} \sigma(T) = \operatorname{cap} T$ by [5]. Hence $\operatorname{cap}(T, x) = \operatorname{cap} T$ for every $y \in Y$.

An operator $T \in B(X)$ is called quasialgebraic if and only if $\operatorname{cap} T = 0$. Similarly T is called locally quasialgebraic if $\operatorname{cap}(T, x) = 0$ for every $x \in X$.

It follows from Corollary 13 that these two notions are equivalent (see [8]). This gives a positive answer to a problem of Halmos [5].

Theorem 14. An operator is quasialgebraic if and only if it is locally quasialgebraic.

Theorem 14 is an analogy to the well-known result of Kaplansky: an operator is algebraic (i.e. $p(T) = 0$ for some non-zero polynomial p) if and only if it is locally algebraic (i.e. for every $x \in X$ there exists a polynomial $p_x \neq 0$ such that $p_x(T)x = 0$).

III. n-tuples of commuting operators

The results of previous section admit a generalization for n-tuples of commuting operators.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X We denote by $\sigma(T) \subset \mathbf{C}^n$ the Harte spectrum [6] of T, i.e. $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n$ does not belong to $\sigma(T)$ if and only if there exist operators $L_1, \dots, L_n, R_1, \dots, R_n \in B(X)$ such that

$$\sum_{i=1}^n L_i(T_i - \lambda_i) = I = \sum_{i=1}^n (T_i - \lambda_i) R_i.$$

Denote further by $\sigma_{\pi e}(T)$ the essential approximate point spectrum of the n-tuple T, i.e. $\lambda = (\lambda_1, \dots, \lambda_n) \in \sigma_{\pi e}(T)$ if and only if

$$\inf \left\{ \sum_{i=1}^n \|(T_i - \lambda_i)x\| : x \in M, \|x\| = 1 \right\} = 0$$

for every subspace M of finite codimension.

The following result is a generalization of Theorem 12 for n-tuples of commuting operators (see [14]):

Theorem 15. Let $T = (T_1, \dots, T_n) \in B(X)^n$ be a mutually commuting n-tuple of operators. Let $x \in X$ and $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \leq \varepsilon$ and

$$\|p(T)y\| \geq \frac{C}{(1 + \deg p)^{2n+\varepsilon}}\, r_e(p(T))$$

for every polynomial p with n variables.

Every polynomial p in n complex variables with $\deg p \leq k$ can be written in the form

$$p(z) = \sum_{|\alpha| \leq k} c_\alpha(p) z^\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is an n-tuple of non-negative integers, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, the coefficients $c_\alpha(p)$ are complex, $z = (z_1, \dots, z_n) \in C^n$ and $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$.

The notion of capacity of commuting n-tuples of operators was introduced by Stirling [16]:

Denote by $\mathcal{P}_k^1(n)$ the set of all "monic" polynomials $p(z) = \sum_{|\mu| \le k} c_\mu(p) z^\mu$ of degree k in n variables with $\sum_{|\mu|=k} |c_\mu(p)| = 1$.

Let $T = (T_1, \dots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. The joint capacity of T was defined in [16] by

$$\operatorname{cap} T = \liminf_{k \to \infty} \operatorname{cap}_k(T)^{1/k} \qquad \text{where} \qquad \operatorname{cap}_k(T) = \inf \left\{ \|p(T)\| : p \in \mathcal{P}_k^1(n) \right\}$$

(in fact the liminf in the definition of $\operatorname{cap} T$ can be replaced by limit, see [12]]).

For a compact subset $K \subset C^n$ define the corresponding capacity by

$$\operatorname{cap} K = \lim_{k \to \infty} \operatorname{cap}_k(K)^{1/k} \qquad \text{where} \qquad \operatorname{cap}_k(K) = \inf \left\{ \|p\|_K : p \in \mathcal{P}_k^1(n) \right\}.$$

This capacity was studied in [16] and called the "homogeneous Tshebyshev constant" of K.

By [16], $\operatorname{cap} \sigma(T) \le \operatorname{cap} T \le 2^n \operatorname{cap} \sigma(T)$. Actually, the equality holds here, see [12].

Theorem 16. Let $T = (T_1, \dots, T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Then $\operatorname{cap} T = \operatorname{cap} \sigma(T)$.

Theorem 17 (see [13])**.** Let T be an n-tuple of mutually commuting operators in a Banach space X. Then $\sigma(T) - \hat{\sigma}_{\pi e}(T)$ consist of at most countable isolated joint eigenvalues, where $\hat{\sigma}_{\pi e}(T)$ denotes the polynomially convex hull of $\sigma_{\pi e}(T)$. In particular, $\operatorname{cap} \sigma(T) = \operatorname{cap} \sigma_{\pi e}(T)$ (actually all reasonable joint spectra have the same capacity).

Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators in a Banach space X and let $x \in X$. We define the local capacity $\operatorname{cap}(T, x)$ by

$$\operatorname{cap}(T, x) = \limsup_{k \to \infty} \operatorname{cap}_k(T, x)^{1/k}$$

where

$$\operatorname{cap}_k(T, x) = \inf \left\{ \|p(T)x\| : p \in \mathcal{P}_k^1(n) \right\}.$$

Clearly $\operatorname{cap}(T, x) \le \operatorname{cap} T$ for every $x \in X$.

Theorem 18 (see [14])**.** Let T be an n-tuple of mutually commuting operators in a Banach space X. Then the set of all $y \in X$ with $\operatorname{cap}(T, y) = \operatorname{cap} T$ is dense in X.

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