On bounded local resolvents

J. Bračič and V. Müller

ABSTRACT. It is known that each normal operator on a Hilbert space with non-empty interior of the spectrum admits vectors with bounded local resolvent. We generalize this result for Banach space operators with the decomposition property (δ) (in particular for decomposable operators). Moreover, the same result holds for operators with interior points in the localizable spectrum.

Let X be a complex Banach space and $B(X)$ the Banach algebra of all bounded linear operators on X. Let $T \in B(X)$. It is well known that the resolvent mapping $(T - z)^{-1}$, which is defined and analytic on the resolvent set $\rho(T)$, is unbounded. On the other hand, the behavior of local resolvent functions may be quite different. In [BG], Bermúdez and González [BG] have shown that a normal operator N on a separable Hilbert space has a non-trivial bounded local resolvent function if and only if the interior of the spectrum of N is not empty, i.e., $\operatorname{Int} \sigma(N) \neq \emptyset$. Neumann [N] extended this result to non-separable spaces, and proved a similar result for multiplication operators induced by a given continuous function on the Banach algebra $C(\Omega)$ of all continuous complex-valued functions on a compact Hausdorff space Ω.

In this article we show that there is a quite large class of bounded operators on a complex Banach space that have non-trivial bounded local resolvent functions. In particular, every decomposable operator T with $\operatorname{Int} \sigma(T) \neq \emptyset$ has this property. On the other hand, there is a decomposable operator T with $\operatorname{Int} \sigma(T) = \emptyset$, which admits a local resolvent funcion that is not only bounded but can be even continuously extended to the whole complex plane.

Before we state our main results we are going to introduce some notation and terminology from local spectral theory (the reader is referred to [LN] for details).

An operator $T \in B(X)$ is said to have the single-valued extension property (SVEP) if, for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f: U \to X$ of the equation

$$(T - z)f(\lambda) = 0 \qquad (z \in U)$$

is the function $f \equiv 0$.

The local resolvent set $\rho_T(x)$ of an operator T with SVEP at $x \in X$ is defined as the set of all $w \in \mathbb{C}$, for which there exists an analytic function $f: U \to X$ on an open neighbourhood U of w such that $(T - z)f(z) = x$ for all $z \in U$. Let $f(z) = \sum_{i=1}^{\infty} x_i (z - w)^i \quad (z \in U)$ be the Taylor expansion of f. Comparing the coefficients, it is easy to see that $w \in \rho_T(x)$ if and only if there are vectors $x_1, x_2, \ldots \in X$ such that $(T - w)x_{i+1} = x_i \quad (i \geq 1)$, $(T - w)x_1 = x$ and $\sup_i \|x_i\|^{1/i} < \infty$.

The research was supported by grant No. 04-2003-04, Programme Kontakt of Czech and Slovenian Ministries of Education. The second author was also supported by grant No. 201/03/0041 of GA ČR.

Keywords and phrases: local spectrum, bounded local resolvent, localizable spectrum.

Mathematics Subject Classification: primary 47A11, 47A10.

The set $\sigma_T(x) = \mathbb{C} \setminus \rho_T(x)$ is called the local spectrum of T at x. It is well known that, for any $x \in X$, the local spectrum of T at x is contained in the spectrum $\sigma(T)$, or, equivalently, $\rho(T) \subseteq \rho_T(x)$. It is well known that $\sigma_T(x)$ is always closed; if $x \neq 0$ then $\sigma_T(x)$ is also nonempty. It is easy to see that $\sigma_T(x+y) \subset \sigma_T(x) \cup \sigma_T(y)$ for all $x, y \in X$. Moreover, if $\sigma_T(x) \cap \sigma_T(y) = \emptyset$, then $\sigma_T(x+y) = \sigma_T(x) \cup \sigma_T(y)$.

If T has SVEP, then, for every $x \in X$, there exists a unique analytic function $R_T(\cdot, x) : \ \rho_T(x) \to X$ such that $(T-z)R_T(z,x) = x$ for all $z \in \rho_T(x)$. This function is called the local resolvent function of T at x, and satisfies $R_T(z,x) = (T-z)^{-1}x$ for all $z \in \rho(T)$.

An operator $T \in B(X)$ is said to have the decomposition property (δ) if, given an arbitrary open cover $\{U_1, U_2\}$ of $\mathbb{C}$, every $x \in X$ admits a decomposition $x = u_1 + u_2$ where u_k $(k = 1,2)$ satisfies $u_k = (T-z)f_k(z)$ for all $z \in \mathbb{C} \setminus \overline{U}_k$ and some analytic function $f_k : \mathbb{C} \setminus \overline{U}_k \to X$. If for every open cover $\{U_1, U_2\}$ of $\mathbb{C}$ there exists a pair of closed linear subspaces Y_1 and Y_2 in X such that they are invariant for $T \in B(X)$, their sum is X, and the spectrum of the restricted operator $T|Y_k$ is contained in U_k $(k = 1,2)$, then T is said to be decomposable.

Denote by $\operatorname{Im} S$ the range of an operator $S \in B(X)$.

Our first theorem says that an operator on a Banach space that has nice spectral properties and whose spectrum has nonempty interior admits a non-trivial bounded local resolvent function.

Theorem 1. Let $T \in B(X)$ have SVEP and the decomposition property (δ). Assume that there exists a nonempty open set $U \subset \sigma(T)$. Then there exists $x \in X$ such that $\sigma_T(x) = \overline{U}$ and the local resolvent function $R_T(z,x)$ is bounded on $\rho_T(x)$.

Proof. Choose a sequence $(\lambda_n)_{n=1}^{\infty} \subset U$ which is dense in $\overline{U}$ and such that $\lambda_i \neq \lambda_j \quad (i \neq j)$. We shall construct a sequence of vectors $(x_n)_{n=1}^{\infty} \subset X$ such that $\lambda_n \in \sigma_T(x_n) \subset U$, $x_n \notin \operatorname{Im}(T-\lambda_n)$, and $x_n \in \operatorname{Im}(T-\lambda_j)$, for all $1 \leq j < n$.

Let $n \in \mathbb{N}$. The property SVEP implies that $\operatorname{Im}(T-\lambda_n) \neq X$ ([LN], Proposition 1.3.2 (f)). Choose $u \in X \setminus \operatorname{Im}(T-\lambda_n)$. Let V and V' be open sets such that

$$\lambda_n \in V \subset \overline{V} \subset V' \subset \overline{V'} \subset U.$$

Consider the open cover $\{V', \mathbb{C} \setminus \overline{V}\}$ of the complex plane. Since T has property (δ), there are $v, w \in X$ and analytic functions $f : \mathbb{C} \setminus \overline{V'} \to X$ and $g : V \to X$ such that $u = v + w$, $v = (T-z)f(z) \quad (z \in \mathbb{C} \setminus \overline{V'})$, and $w = (T-z)g(z) \quad (z \in V)$. Therefore $w = (T-\lambda_n)g(\lambda_n) \in \operatorname{Im}(T-\lambda_n)$. For $n = 1$, let $x_1 = v$. For $n \geq 2$, let $x_n = (T-\lambda_1)\cdots(T-\lambda_{n-1})v$. Then $\sigma_T(x_n) \subset \sigma_T(v) \subset \overline{V'} \subset U$ and $x_n \in \operatorname{Im}(T-\lambda_j)$ for all $j < n$.

Note that $u \notin \operatorname{Im}(T-\lambda_n)$, $w \in \operatorname{Im}(T-\lambda_n)$, and so $v \notin \operatorname{Im}(T-\lambda_n)$. Since the polynomials $z-\lambda_n$ and $(z-\lambda_1)\cdots(z-\lambda_{n-1})$ are relatively prime, there are polynomials $q_1(z)$ and $q_2(z)$ such that

$$(z-\lambda_n)q_1(z) + (z-\lambda_1)\cdots(z-\lambda_{n-1})q_2(z) = 1.$$

Thus

$$(T-\lambda_n)q_1(T)v + (T-\lambda_1)\cdots(T-\lambda_{n-1})q_2(T)v = v.$$

Hence $q_2(T)x_n = (T-\lambda_1)\cdots(T-\lambda_{n-1})q_2(T)v \notin \operatorname{Im}(T-\lambda_n)$, and so $x_n \notin \operatorname{Im}(T-\lambda_n)$.

Since $\sigma_T(x_n) \subset U$, we have $\sup_{z \notin \overline{U}} \|R_T(z, x_n)\| < \infty$.

Now we construct inductively a sequence of positive numbers $(\alpha_n)_{n=1}^{\infty}$ such that

$$\begin{aligned}
&\|\alpha_n x_n\| \leq 2^{-n}, \\
&\sup_{z \notin \overline{U}} \|R_T(z, \alpha_n x_n)\| \leq 2^{-n}, \\
&\sum_{i=1}^{n} \alpha_n x_n \notin \operatorname{Im}(T - \lambda_n), \\
&\alpha_n x_n \in 2^{-n}(T - \lambda_j) B_X \quad (j < n),
\end{aligned} \tag{1}$$

where B_X denotes the closed unit ball in X.

It is obvious that there exists a positive number α_1 that satisfies (1). Suppose that the numbers $\alpha_1, \ldots, \alpha_{n-1}$ satisfying (1) have already been constructed. Since $x_n \notin \operatorname{Im}(T - \lambda_n)$, there is at most one $\mu > 0$ such that $\sum_{i=1}^{n-1} \alpha_i x_i + \mu x_n \in \operatorname{Im}(T - \lambda_n)$. Thus (1) is satisfied for all positive α_n which are small enough.

Let the numbers α_n be constructed in the above described way. Set $x = \sum_{i=1}^{\infty} \alpha_i x_i$. For $z \notin \overline{U}$ we have

$$\sum_{i=1}^{\infty} \|R_T(z, \alpha_i x_i)\| \leq \sum_{i=1}^{\infty} 2^{-i} = 1.$$

Consequently, $\sigma_T(x) \subset \overline{U}$ and $\sup_{z \notin \overline{U}} \|R_T(z, x)\| \leq 1$. It remains to show that $\sigma_T(x) = \overline{U}$.

For each $n \in \mathbb{N}$ we have $\sum_{i=1}^{n} \alpha_i x_i \notin \operatorname{Im}(T - \lambda_n)$ and

$$\sum_{i=n+1}^{\infty} \alpha_i x_i \in \sum_{i=n+1}^{\infty} 2^{-i}(T - \lambda_n) B_X \subset 2^{-n}(T - \lambda_n) B_X \subset \operatorname{Im}(T - \lambda_n).$$

Hence $x \notin \operatorname{Im}(T - \lambda_n)$, and therefore $\lambda_n \in \sigma_T(x)$. We conclude that $\sigma_T(x) = \overline{U}$ and consequently that the local resolvent function of T at x is bounded. □

An immediate consequence of the previous theorem is the following corollary.

Corollary 2. Let $T \in B(X)$ be a decomposable operator. If $\operatorname{Int} \sigma(T) \neq \emptyset$, then there exists a nonzero $x \in X$ such that $R_T(z, x)$ is bounded on $\rho_T(x)$.

In the proof of Theorem 1 we have not used the full strength of property (δ). In fact it is sufficient that the points of the set U are separated by local spectra. For this purpose it can be used the concept of localizable spectrum, see [EP].

Let $T \in B(X)$ be an operator with SVEP. The localizable spectrum $\sigma_{loc}(T)$ of T is the set of all complex numbers λ with the following property: for each open neighbourhood V of λ there exists a nonzero vector $x \in X$ such that $\sigma_T(x) \subset V$.

Clearly $\sigma_{loc}(T)$ is a closed subset of $\sigma(T)$. For operators with property (δ) it is easy to see that $\sigma_{loc}(T) = \sigma(T)$. On the other hand, for an arbitrary operator $T \in B(X)$, the localizable spectrum $\sigma_{loc}(T)$ is always contained in the approximate point spectrum $\sigma_{ap}(T)$. Indeed, let $\lambda \in \sigma_{loc}(T)$. For each open neighbourhood V there is a nonzero vector $x \in X$ with $\sigma_T(x) \subset V$. Take $\mu \in \partial\sigma_T(x)$. By Theorem

3.1.12 in [LN], $\mu \in \sigma_{ap}(T)$. Since V was an arbitrary neighbourhood of λ, we conclude that $\lambda \in \sigma_{ap}(T)$. In particular, this observation implies the well known fact that $\sigma_{ap}(T) = \sigma(T)$ whenever T has property (δ).

On the other hand, there exist operators with empty localizable spectrum. Namely, let $T \in B(X)$ be an operator with more than two points in the spectrum and with the property that $\sigma_T(x) = \sigma(T)$ for all nonzero vectors x (the existence of such an operator follows, for instance, from Proposition 1.6.9 in [LN]). It is clear that $\sigma_{loc}(T)$ is empty.

Denote by $\sigma_p(T)$ the point spectrum of an operator $T \in B(X)$.

Lemma 3. Let $T \in B(X)$ be an operator with SVEP. Then for each $\lambda \in \operatorname{Int} \sigma_{loc}(T)$ and an open neighbourhood V of λ there is a vector $x \in X$ such that $\lambda \in \sigma_T(x) \subset \overline{V}$.

Proof. Choose a positive number r_0 such that $\{z \in \mathbb{C} : |z - \lambda| \le r_0\} \subset V \cap \operatorname{Int} \sigma_{loc}(T)$ and let $(r_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that $r_0 > r_1 > r_2 > \cdots$ and $\lim_{n\to\infty} r_n = 0$. For $n \in \mathbb{N}$ let $V_n = \{z \in \mathbb{C} : r_n < |z - \lambda| < r_{n-1}\}$. Since $V_n \subset \sigma_{loc}(T)$ for each n, there exist, by the assumption, unit vectors $x_n \in X$ such that $\sigma_T(x_n) \subset V_n \quad (n \in \mathbb{N})$.

Choose positive numbers $\alpha_1, \alpha_2, \ldots$ such that $\alpha_n \le 2^{-n}$ and

$$\sup\bigl\{\|R_T(z, \alpha_n x_n)\| : |z - \lambda| \ge r_{n-1}\bigr\} \le 2^{-n}.$$

Set $x = \sum_{i=1}^{\infty} \alpha_i x_i$. Clearly

$$\sup\bigl\{\|R_T(z, x)\| : z \in \mathbb{C} \setminus V\bigr\} \le \sum_{i=1}^{\infty} \sup\bigl\{\|R_T(z, \alpha_i x_i)\| : z \in \mathbb{C} \setminus V\bigr\} \le \sum_{i=1}^{\infty} 2^{-i} < \infty,$$

and so $\sigma_T(x) \subset \overline{V}$.

For each n we have in the same way that

$$\sup\Bigl\{\Bigl\|R_T\Bigl(z, \sum_{i=n+1}^{\infty} \alpha_i x_i\Bigr)\Bigr\| : |z - \lambda| \ge r_n\Bigr\} < \infty,$$

and so $\sigma_T(\sum_{i=n+1}^{\infty} \alpha_i x_i) \subset \{z : |z - \lambda| \le r_n\}$. Since the local spectra of the elements $\alpha_1 x_1, \ldots, \alpha_n x_n$ and $\sum_{i=n+1}^{\infty} \alpha_i x_i$ are mutually disjoint, we have

$$\sigma_T(x) = \bigcup_{i=1}^{n} \sigma_T(x_i) \cup \sigma_T\Bigl(\sum_{i=n+1}^{\infty} \alpha_i x_i\Bigr) \supset \sigma_T(x_n).$$

Thus there exists a number $\mu_n \in \sigma_T(x_n)$ which is also in $\sigma_T(x) \cap V_n$. Since $\mu_n \to \lambda$, we conclude that $\lambda \in \sigma_T(x)$. □

Theorem 4. Let $T \in B(X)$, let the point spectrum of T have empty interior and assume that $U \subset \sigma_{loc}(T)$ is a nonempty open subset. Then there exists $x \in X$ such that $\sigma_T(x) = \overline{U}$ and $R_T(z, x)$ is bounded on $\rho_T(x)$. Moreover, for every $u \in X$ with $\sigma_T(u) \subset U$ and every $\varepsilon > 0$ there exists $x \in X$ such that $\|x - u\| \le \varepsilon$, $\sigma_T(x) = \overline{U}$, and the local resolvent function of T at x is bounded.

Proof. Note first that T has SVEP since the interior of the point spectrum is empty. By the same reason we can choose a dense sequence $(\lambda_n)_{n=1}^\infty \subset U$ such that the kernel of $T-\lambda_n$ is trivial for each n. Moreover, we can assume that $\lambda_i \neq \lambda_j \quad (i \neq j)$.

Let $\varepsilon > 0$ and let $u \in X$ satisfy $\sigma_T(u) \subset U$. By assumption, we can find a vector $x_1 \in X$ with $\|x_1\| = 1$ and $\lambda_1 \in \sigma_T(x_1) \subset U$. Similarly, for each $n \geq 2$, there exists a vector $x_n \in X$ of norm one such that $\lambda_n \in \sigma_T(x_n) \subset U \setminus \{\lambda_1, \dots, \lambda_{n-1}\}$.

Now we construct inductively a sequence $(\alpha_n)_{n=1}^\infty$ of non-negative numbers, a subset $M \subset \mathbb{N}$, for each $n \in M$ a nonnegative integer a_n, and, for every $n \in \mathbb{N} \setminus M$ and $k \in \mathbb{N}$, a positive integer $m(n,k)$ such that the following conditions will be fulfilled.

(i) $\quad \|\alpha_n x_n\| \leq 2^{-n}\varepsilon$;

(ii) $\quad \sup_{z \notin \overline{U}} \|R_T(z, \alpha_n x_n)\| \leq 2^{-n}$;

(iii) $\quad \lambda_n \in \sigma_T\Big(u + \sum_{i=1}^n \alpha_i x_i\Big)$;

(iv) $\quad \alpha_n x_n \in 2^{-n}(T-\lambda_j)^{a_j+1} B_X \qquad (j < n, j \in M)$;

(v) $\quad \big\|(T-\lambda_j)^{-m(j,k)} \alpha_n x_n\big\| \leq 2^{-n} k^{m(j,k)} \qquad (j < n, j \notin M, k \in \mathbb{N})$;

(vi) $\quad n \in M \Leftrightarrow u + \sum_{i=1}^n \alpha_i x_i \notin \bigcap_{k=1}^\infty \mathrm{Im}\,(T-\lambda_j)^k$,
and, if $n \in M$, then $a_n = \max\{k : u + \sum_{i=1}^n \alpha_i x_i \in \mathrm{Im}\,(T-\lambda_n)^k\}$;

(vii) $\quad \Big\|(T-\lambda_n)^{-m(n,k)}\Big(u + \sum_{i=1}^n \alpha_i x_i\Big)\Big\| \geq k^{m(n,k)} \qquad (n \notin M, k \in \mathbb{N})$.

Let $n \geq 1$ and suppose that the numbers $\alpha_1, \dots, \alpha_{n-1}$, the set $M \cap \{1, \dots, n-1\}$ and the numbers $a_j \quad (j \leq n-1, j \in M)$ and $m(j,k) \quad (j \leq n-1, j \in \mathbb{N} \setminus M, k \in \mathbb{N})$ satisfying (i)-(vii) have already been constructed. We distinguish two cases:

(a) If $\lambda_n \in \sigma_T\Big(u + \sum_{i=1}^{n-1} \alpha_i x_i\Big)$, set $\alpha_n = 0$. Then (i)-(v) are satisfied trivially.

(b) Suppose that $\lambda_n \notin \sigma_T\Big(u + \sum_{i=1}^{n-1} \alpha_i x_i\Big)$. Since $\lambda_n \in \sigma_T(x_n)$, (iii) is satisfied for each positive α_n. Since $\sigma_T(x_n) \subset U \setminus \{\lambda_1, \dots, \lambda_{n-1}\}$, we have $x_n \in \bigcap_k \mathrm{Im}\,(T-\lambda_j)^k$ for all $j < n$. Thus (i), (ii) and (iv) are satisfied for all $\alpha_n > 0$ which are small enough.

For each $j < n$, $j \notin M$, we have $\lambda_j \notin \sigma_T(x_n)$, and therefore $x_n \in \bigcap_{k=1}^\infty \mathrm{Im}\,(T-\lambda_j)^k$ and

$$\sup_m \big\|(T-\lambda_j)^{-m} x_n\big\|^{1/m} < \infty.$$

Thus there is a positive constant c_j such that $\|(T-\lambda_j)^{-m} x_n\| \leq c_j^m$ for all $m \geq 1$. Hence for $\alpha_n > 0$ small enough we have

$$\|(T-\lambda_j)^{-m(j,k)} \alpha_n x_n\| \leq \alpha_n c_j^{m(j,k)} \leq 2^{-n} k^{m(j,k)}$$

for all $k \in \mathbb{N}$. Consequently, (v) is satisfied for all $\alpha_n > 0$ which are small enough.

In both cases (a) and (b) we can choose the number $\alpha_n \geq 0$ satisfying (i) – (v).

We include the number n into the set M if and only if

$$u + \sum_{i=1}^n \alpha_i x_i \notin \bigcap_{k=1}^\infty \mathrm{Im}\,(T-\lambda_n)^k.$$

In this case we define $a_n = \max\Big\{k : u + \sum_{i=1}^n \alpha_i x_i \in \mathrm{Im}\,(T-\lambda_n)^k\Big\}$.

Suppose that $n \notin M$, that is, $u + \sum_{i=1}^n \alpha_i x_i \in \bigcap_k \mathrm{Im}\,(T-\lambda_n)^k$. Since $T-\lambda_n$ is injective and $\lambda_n \in \sigma_T\Big(u + \sum_{i=1}^n \alpha_i x_i\Big)$, we have

$$\sup_m \Big\|(T-\lambda_n)^{-m}\Big(u + \sum_{i=1}^n \alpha_i x_i\Big)\Big\|^{1/m} = \infty.$$

Therefore for each $k \in \mathbb{N}$ there is an $m(n,k) \in \mathbb{N}$ such that

$$\Big\|(T-\lambda_n)^{-m(n,k)}\Big(u + \sum_{i=1}^n \alpha_i x_i\Big)\Big\| \geq k^{m(1,k)}.$$

Let the sequence $(\alpha_n)_{n=1}^\infty$ be constructed in the above described way. Set $x = u + \sum_{i=1}^\infty \alpha_i x_i$. Clearly $\|x-u\| = \Big\|\sum_{i=1}^\infty \alpha_i x_i\| \leq \sum_{i=1}^\infty 2^{-i}\varepsilon = \varepsilon$. Furthermore,

$$\begin{aligned}\sup_{z\notin \overline{U}} \|R_T(z,x)\| &\leq \sup_{z\notin \overline{U}} \|R_T(z,u)\| + \sum_{i=1}^\infty \sup_{z\notin \overline{U}} \|R_T(z,\alpha_i x_i)\| \\ &\leq \sup_{z\notin \overline{U}} \|R_T(z,u)\| + \sum_{i=1}^\infty 2^{-i} < \infty.\end{aligned}$$

Hence $\sigma_T(x) \subset \overline{U}$ and the local resolvent $R_T(z,x)$ is bounded on $\mathbb{C}\setminus\overline{U}$.

It is sufficient to show that $\sigma_T(x) = \overline{U}$. Let $n \in \mathbb{N}$. If $n \in M$ then $u + \sum_{i=1}^n \alpha_i x_i \notin \mathrm{Im}\,(T-\lambda_n)^{a_n+1}$ and, by (iv),

$$\sum_{i=n+1}^\infty \alpha_i x_i \in \sum_{i=n+1}^\infty 2^{-i}(T-\lambda_n)^{a_n+1}B_X \subset (T-\lambda_n)^{a_n+1}B_X.$$

Consequently, $x \notin \mathrm{Im}\,(T-\lambda_n)^{a_n+1}$, and so $\lambda_n \in \sigma_T(x)$.

Let $n \in \mathbb{N}\setminus M$. Then the kernel of $T-\lambda_n$ is trivial and

$$\begin{aligned}&\sup_m \Big\|(T-\lambda_n)^{-m}x\Big\|^{1/m} \geq \sup_k \Big\|(T-\lambda_n)^{-m(n,k)}x\Big\|^{1/m(n,k)} \\ &\geq \sup_k \Big(\Big\|(T-\lambda_n)^{-m(n,k)}\Big(u+\sum_{i=1}^n \alpha_i x_i\Big)\Big\| - \Big\|(T-\lambda_n)^{-m(n,k)}\sum_{i=n+1}^\infty \alpha_i x_i\Big\|\Big)^{1/m(n,k)} \\ &\geq \sup_k \Big(k^{m(n,k)} - \sum_{i=n+1}^\infty 2^{-i}k^{m(n,k)}\Big)^{1/m(n,k)} \geq \sup_k 2^{-1/m(n,k)}\cdot k = \infty.\end{aligned}$$

Hence $\lambda_n \in \sigma_T(x)$. Thus $\sigma_T(x) = \overline{U}$ and $R_T(z,x)$ is bounded on $\rho_T(x)$. □

Problem. Is it possible to replace in Theorem 4 the assumption that the point spectrum of T has empty interior by the condition SVEP?

We have seen that there is quite large class of operators that admit a nontrivial bounded local resolvent function. In the opposite direction we have the following result.

Let X be a Banach space. We consider X to be canonically embedded into its second dual X^{**}.

Theorem 5. Suppose that $T \in B(X)$ has SVEP. If the local resolvent function of T at $x \in X$ is bounded, then

$$x \in \bigcap_{z \in \partial\sigma(T)} \mathrm{Im}\,(T^{**} - z). \tag{2}$$

Proof. Let $\lambda \in \partial\sigma(T)$. If $\lambda \notin \partial\sigma_T(x)$, then $\lambda \notin \sigma_T(x)$, which gives $x \in \mathrm{Im}\,(T-\lambda) \subset \mathrm{Im}\,(T^{**}-\lambda)$.

Suppose now that $\lambda \in \partial\sigma_T(x)$. Then there is a sequence of complex numbers $\lambda_n \notin \sigma_T(x)$ converging to λ. Since $R_T(z,x)$ is bounded, the vectors $x_n = R_T(\lambda_n, x)$ form a bounded sequence in X. Let $u \in X^{**}$ be a w^*-accumulation point of this sequence. Then $\liminf_{n\to\infty} |\langle v^*, u - x_n\rangle| = 0$ for all $v^* \in X^*$. Note that $(T-\lambda_n)x_n = x$ for all n. Thus, for an arbitrary $x^* \in X^*$, we have

$$\begin{aligned}
0 &= \liminf_{n\to\infty}\big|\big\langle (T^*-\lambda)x^*, u - x_n\big\rangle\big| = \liminf_{n\to\infty}\big|\big\langle x^*, (T^{**}-\lambda)u - (T-\lambda)x_n\big\rangle\big| \\
&= \liminf_{n\to\infty}\big|\big\langle x^*, (T^{**}-\lambda)u - (T-\lambda_n)x_n + (\lambda-\lambda_n)x_n\big\rangle\big| \\
&= \liminf_{n\to\infty}\big|\big\langle x^*, (T^{**}-\lambda)u - x + (\lambda-\lambda_n)x_n\big\rangle\big| = |\langle x^*, (T^{**}-\lambda)u - x\rangle|.
\end{aligned}$$

Hence $\langle x^*, (T^{**}-\lambda)u - x\rangle = 0$. Since $x^* \in X^*$ was arbitrary, we have $x = (T^{**}-\lambda)u \in \mathrm{Im}\,(T^{**}-\lambda)$. This completes the proof of Theorem 5. □

Note that, by Example 1 in [BG], (2) cannot be replaced by $x \in \bigcap_{z\in\partial\sigma(T)} \mathrm{Im}\,(T-z)$. Of course, if X is reflexive, then $\bigcap_{z\in\partial\sigma(T)} \mathrm{Im}\,(T-z) = \bigcap_{z\in\partial\sigma(T)} \mathrm{Im}\,(T^{**}-z)$.

Corollary 6. Let $T \in B(X)$ have SVEP. Then the set of all vectors $x \in X$ with bounded local resolvent is of the first category in X.

Proof. Denote by M the set of all vectors $x \in X$ with bounded local resolvent. Choose $\lambda \in \partial\sigma(T)$. Then $M \subset \mathrm{Im}\,(T^{**}-\lambda) \cap X$. It is enough to show that $\mathrm{Im}\,(T^{**}-\lambda) \cap X$ is of the first category in X.

Let $X_0 = (T^{**}-\lambda)^{-1}X$. Clearly X_0 is a closed subspace of X^{**} and $\mathrm{Im}\,(T^{**}-\lambda) \cap X = (T^{**}-\lambda)X_0$. It is sufficient to show that $(T^{**}-\lambda)X_0 \neq X$. Suppose on the contrary that $(T^{**}-\lambda)X_0 = X$. By the open mapping theorem, there is a constant $c > 0$ such that for each $x \in B_X$ there is an $x^{**} \in X_0$ with $(T^{**}-\lambda)x^{**} = x$ and $\|x^{**}\| \le c$.

Let $u^* \in X^*$, $\|u^*\| = 1$. Then $1 = \|u^*\| = \sup_{x\in B_X} |\langle u^*, x\rangle|$. For $x \in B_X$ let $x^{**} \in X_0$ be the element described above. Then

$$|\langle u^*, x\rangle| = |\langle u^*, (T^{**}-\lambda)x^{**}\rangle| = |\langle (T^*-\lambda)u^*, x^{**}\rangle| \le c \cdot \|(T^*-\lambda)u^*\|.$$

Thus $\|(T^*-\lambda)u^*\| \ge c^{-1}$ and $(T^*-\lambda)$ is bounded below. This is a contradiction with the assumption that $\lambda \in \partial\sigma(T) = \partial\sigma(T^*)$. □

By Theorem 1.5.7 in [LN], we have $\bigcap_{z\in\mathbb{C}} \operatorname{Im}(N-z)=\{0\}$, for a normal operator N on a Hilbert space. Thus, a combination of Theorems 1 and 5 gives the following corollary, see [BG], [N].

Corollary 7. A normal operator N on a Hilbert space has a non-trivial bounded local resolvent function if and only if $\operatorname{Int}\sigma(N)\neq\emptyset$.

An analogous statement for decomposable operators is not true. An example of a decomposable operator whose spectrum has no interior points but which admits bounded local resolvent functions was constructed in [BG], Example 1 (note that the operator constructed there is decomposable since it is an invertible isometry). We give below another example of this kind. Moreover, the example has an additional property that the local resolvent is not only bounded but admits a continuous extension to the whole complex plane.

The authors are indebted to Dan Timotin for the main argument in the following construction.

Example 8. Let m denote the Lebesgue measure in $\mathbb{C}$. For $\lambda\in\mathbb{C}$ and $r>0$ let $D(\lambda,r)=\{z\in\mathbb{C}:|z-\lambda|<r\}$.

Let $M\subset\mathbb{C}$ be a compact set such that $\operatorname{Int}M=\emptyset$ and $m(T)>0$ (for example, set $M=\overline{D(0,1)}\setminus\bigcup_n D(\lambda_n,2^{-n})$, where (λ_n) is a dense sequence in $D(0,1)$).

Let μ be the restriction of m to M, $X=\ell_1(\mu)$ and let $T\in B(X)$ be the multiplication operator defined by $(Tf)(z)=zf(z)\quad(z\in M, f\in X)$. Then T is decomposable and $\sigma(T)=\operatorname{supp}\mu\subset M$. So $\operatorname{Int}\sigma(T)=\emptyset$. Let $g\in X$ be the constant function equal to 1. For $\lambda\in\mathbb{C}$ define $H(\lambda)\in X$ by $H(\lambda)=\frac{1}{z-\lambda}$.

For each $\lambda\in\mathbb{C}$ we have

$$\|H(\lambda)\|_X=\int\frac{\mathrm{d}\,\mu(z)}{|z-\lambda|}\le\int_{D(\lambda,1)}\frac{\mathrm{d}\,m(z)}{|z-\lambda|}+\int_{M\setminus D(\lambda,1)}\frac{\mathrm{d}\,\mu(z)}{|z-\lambda|}\le 2\pi+m(M).$$

Hence $H:\mathbb{C}\to X$ is a well-defined bounded function. Since $\sigma_p(T)=\emptyset$, H is an extension of the local resolvent $R_T(\cdot,g)$. Moreover, the function H is continuous. Let $\lambda,\lambda'\in\mathbb{C}$. Let $a=\frac{\lambda+\lambda'}{2}$, $\varepsilon=\frac{\lambda-\lambda'}{2}$ and $R=\operatorname{diam}M$. Then

$$\begin{aligned}
\|H(\lambda)-H(\lambda')\|_X&=\int\Big|\frac{1}{z-\lambda}-\frac{1}{z-\lambda'}\Big|\mathrm{d}\,\mu(z)\le\int_{D(a,R)}\Big|\frac{1}{z-\lambda}-\frac{1}{z-\lambda'}\Big|\mathrm{d}\,m(z)\\
&\le\int_{|z|\le R}\Big|\frac{1}{z-\varepsilon}-\frac{1}{z+\varepsilon}\Big|\mathrm{d}\,m(z)\\
&\le\int_{|z|\le 2|\varepsilon|}\Big|\frac{1}{z-\varepsilon}-\frac{1}{z+\varepsilon}\Big|\mathrm{d}\,m(z)+2|\varepsilon|\int_{2|\varepsilon|\le|z|\le R}\frac{\mathrm{d}\,m(z)}{|z^2-\varepsilon^2|}\\
&\le 2\int_{|z|\le 3|\varepsilon|}\frac{\mathrm{d}\,m(z)}{|z|}+2|\varepsilon|\cdot\frac{4}{3}\int_{2|\varepsilon|\le|z|\le R}\frac{\mathrm{d}\,m(z)}{|z^2|}\le 12\pi|\varepsilon|+\frac{8|\varepsilon|}{3}\int_{2|\varepsilon|}^{R}\frac{2\pi\mathrm{d}\,r}{r}\\
&=12\pi|\varepsilon|+\frac{16\pi|\varepsilon|}{3}(\ln R-\ln|\varepsilon|)\to 0
\end{aligned}$$

as $|\varepsilon|\to 0$. Hence H is continuous.

Problem. Does there exist an operator $T \in B(X)$ with $\operatorname{Int} \sigma(T) = \emptyset$ and a nonzero vector $x \in X$ such that the local resolvent function $R_T(\cdot, x)$ admits a smooth extension to the complex plane? It is well-known, that it is not possible to have an analytic extension.

We have seen that the set of all vectors with bounded local resolvent is always of the first category. However, this set can be dense.

Theorem 9. Let N be a normal operator on a Hilbert space H. Then the following two conditions are equivalent:
(i) the set of all vectors with bounded local resolvent is dense in H;
(ii) $E(\partial\sigma(N)) = \{0\}$, where $E(\cdot)$ denotes the spectral measure of N.

Proof. (i)$\Rightarrow$(ii): Let $H_1 = E(\partial\sigma(N))$ and $H_2 = H \ominus H_1 = E(\operatorname{Int} \sigma(N))$. Suppose that $H_1 \neq \{0\}$. Let $x = x_1 + x_2$ with $x_i \in H_i \quad (i = 1, 2)$ and $x_1 \neq 0$. Then the local resolvent of x is not bounded. Namely, if it were bounded, we would have $x \in \bigcap_{z\in\partial\sigma(N)} \operatorname{Im}(N - z)$, by Theorem 5. This would imply $x_1 \in \bigcap_{z\in\partial\sigma(N)} (N - z)H_1$, and therefore $x_1 = 0$, which is a contradiction. Hence the set of all vectors $x \in H$ with bounded local resolvent is not dense.

(ii)$\Rightarrow$(i): Let M be the set of all vectors $x \in H$ with bounded local resolvent. By Theorem 4, we have $\overline{M} \supset \{x \in H : \sigma_N(x) \subset \operatorname{Int} \sigma(N)\}$. However, this set is dense in H. $\square$

References

[BG] T. Bermúdez and M. González, *On the boundedness of the local resolvent function*, Integral Equations Operator Theory 34 (1999), 1-8.
[EP] J. Eschmeier, B. Prunaru, *Invariant subspaces and localizable spectrum*, Int. Eq. Operator Theory 42 (2002), 461–471.
[LN] K. B. Laursen and M. M. Neumann, *An Introduction to Local Spectral Theory*, London Math. Soc. Monographs 20, Clarendon Press, Oxford, 2000.
[N] M. M. Neumann, *On local spectral properties of operators on Banach spaces*, Rend. Circ. Mat. Palermo (2) Suppl. 56 (1998), 15-25.

IMFM
University of Ljubljana
Jadranska ul. 19, 1111 Ljubljana
Slovenia

janko.bracic@fmf.uni-lj.si

Mathematical Institute
Czech Academy of Sciences
Žitná 25, 115 67 Prague 1
Czech Republic

muller@math.cas.cz