On the semi-Browder spectrum

Vladimír Kordula, Vladimír Müller and Vladimir Rakočević

Abstract. An operator in a Banach space is called upper (lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (descent). We extend this notion to n-tuples of commuting operators and show that this notion defines a joint spectrum. Further we study relations between semi-Browder and (essentially) semi-regular operators.

Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators in a complex Banach space X and by I the identity operator in X. For T in $\mathcal{L}(X)$ denote by $N(T) = \{x \in X : Tx = 0\}$ and $R(T) = \{Tx : x \in X\}$ its kernel and range, respectively. Denote further $R^\infty(T) = \bigcap_{k=0}^\infty R(T^k)$ and $N^\infty(T) = \bigcup_{k=0}^\infty N(T^k)$.

The sets of all upper (lower) semi-Fredholm operators in X will be denoted by $\Phi_+(X)$ and $\Phi_-(X)$. Recall that $T \in \Phi_+(X)$ if and only if $\dim N(T) < \infty$ and $R(T)$ is closed; $T \in \Phi_-(X)$ if and only if $\operatorname{codim} R(T) < \infty$ (then $R(T)$ is closed automatically). The ascent and descent of T are defined by $a(T) = \min\{n : N(T^n) = N(T^{n+1})\}$ and $d(T) = \min\{n : R(T^n) = R(T^{n+1})\}$.

We say that an operator $T \in \mathcal{L}(X)$ is upper (lower) semi-Browder if it is upper (lower) semi-Fredholm and has a finite ascent (descent). The set of all upper (lower) semi-Browder operators in X will be denoted by $\mathcal{B}_+(X)$ and $\mathcal{B}_-(X)$. Semi-Browder operators were studied by many authors, see e.g. [4], [12], [14], [18], [20], [21], [22], [24]. The name was introduced in [6].

We extend the notion of semi-Browder operators to n-tuples of commuting operators. We discuss the lower semi-Browder case; the upper case is dual.

Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We use the standard multiindex notation. Denote by $\mathbb{Z}_+$ the set of all non-negative integers. If $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{Z}_+^n$ then denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $T^\alpha = T_1^{\alpha_1} \cdots T_n^{\alpha_n}$.

For $k = 0, 1, 2, ...$, denote $M_k(T) = R(T_1^k) + \cdots + R(T_n^k)$ and let $M_k'(T)$ be the smallest subspace of X containing the set $\bigcup\{R(T^\alpha) : \alpha \in Z_+^n \text{ and } |\alpha| = k\}$. Clearly $X = M_0(T) \supset M_1(T) \supset M_2(T) \supset \cdots$ and $X = M_0'(T) \supset M_1'(T) \supset M_2'(T) \supset \cdots$. Further

$$M'_{n(k-1)+1}(T) \subset M_k(T) \subset M_k'(T). \tag{1}$$

Indeed, if $\alpha = (\alpha_1, ..., \alpha_n) \in Z_+^n$ and $|\alpha| = n(k-1)+1$ then there exists i, $1 \le i \le n$ such that $\alpha_i \ge k$, so that $R(T^\alpha) \subset R(T_i^k) \subset M_k(T)$. This proves the first inclusion of (1) and the second inclusion is clear.

Denote $R^\infty(T) = \bigcap_{k=0}^\infty M_k(T) = \bigcap_{k=0}^\infty M_k'(T)$.

AMS Subject Classification (1991): 47A10, 47A53, 47A55

The first two authors were supported by the grant No. 119106 of the Academy of Sciences, Czech Republic. The work of the third-named author was supported by the Science Fund of Serbia, grant number 04M03, through Matematički Institut.

If $M'_k(T) = M'_{k+1}(T)$ for some k then it is easy to see by induction that $M'_m(T) = M'_k(T)$ for every $m \geq k$, so that $R^\infty(T) = M'_k(T)$.

As usual we say that an n-tuple $T = (T_1, ..., T_n)$ of mutually commuting operators in X is lower semi-Fredholm ($T \in \Phi_-^{(n)}(X)$) if

$$\operatorname{codim} M_1(T) = \operatorname{codim}(R(T_1) + \cdots + R(T_n)) < \infty.$$

Clearly $T = (T_1, \dots, T_n)$ is lower semi-Fredholm if and only if the operator $\hat{T} : X^n \to X$ defined by $\hat{T}(x_1, \dots, x_n) = T_1x_1 + \cdots + T_nx_n$ is lower semi-Fredholm.

We say that $T = (T_1, \dots, T_n)$ is semi-Browder if $\operatorname{codim} R^\infty(T) < \infty$. The set of all lower semi-Browder n-tuples will be denoted by $\mathcal{B}_-^{(n)}(X)$. Clearly $\Phi_-^{(n)}(X) \subset \mathcal{B}_-^{(n)}(X)$.

Define

$$\sigma_{\Phi_-}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \Phi_-^{(n)}(X)\},$$

and

$$\sigma_{\mathcal{B}_-}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \mathcal{B}_-^{(n)}(X)\}.$$

It is well known that σ_{Φ_-} satisfies the spectral mapping property [1]. In particular, $(T_1, ..., T_n) \in \Phi_-^{(n)}(X)$ if and only if $(T_1^k, ..., T_n^k) \in \Phi_-^{(n)}(X)$. Thus $\operatorname{codim} M_1(T) < \infty$ implies $\operatorname{codim} M_k(T) < \infty$ for every k.

Theorem 1. Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. The following statements are equivalent:

(a) $T \in \mathcal{B}_-^{(n)}(X)$.

(b) $T \in \Phi_-^{(n)}(X)$ and there exists k such that $M'_k(T) = M'_{k+1}(T)$.

(c) $T \in \Phi_-^{(n)}(X)$ and there exists k such that $M_k(T) = M_{k+1}(T)$.

(d) There exists a subspace $Y \subset X$ invariant with respect to every T_i $(i = 1, \dots, n)$ such that $\operatorname{codim} Y < \infty$ and $T_1Y + \cdots + T_nY = Y$. It is possible to take $Y = R^\infty(T)$.

Proof. (c) $\Rightarrow$ (b): Let $M_k(T) = M_{k+1}(T)$ for some k. Using the same argument as in the proof of (1) it is possible to show that $M'_{n(k-1)+1}(T) = M'_{n(k-1)+2}(T)$.

(b) $\Rightarrow$ (a): Let $M'_k(T) = M'_{k+1}(T)$ for some k. Then $M_k(T) \subset M'_k(T) = R^\infty(T)$. Further $T \in \Phi_-^{(n)}(X)$ implies $\operatorname{codim} M_k(T) < \infty$, so that $T \in \mathcal{B}_-^{(n)}(X)$.

(a) $\Rightarrow$ (d): Set $Y = R^\infty(T)$. Clearly Y is invariant with respect to T_i $(i = 1, \dots, n)$, $\operatorname{codim} Y < \infty$ and $Y = M_k(T) = M_{k+1}(T)$ for some k. If $y \in Y$ then for some $x_1, \dots, x_n \in X$ we have

$$y = \sum_{i=1}^{n} T_i^{k+1}x_i = \sum_{i=1}^{n} T_i(T_i^k x_i) \in T_1Y + \cdots + T_nY.$$

(d) $\Rightarrow$ (c): Since $M_1(T) \supset M_1(T|_Y) = Y$ we have $\operatorname{codim} M_1(T) < \infty$ so that $T \in \Phi_-^{(n)}(X)$. Further $M'_1(T|_Y) = M'_0(T|_Y) = Y$ implies $R^\infty(T|_Y) = Y$ and $M_k(T) \supset M_k(T|_Y) \supset Y$ for every k. Thus the sequence $M_k(T)$ is constant for k big enough.

Corollary 2. Let $T = (T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that $(T_1 - \lambda_1, ..., T_n - \lambda_n) \in \mathcal{B}_-^{(n)}(X)$ for all complex numbers $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^n |\lambda_i| < \epsilon$. Moreover $\operatorname{codim} R^\infty(T_1 - \lambda_1, ..., T_n - \lambda_n) \leq \operatorname{codim} R^\infty(T_1, ..., T_n)$.

Proof. Denote $Y = R^\infty(T)$. Then $\operatorname{codim} Y < \infty$ and $T_1Y + \cdots + T_nY = Y$. There exists $\epsilon > 0$ such that $(T_1-\lambda_1)Y+\cdots+(T_n-\lambda_n)Y = Y$ if $\lambda_1, \dots, \lambda_n \in \mathbb{C}$, $\sum_{i=1}^n |\lambda_i| < \epsilon$, so that $R^\infty(T_1 - \lambda_1, ..., T_n - \lambda_n) \supset Y = R^\infty(T_1, ..., T_n)$.

Proposition 3. Suppose $T_1, ..., T_n, S_1, ..., S_n$ are mutually commuting operators in X such that $\sum_{i=1}^n T_iS_i = I$. Then $(T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$.

Proof. Clearly $M_1(T_1, ..., T_n) = X = M_0(T_1, ..., T_n)$ so that $(T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$.

Corollary 4. $\sigma_{\mathcal{B}_-}(T)$ is a compact subset of $\mathbb{C}^n$.

Proof. $\sigma_{\mathcal{B}_-}(T)$ is closed by Corollary 2. Further $\sigma_{\mathcal{B}_-}(T) \subset \sigma^{<T>}(T)$ where $< T >$ denotes the smallest closed subalgebra of $\mathcal{L}(X)$ containing $T_1, ..., T_n$ and the identity operator and $\sigma^{<T>}(T)$ denotes the spectrum in the commutative Banach algebra $< T >$. Thus $\sigma_{\mathcal{B}_-}(T)$ is bounded and hence compact.

Lemma 5. Let $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators in a Banach space X. Suppose $\operatorname{codim} R^\infty(T_1, ..., T_n) = \infty$ and let $k \in \mathbb{N}$. Then there exists a complex number λ such that

$$\operatorname{codim}\left[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1} - \lambda)^k)\right] \geq k. \tag{2}$$

Proof. Using condition (c) of Theorem 1 we can distinguish two cases:

(a) $(T_1, ..., T_n) \notin \Phi_-^{(n)}(X)$ so that $(0, ..., 0) \in \sigma_{\Phi_-}(T_1, ..., T_n)$. By the projection property for σ_{Φ_-} there exists $\lambda \in \mathbb{C}$ such that $(0, ..., 0, \lambda) \in \sigma_{\Phi_-}(T_1, ..., T_n, T_{n+1})$, i.e., $\operatorname{codim}[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1} - \lambda)^k)] = \infty$. Hence we have (2).

(b) $\operatorname{codim} M_m(T) < \infty$ and $M_m(T) \neq M_{m+1}(T)$ for every $m \geq 1$ where $T = (T_1, ..., T_n)$.

Fix $k \in \mathbb{N}$. Then there exists $i, 1 \leq i \leq n$ such that $R(T_i^{k-1}) \not\subset M_k(T)$ (otherwise $M_{k-1}(T) = M_k(T)$). Denote $Y = X/M_k(T)$, so that $\dim Y < \infty$ and let $S : Y \mapsto Y$ be defined by $S(x + M_k(T)) = T_ix + M_k(T)$. Clearly $S^k = 0$ and $S^{k-1} \neq 0$.

Consider the operator $U : Y \mapsto Y$ defined by $U(x + M_k(T)) = T_{n+1}x + M_k(T)$. Clearly $US = SU$. Let Z be a subspace of Y satisfying $Z \oplus N(S^{k-1}) = Y$. In this decomposition U can be written as

$$U = \begin{pmatrix} U_{11} & 0 \\ U_{12} & U_{22} \end{pmatrix}.$$

Choose a complex number λ such that $U_{11} - \lambda$ is singular, i.e., there exists a non-zero $z \in Z$ with $(U - \lambda)z \in N(S^{k-1})$. Since $z \in N(S^k) \setminus N(S^{k-1})$ we have

$$S^{k-m}z \in N(S^m) \setminus N(S^{m-1}) \qquad (m = 1, \dots, k).$$

Further

$$(U - \lambda)S^{k-m}z = S^{k-m}(U - \lambda)z \in S^{k-m}N(S^{k-1}) \subset N(S^{m-1}).$$

For $m = 1, \dots, k$ we have

$$\dim\big[N(S^m)/(U-\lambda)^m N(S^m)\big] = \dim N\big((U-\lambda)^m|_{N(S^m)}\big) \ge \dim N\big((U-\lambda)^m|_M\big),$$

where $M = N(S^{m-1}) \vee \{S^{k-m}z\}$ and $(U-\lambda)^m M \subset (U-\lambda)^{m-1}N(S^{m-1})$. Further

$$\begin{aligned}&\dim N\big((U-\lambda)^m|_M\big) = \dim\big[M/(U-\lambda)^m M\big]\\ &\ge \dim\big[M/(U-\lambda)^{m-1}N(S^{m-1})\big] = \dim\big[N(S^{m-1})/(U-\lambda)^{m-1}N(S^{m-1})\big] + 1,\end{aligned}$$

since $S^{k-m}z \notin N(S^{m-1})$. Thus, by induction,

$$\dim\big[N(S^m)/(U-\lambda)^m N(S^m)\big] \ge m \qquad (m = 1, \dots, k).$$

In particular $\dim(Y/(U-\lambda)^k Y) \ge k$. Consequently

$$\operatorname{codim}\big[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1}-\lambda)^k)\big] \ge k.$$

Corollary 6. Let $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators in a Banach space X. Suppose that $\operatorname{codim} R^\infty(T_1, ..., T_n) = \infty$. Then there exists $\lambda \in \mathbb{C}$ such that

$$\operatorname{codim} R^\infty(T_1, ..., T_n, T_{n+1} - \lambda) = \infty.$$

Proof. For each $k \ge 1$ we can find $\lambda_k \in \mathbb{C}$ such that

$$\begin{aligned}&\operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda_k)\\ \ge&\operatorname{codim}\,[R(T_1^k) + \cdots + R(T_n^k) + R((T_{n+1}-\lambda_k)^k)] \ge k.\end{aligned}$$

Clearly $\lambda_k \in \sigma(T_{n+1})$ for every k. Thus we may assume (by passing to a subsequence, if necessary) that the sequence $\{\lambda_k\}$ is convergent, $\lambda_k \to \lambda \in \sigma(T_{n+1})$. We have

$$\lim_{k\to\infty} \operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda_k) = \infty.$$

By Corollary 2 this implies that $\operatorname{codim} R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda) = \infty$.

Corollary 7. If $T_1, ..., T_n, T_{n+1}$ be mutually commuting operators, then

$$\sigma_{\mathcal{B}_-}(T_1, ..., T_n) = P\sigma_{\mathcal{B}_-}(T_1, ..., T_{n+1}),$$

where $P : \mathbb{C}^{n+1} \mapsto \mathbb{C}^n$ is the projection onto the first n coordinates.

Proof. The inclusion $\subset$ was proved in Corollary 6. If $(T_1, ..., T_n) \in {\mathcal{B}_-}^{(n)}(X)$ then clearly

$$R^\infty(T_1, \dots, T_n, T_{n+1} - \lambda) \supset R^\infty(T_1, \dots, T_n),$$

so that $(T_1, \dots, T_n, T_{n+1} - \lambda) \in \mathcal{B}_-^{(n+1)}(X)$ for every $\lambda \in \mathbb{C}$. This proves the second inclusion.

Corollary 8. $\sigma_{\mathcal{B}_-}$ is a subspectrum in the sense of Żelazko [25]. Consequently, by [17], $\sigma_{\mathcal{B}_-}$ satisfies the spectral mapping property:

$$f\sigma_{\mathcal{B}_-}(T) = \sigma_{\mathcal{B}_-}f(T)$$

for every n-tuple $T = (T_1, ..., T_n)$ of mutually commuting operators and every m- tuple $f = (f_1, \ldots, f_m)$ of functions analytic in a neighbourhood of the Taylor spectrum of $(T_1, ..., T_n)$.

The following lemma is a well–known stability result for semi-Fredholm operators.

Lemma 9. Let $T = (T_1, ..., T_n) \in \Phi_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that

$$\operatorname{codim} M_1(S) \leq \operatorname{codim} M_1(T)$$

for every commuting n-tuple $S = (S_1, ..., S_n) \in \mathcal{L}(X)^n$ with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$.

The previous lemma enables to generalize the result of [12] for n-tuples of operators.

Theorem 10. Let $T = (T_1, ..., T_n) \in \mathcal{B}_-^{(n)}(X)$. Then there exists $\epsilon > 0$ such that $S \in \mathcal{B}_-^{(n)}(X)$ for every commuting n-tuple $S = (S_1, ..., S_n) \in \mathcal{L}(X)^n$ with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$.

Proof. Choose k such that $M_k(T) = R^\infty(T)$ and $\operatorname{codim} R^\infty(T) \leq k$. Then $(T_1^{k+1}, \ldots, T_n^{k+1}) \in \Phi_-^{(n)}(X)$. By the previous lemma there exists $\epsilon > 0$ with the following property: if $S = (S_1, ..., S_n)$ is a commuting n-tuple of operators in X with $\sum_{i=1}^n \|S_i - T_i\| < \epsilon$ then $(S_1^{k+1}, \ldots, S_n^{k+1}) \in \Phi_-^{(n)}(X)$ and

$$\begin{aligned}
&\operatorname{codim} M_1(S_1^{k+1}, ..., S_n^{k+1}) \leq \operatorname{codim} M_1(T_1^{k+1}, \ldots, T_n^{k+1}) \\
=&\operatorname{codim} M_{k+1}(T) = \operatorname{codim} R^\infty(T) \leq k.
\end{aligned}$$

Since $M_1(S) \supset M_2(S) \supset \cdots \supset M_{k+1}(S)$ and $\operatorname{codim} M_{k+1}(S) \leq k$, there exists $j \leq k$ such that $M_j(S) = M_{j+1}(S)$. Consequently $S \in \mathcal{B}_-^{(n)}(X)$.

From the general theory of joint spectrum it is easy to deduce the following consequences:

(a) The mapping $(T_1, \ldots, T_n) \mapsto \sigma_{\mathcal{B}_-}(T_1, \ldots, T_n)$ is upper semi-continuous. In particular, if $T_1 \in \mathcal{L}(X)$ and U is a neighbourhood of $\sigma_{\mathcal{B}_-}(T_1)$, then $\sigma_{\mathcal{B}_-}(S_1) \subset U$ for every operator S_1 close enough to T_1.

(b) $\sigma_{\mathcal{B}_-}$ is continuous on commuting elements, see [11], Theorem 1.9. More precisely, if $\{T_k\}_{k=1}^\infty \subset \mathcal{L}(X)$, $T \in \mathcal{L}(X)$, $\lim T_k = T$ and $T_kT = TT_k$, $k = 1, 2, \ldots$, then $\lambda \in \sigma_{\mathcal{B}_-}(T)$ if and only if there exist $\lambda_k \in \sigma_{\mathcal{B}_-}(T_k)$ such that $\lambda_k \to \lambda$.

(c) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then (cf. [11], Proposition 1.8)

$$\delta\big(\sigma_{\mathcal{B}_-}(T), \sigma_{\mathcal{B}_-}(S)\big) \leq r_e(T - S),$$

where δ denotes the Hausdorff distance and r_e the essential spectral radius,

$$r_e(T) = \max\{|\lambda|, T - \lambda \quad \text{is not Fredholm}\} = \max\{|\lambda|, T - \lambda \notin \mathcal{B}_-(X)\},$$

see [7].

(d) Let $T, S \in \mathcal{L}(X)$, $TS = ST$. Then

$$TS \in \mathcal{B}_-(X) \Longleftrightarrow T, S \in \mathcal{B}_-(X),$$

see [6] and [16], Theorem 2.1.

(e) Let T and Q be commuting operators acting in X, let $T \in \mathcal{B}_-(X)$ and let Q be a quasinilpotent. Then $T + Q \in \mathcal{B}_-(X)$, see e.g. [11], Remark after Theorem 1.9, [18], Theorem 4.1 or [21], Corollary 2.

Analogously we can define the upper semi-Browder n-tuples. Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. We say that T is upper semi-Fredholm ($T \in \Phi_+^{(n)}(X)$) if the mapping $\tilde{T} : X \mapsto X^n$ defined by $\tilde{T}x = (T_1 x, \dots, T_n x)$ is upper semi-Fredholm. We say that T is upper semi-Browder ($T \in \mathcal{B}_+^{(n)}(X)$) if $T \in \Phi_+^{(n)}(X)$ and $\dim N^\infty(T) < \infty$, where

$$N^\infty(T) = \bigcup_{k=1}^{\infty} \bigl[N(T_1^k) \cap \cdots \cap N(T_n^k)\bigr].$$

Denote $T^* = (T_1^*, \dots, T_n^*) \in \mathcal{L}(X^*)^n$.

Theorem 11. Let $T = (T_1, ..., T_n)$ be an n-tuple of mutually commuting operators in a Banach space X. Then

$$T \in \mathcal{B}_-^{(n)}(X) \Longleftrightarrow T^* \in \mathcal{B}_+^{(n)}(X^*)$$

and

$$T \in \mathcal{B}_+^{(n)}(X) \Longleftrightarrow T^* \in \mathcal{B}_-^{(n)}(X^*).$$

Proof. The corresponding equivalences are well-known for semi-Fredholm n-tuples. Further it is easy to check that

$$N(T_1^k) \cap \cdots \cap N(T_n^k) = {}^\perp\bigl[R(T_1^{*k}) + \cdots + R(T_n^{*k})\bigr].$$

and

$$\bigl[R(T_1^k) + \cdots + R(T_n^k)\bigr]^\perp = N(T_1^{*k}) \cap \cdots \cap N(T_n^{*k}).$$

The statement of Theorem 11 is now an easy consequence of these identities.

For a commuting n-tuple $T = (T_1, ..., T_n) \in \mathcal{L}(X)^n$ we define the upper semi-Browder spectrum of T by

$$\sigma_{\mathcal{B}_+}(T) = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n, (T_1 - \lambda_1, \dots, T_n - \lambda_n) \notin \mathcal{B}_+^{(n)}(X)\}.$$

By the previous theorem it is easy to see that $\sigma_{\mathcal{B}_+}$ satisfies the same properties as $\sigma_{\mathcal{B}_-}$.

Define further the Browder spectrum $\sigma_{\mathcal{B}}$ of a commuting n-tuple $T = (T_1, ..., T_n)$ by

$$\sigma_{\mathcal{B}}(T) = \sigma_{\mathcal{B}_-}(T) \cup \sigma_{\mathcal{B}_+}(T).$$

For a single operator T_1 this definition coincides with the usual definition of the Browder spectrum of T_1 as the union of $\sigma_e(T_1)$ and the limit points of $\sigma(T_1)$, where $\sigma_e(T_1)$ denotes the essential spectrum of T_1, i.e.,

$$\sigma_e(T_1) = \{\lambda \in \mathbb{C}, T - \lambda \quad \text{is not Fredholm}\}$$

and $\sigma(T_1)$ denotes the ordinary spectrum of T_1. Again it is easy to see that $\sigma_{\mathcal{B}}$ satisfies all properties proved for $\sigma_{\mathcal{B}_-}$.

Remark. The possibility of extending the Browder spectrum to commuting n- tuples was proved in [3]. Our extension

$$\sigma_{\mathcal{B}}(T_1, ..., T_n) = \sigma_{\mathcal{B}_-}(T_1, ..., T_n) \cup \sigma_{\mathcal{B}_+}(T_1, ..., T_n)$$

exhibits similar properties as the spectrum

$$\sigma_b(T_1, ..., T_n) = \sigma_{Te}(T_1, ..., T_n) \cup \big(\sigma_T(T_1, ..., T_n)\big)'$$

defined there. (Here σ_t and σ_{te} denote the Taylor and and the essential Taylor spectrum and M' denotes the set of all limit points of a set M.) However these extensions differ for $n \geq 2$, an example will be given later.

The semi-Fredholm and semi-Browder operators are closely related with semi-regular and essentially semi-regular operators which (under various names) were intensively studied, see e. g. [5], [9], [10], [11], [13], [15], [16], [19] and [23]. An operator $T \in \mathcal{L}(X)$ is called semi-regular if it has closed range and $N(T) \subset R^\infty(T)$. T is essentially semi-regular if $R(T)$ is closed and $\dim\big[N(T)/(N(T) \cap R^\infty(T))\big] < \infty$.

From a number of equivalent properties of essentially semi-regular operators we point out the following Kato decomposition (see [16, Theorem 3.1], [19, Theorem 2.1]).

Proposition 12. An operator $T \in \mathcal{L}(X)$ is essentially semi-regular if and only if $R(T)$ is closed and there exist closed subspaces $X_1, X_2 \subset X$ invariant with respect to T such that $X = X_1 \oplus X_2$, $\dim X_1 < \infty$, $T|_{X_1}$ is nilpotent and $T|_{X_2}$ is semi-regular.

If $T \in \mathcal{L}(X)$ is a lower semi-Browder operator then the space X_2 in the Kato decomposition is determined uniquely and $X_2 = R^\infty(T)$. Thus $T|_{X_2}$ is onto. The analogous statement for n-tuples of commuting operator is not true.

Example. Denote by H the Hilbert space with an orthonormal basis $\{e_{i,j} : i, j \in Z, i \geq 0 \quad \text{or} \quad j \geq 0\} \cup \{e_{-1,-1}\}$. Define operators $T_1, T_2 \in \mathcal{L}(X)$ by

$$\begin{aligned} T_1 e_{i,j} &= e_{i+1,j}, \\ T_2 e_{i,j} &= e_{i,j+1}. \end{aligned}$$

We list some properties of the pair (T_1, T_2) :

(a) T_1 and T_2 are commuting isometries so that $(T_1, T_2) \in \mathcal{B}_+^{(n)}(X)$.

(b) Denote

$$Y = \vee\{e_{i,j} : i, j \in Z, i \geq 0 \quad \text{or} \quad j \geq 0\} = \{e_{-1,-1}\}^\perp.$$

Then $T_iY \subset Y \quad (i = 1,2)$, $T_1Y + T_2Y = Y$ and $\operatorname{codim} Y = 1$. Thus $(T_1, T_2) \in \mathcal{B}_-^{(n)}(X)$.

(c) Denote by σ_t the Taylor spectrum. Then $(0,0) \in \sigma_t(T_1, T_2)$. Indeed, $e_{-1,-1} \notin T_1H + T_2H$ so that $T_1H + T_2H \neq H$.

(d) $(0,0)$ is a limit point of the Taylor spectrum of (T_1, T_2). Indeed, if $(0,0)$ were an isolated point of $\sigma_t(T_1, T_2)$ then, using the Taylor functional calculus, it would be possible to decompose H as $H = H_1 \oplus H_2$ where $T_iH_j \subset H_j \quad (i,j = 1,2)$, $\sigma_t(T_1|_{H_1}, T_2|_{H_1}) = \{0,0\}$ and $\{0,0\} \notin \sigma_t(T_1|_{H_2}, T_2|_{H_2})$. Since T_1 and T_2 are commuting isometries it would mean that the approximate point spectrum

$$\begin{aligned}&\sigma_\pi(T_1|_{H_1}, T_2|_{H_1})\\ &\quad = \big\{(\lambda_1, \lambda_2) \in \mathbb{C}^2 : \inf\{\|(T_1 - \lambda_1)x\| + \|(T_2 - \lambda_2)x\|, x \in H_1, \|x\| = 1\} = 0\big\}\end{aligned}$$

is empty. Thus $H_1 = \{0\}$, a contradiction with the fact that

$$(0,0) \in \sigma_t(T_1|_{H_1}, T_2|_{H_1}).$$

(e) We have

$$(0,0) \in \sigma_t(T_1, T_2)' \subset \sigma_b(T_1, T_2)$$

and

$$(0,0) \notin \sigma_{\mathcal{B}}(T_1, T_2) = \sigma_{\mathcal{B}_+}(T_1, T_2) \cup \sigma_{\mathcal{B}_-}(T_1, T_2).$$

Thus the joint spectra $\sigma_{\mathcal{B}}$ and σ_b are different.

(f) In the same way as in (d) it is possible to show that there is no (not necessarily orthogonal) decomposition $H = H_1 \oplus H_2$ such that $T_iH_j \subset H_j \quad (i,j = 1,2)$, $T_1|_{H_1}$ and $T_2|_{H_1}$ are nilpotent and $T_1H_2 + T_2H_2 = H_2$. Thus there is no analogy to the Kato decomposition of a single semi-Browder operator.

Problem. Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators in a Banach space X. Denote by σ_δ the defect spectrum of T, i. e.,

$$\sigma_\delta(T) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : (T_1 - \lambda_1)X + \cdots + (T_n - \lambda_n)X \neq X\}.$$

Using Theorem 1 it is possible to obtain

$$\sigma_{\Phi_-}(T) \cup \sigma_\delta(T)' \subset \sigma_{\mathcal{B}}(T).$$

For $n = 1$ the opposite inclusion also takes place. It is an open problem whether $\sigma_{\Phi_-}(T) \cup \sigma_\delta(T)' = \sigma_{\mathcal{B}}(T)$ for $n \geq 2$.

Proposition 13. Let T be an essentially semi-regular operator on a Banach space X. Then $R^\infty(T)$ is closed, $TR^\infty(T) = R^\infty(T)$ and the operator $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ induced by T is upper semi-Browder.

Proof. Set $M = R^\infty(T)$. Let $X = X_1 \oplus X_2$ be the Kato decomposition of T (see Proposition 12) and denote $T_i = T|_{X_i} \quad (i = 1,2)$. Clearly $M = R^\infty(T_2) \subset X_2$. It is well-known that M is closed and $TM = M$, see e.g. [16], Lemma 1.4. Let $k \geq 1$ and

$x = x_1 \oplus x_2 \in X$ satisfy $T^k x \in M$. Then $T_2^k x_2 \in M$ so that $x_2 \in M$, see [16, Lemma 1.4]. Thus $x \in X_1 + M$ and $\dim N(\tilde{T}^k) \le \dim X_1$. Consequently $\dim N^\infty(\tilde{T}) \le \dim X_1 < \infty$.

Let $\pi : X \mapsto X/M$ be the canonical projection. As $M \subset R(T)$ and $R(\tilde{T}) = \{Tx + M, x \in X\} = \pi R(T)$, the range of $\tilde{T}$ is closed. Thus $\tilde{T}$ is upper semi-Browder.

Theorem 14. Let T be an operator on a Banach space X. Then the following conditions are equivalent:

(a) T is essentially semi-regular,

(b) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Fredholm and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Fredholm,

(c) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,

(d) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is surjective and the induced operator $\tilde{T} : X/M \mapsto X/M$ is upper semi-Browder,

(e) there exists a closed subspace M of X such that $TM \subset M$, $T|_M$ is lower semi-Browder and the induced operator $\tilde{T} : X/M \mapsto X/M$ is bounded below.

Proof. By Proposition 13, $(a) \Rightarrow (d)$. The implications $(d) \Rightarrow (c) \Rightarrow (b)$ are straightforward.

$(b) \Rightarrow (a)$: First we show that $R(T)$ is closed. Let $\pi : X \mapsto X/M$ be the canonical projection. If $y \in R(T)$, $y = Tx$ for some $x \in X$, then $\pi y = Tx + M = \tilde{T}(x + M) \in R(\tilde{T})$, so that $R(T) \subset \pi^{-1} R(\tilde{T})$. Let $y \in X$ and $\pi y \in R(\tilde{T})$, i.e., $y + M = Tx + M$ for some $x \in X$. Then $y \in R(T) + M = R(T) + (F + TM) \subset R(T) + F$ for some finite dimensional subspace F of M. Thus $\pi^{-1}(R(\tilde{T})) \subset R(T) + F \subset \pi^{-1}(R(\tilde{T})) + F$. Further $\pi^{-1}(R(\tilde{T})) + F$ is closed since π is continuous, $R(\tilde{T})$ is closed and F finite dimensional. Hence $R(T) + F$ is closed, and so $R(T)$ is closed.

As $\pi N(T) \subset N(\tilde{T})$ and $\dim N(\tilde{T})$ is finite dimensional, there exists a finite dimensional subspace $G_1 \subset N(T)$ such that $N(T) \subset G_1 + N(T|_M)$. The operator $T|_M$ is lower semi-Fredholm and consequently essentially semi-regular, i.e., there exists a finite dimensional subspace G_2 of M such that $N(T|_M) \subset G_2 + R^\infty(T|_M)$. Thus

$$N(T) \subset G_1 + N(T|_M) \subset G_1 + G_2 + R^\infty(T|_M) \subset (G_1 + G_2) + R^\infty(T),$$

and T is essentially semi-regular.

$(a) \Rightarrow (e)$: Let $X = X_1 \oplus X_2$ be the Kato decomposition of T, i.e., $\dim X_1 < \infty$, $TX_1 \subset X_1$, $TX_2 \subset X_2$, $T|_{X_1}$ is nilpotent and $T_2 = T|_{X_2}$ is semi-regular. Set $M = X_1 \oplus R^\infty(T_2) = X_1 \oplus R^\infty(T)$. Clearly, M is closed and since $TR^\infty(T) = R^\infty(T)$, we have $T|_M$ is a lower semi-Browder operator.

Let $\tilde{T} : X/M \mapsto X/M$ be the operator induced by T. If $x = x_1 \oplus x_2$ satisfies $Tx \in M$ then $T_2 x_2 \in R^\infty(T_2)$, so that $x_2 \in R^\infty(T_2)$ and $x \in M$. Hence $N(\tilde{T}) = \{0\}$.

We show that $R(\tilde{T})$ is closed. Let $x, x_k \in X$ $(k = 1, 2, \ldots)$ and let $Tx_k + M \to x + M$ in the topology of X/M. Then $x \in \overline{R(T) + M} = R(T) + M$ since $M \subset R(T) + X_1$. Consequently $x + M \in R(\tilde{T})$. Hence $R(\tilde{T})$ is closed and $\tilde{T}$ is bounded below.

$(e) \Rightarrow (b)$: Clear.

It is well-known that if $T \in \mathcal{L}(X)$ is essentially semi-regular and K is compact operator commuting with T then $T + K$ is also essentially semi-regular [5], Theorem

5.9. Now we can prove a sharper result. Let us denote by

$$r_+(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_+(X) \quad \text{for} \quad |\lambda| < \epsilon\}$$

and

$$r_-(T) = \sup\{\epsilon \geq 0 : T - \lambda I \in \Phi_-(X) \quad \text{for} \quad |\lambda| < \epsilon\}$$

the semi-Fredholm radii of T. An operator $T \in \mathcal{L}(X)$ is upper (lower) semi-Fredholm if and only if $r_+(T) > 0 \quad (r_-(T) > 0)$.

Lemma 15. Let A be an operator on a Banach space X and let M be a closed subspace of X such that $AM \subset M$. Then $r_e(A|_M) \leq r_e(A)$ and $r_e(\tilde{A}) \leq r_e(A)$ where $\tilde{A} : X/M \mapsto X/M$ is the operator induced by A.

Proof. Let $A \in \mathcal{L}(X)$ be a Fredholm operator and let $AM \subset M$. Then $R(A|_M)$ is closed (see [2], Lemma 4.3.1) and $\dim N(A|_M) \leq N(A) < \infty$. Thus $A|_M$ is upper semi-Fredholm. Further, $\operatorname{codim} R(\tilde{A}) \leq \operatorname{codim} R(A) < \infty$, and hence $\tilde{A}$ is lower semi-Fredholm.

The rest follows from the fact that upper and lower semi-Fredholm spectra contain the boundary of the essential spectrum [7].

Theorem 16. Let $T, S \in \mathcal{L}(X)$, $TS = ST$ and let T be essentially semi-regular. Let $\hat{T} = T|_{R^\infty(T)}$ and let $\tilde{T} : X/R^\infty(T) \mapsto X/R^\infty(T)$ be the operator induced by T. If $r_e(S) < \min\{r_-(\hat{T}), r_+(\tilde{T})\}$ then $T + S$ is essentially semi-regular.

Proof. By Theorem 14, $\hat{T} \in \Phi_-(X)$ and $\tilde{T} \in \Phi_+(X)$. As $TS = ST$, we have $SR^\infty(T) \subset R^\infty(T)$ and we can define the operators $\tilde{S} : X/R^\infty(T) \to X/R^\infty(T)$ and $\hat{S} = S|_{R^\infty(T)}$. Clearly, $\hat{T}\hat{S} = \hat{S}\hat{T}$ and $\tilde{T}\tilde{S} = \tilde{S}\tilde{T}$. By Lemma 15, $r_e(\hat{S}) \leq r_e(S) < r_-(\hat{T})$ and $r_e(\tilde{S}) \leq r_e(S) < r_+(\tilde{T})$. As in [11], Theorem 1.9 it is possible to deduce that $\hat{T} + \hat{S}$ is lower semi-Fredholm and $\tilde{T} + \tilde{S}$ is upper semi-Fredholm. By Theorem 14, $T + S$ is essentially semi-regular.

Corollary 17. Let T be an essentially semi-regular operator on a Banach space X, $S \in \mathcal{L}(X)$, $TS = ST$ and let S be a Riesz operator (i.e., $r_e(S) = 0$). Then $T + S$ is essentially semi-regular.

For $T \in \mathcal{L}(X)$ denote by

$$\sigma_\gamma(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not semi-regular}\}$$

and

$$\sigma_{\gamma e}(T) = \{\lambda \in \mathbb{C} : T - \lambda \quad \text{is not essentially semi-regular}\}.$$

The spectrum $\sigma_\gamma(T)$ and its essential version the set $\sigma_{\gamma e}(T)$ were studied (under various names) by many authors, see e.g., [9], [10], [11], [13], [15], [16], [19] and [23].

Corollary 18. Let $T \in \mathcal{L}(X)$. Then

$$\sigma_{\gamma e}(T) = \bigcap \sigma_\gamma(T + S)$$

where the intersection is taken over all Riesz operators in X commuting with T.

Proof. The inclusion $\supset$ follows from [19, Theorem 3.1]. The opposite inclusion follows from the previous corollary.

Theorem 19. Let X be an infinite dimensional Banach space and $S \in \mathcal{L}(X)$. Then the following conditions are equivalent:

(a) $\sigma_{\gamma e}(T+S) = \sigma_{\gamma e}(T)$ for every $T \in \mathcal{L}(X)$ commuting with S,
(b) S is a Riesz operator.

Proof. $(b) \Rightarrow (a)$: See Corollary 17.

$(a) \Rightarrow (b)$: Take $T=0$. Then $\sigma_{\gamma e}(S) = \sigma_{\gamma e}(0) = \{0\}$. By [19], Corollary 3.4 or [16], Theorem 3.8, $\sigma_e(T) = \{0\}$ so that S is a Riesz operator.

Acknowledgement. The paper was written during the third-named author's stay at the Institute of Mathematics of the Czech Academy of Sciences in Spring 1995. The author would like to thank for warm hospitality and excellent working conditions in Prague.

REFERENCES

[1] J. J. Buoni, R. Harte and T. Wickstead, Upper and lower Fredholm spectra, Proc. Amer. Math. Soc., 66 (1977), 309–314.
[2] S. R. Caradus, W. E. Pfaffenberger and B. Yood, Calkin algebras and algebras of operators on Banach spaces, Marcel Dekker, 1974.
[3] R. E. Curto, A. T. Dash, Browder spectral systems, Proc. Amer. Math. Soc., 103 (1988), 407–413.
[4] S. Grabiner, Ascent, descent, and compact perturbations, Proc. Amer. Math. Soc. 71 (1978), 79–80.
[5] S. Grabiner, Uniform ascent and descent of bounded operators, J. Math. Soc. Japan, 34 (1982), 317–337.
[6] R. Harte, Invertibility and Singularity for Bounded Linear Operators, Marcel Dekker, Inc., New York and Basel, 1988.
[7] R. E. Harte and A. W. Wickstead, Upper and lower Fredholm spectra II, Math. Z., 154 (1977), 253–256.
[8] T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, J. Analyse Math., 6 (1958), 261–322.
[9] V. Kordula, The essential Apostol spectrum and finite dimensional perturbations, Proc. Roy. Irish Acad. (to appear).
[10] V. Kordula and V. Müller, The distance from the Apostol spectrum, Proc. Amer. Math. Soc. (to appear).
[11] V. Kordula and V. Müller, On the axiomatic theory of spectrum, Studia Math. 119 (1996), 109–128.
[12] H. Kroh and P. Volkmann, Störungssätze für Semifredholmoperatoren, Math. Z. 148 (1976), 295–297.
[13] M. Mbekhta, Résolvant généralisé et théorie spectrale, J. Operator Theory 21 (1989), 69–105.

[14] M. Mbekhta and V. Müller, On the axiomatic theory of spectrum II., Studia Math. 119 (1996), 129–147.
[15] M. Mbekhta, A. Ouahab, Contribution à la théorie spectrale généralisé dans les espaces de Banach, C. R. Acad. Sci. Paris 313 (1991), 833–836.
[16] V. Müller, On the regular spectrum, J. Operator Theory 31 (1994), 363–380.
[17] M. Putinar, Functional calculus and the Gelfand transformation, Studia Math. 84 (1984), 83–86.
[18] V. Rakočević, Approximate point spectrum and commuting compact perturbations, Glasgow Math. J. 28 (1986), 193–198.
[19] V. Rakočević, Generalized spectrum and commuting compact perturbations, Proc. Edinb. Math. Soc. 36 (1993), 197–209.
[20] V. Rakočević, Semi-Fredholm operators with finite ascent or descent and perturbations, Proc. Amer. Math. Soc. 123 (1995), 3823–3825.
[21] V. Rakočević, Semi-Browder operators and perturbations, Studia Math. (to appear).
[22] V. Rakočević, Semi-Fredholm operators with finite ascent or descent and corresponding spectra, Proceedings of the conference in Priština, University of Priština, 1994, 79–89.
[23] Ch. Schmoeger, Ein Spektralabbildungssatz, Arch. Math., 55 (1990), 484–489.
[24] T. T. West, A Riesz-Schauder theorem for semi-Fredholm operators, Proc. R. Ir. Acad. 87 (1987), 137–146.
[25] W. Żelazko, Axiomatic approach to joint spectra I., Studia Math., 64 (1979), 249–261.

Institute of Mathematics AV ČR
Žitná 25, 11567 Praha 1
Czech Republic
e-mail: vmuller@mbox.cesnet.cz

University of Niš, Faculty of Philosophy
Department of Mathematics
Ćirila and Metodija 2, 18000 Niš
Yugoslavia - Serbia
e-mail: vrakoc@archimed.filfak.ni.ac.yu