OPERATOR TUPLES AND ANALYTIC MODELS

OVER GENERAL DOMAINS IN $\mathbf{C}^n$

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Abstract. For a class of Hilbert spaces $\mathcal{H}$ of functions analytic on a domain $D \subset \mathbf{C}^n$, we characterize the n-tuples T of commuting Hilbert space operators which can be represented by means of multiplications by the coordinate functions on $\mathcal{H}$. In case $\mathcal{H}$ is a subspace of some L^2-space we thus obtain for such T a normal dilation constructed in terms of the reproducing kernel of $\mathcal{H}$. This generalizes known results of this type and provides new models in a large class of functional Hilbert spaces, including the standard weighted Bergman spaces of analytic functions on bounded symmetric domains.

One of the most important results of the Sz.–Nagy and Foiaş dilation theory is the existence of a model for contractions. In particular, the following results can be proved:

Theorem 0.
(a) *Let T be a Hilbert space contraction with spectrum contained in the open unit disc. Then T is unitarily equivalent to a restriction of the backward shift of infinite multiplicity to an invariant subspace.*
(b) *More generally, a Hilbert space contraction T is unitarily equivalent to a restriction of the backward shift of infinite multiplicity to an invariant subspace if and only if $T^n \to 0$ in the strong operator topology.*

There are many generalizations of these results both for single contractions and for n-tuples of commuting operators. In general there are two approaches to such results. The first of them is the geometric approach, see [CV], [M], [MV], [Po], which makes it possible to give an explicit form of the model, even without the assumption on the spectrum, but only in some special situations like for single contractions, spherical contractions or n-tuples of contractions having the regular dilation.

The second approach is due to Agler [A1], see also [At], who proved the existence of the model using the representation theory and C^*-algebras methods. This approach can be applied in many situations, but — except of the polydisc and the ball case — the model is in general rather inexplicit, expressed by means of a representation.

The aim of this paper is to combine the advantages of these two approaches. We give an explicit form of the model for a wide class of domains. Moreover, the construction is relatively simple.

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Let D be a nonempty open domain in $\mathbf{C}^n$. Write $\widetilde{D} := \{\overline{z} : z \in D\}$. For $f : D \to \mathbf{C}$, set $\widetilde{f}(w) := \overline{f(\overline{w})}$ $(w \in \widetilde{D})$.

Denote by $B(X)$ the set of all bounded linear operators on a Banach space X.

Definition 1. Let D be an open domain in $\mathbf{C}^n$. A Hilbert space $\mathcal{H}$ of functions analytic on D is called a D-space if conditions (a)–(c) below are satisfied:

(a) *$\mathcal{H}$ is invariant under the operators Z_j $(j = 1, \ldots, n)$ of multiplication by the coordinate functions,*

$$(Z_j f)(z) := z_j f(z) \qquad (f \in \mathcal{H}, z = (z_1, \ldots, z_n) \in D).$$

It follows from the next assumption and the closed graph theorem that the operators Z_j are, in fact, bounded.

(b) *For each $z \in D$, the evaluation functional $f \mapsto f(z)$ is continuous on $\mathcal{H}$.*

By the Riesz representation theorem there is $C_z \in \mathcal{H}$ such that $f(z) = \langle f, C_z \rangle$ for all $f \in \mathcal{H}$. Define the function $C(z, w) := C_{\overline{w}}(z)$ $(z \in D, w \in \widetilde{D})$. (The function $C(z, \overline{w})$ is known as the reproducing kernel of $\mathcal{H}$.)

(c) *$C(z, w) \neq 0$ for all $z \in D$, $w \in \widetilde{D}$.*

It is easy to see that C is analytic on $D \times \widetilde{D}$ and

$$C(z, w) = \langle C_{\overline{w}}\,, C_z \rangle = \overline{\langle C_z, C_{\overline{w}} \rangle} = \overline{C(\overline{w}, \overline{z})} \qquad (z \in D,\ w \in \widetilde{D}).$$

The functions C_z $(z \in D)$ are dense in $\mathcal{H}$. Indeed, if $f \in \mathcal{H}$ and $f \perp C_z$ for all $z \in D$ then $f(z) = \langle f, C_z \rangle = 0$. Further

$$Z_j^* C_u = \overline{u_j} C_u \qquad \big(u = (u_1, \ldots, u_n) \in D, j = 1, \ldots, n\big).$$

By computing $\|C_z - C_u\|^2_{\mathcal{H}}$ in terms of the inner product we obtain the continuity of the map

$$D \ni z \mapsto C_z \in \mathcal{H}.$$

In particular, $\mathcal{H}$ is separable.

The next statement is well-known, cf. for example the corresponding result in [B].

Lemma 2. *Let $\mathcal{H}$ be a D-space and $\{\psi_k\}$ an orthonormal basis in $\mathcal{H}$. Then*

$$C(z, w) = \sum_{l=1}^{\infty} \psi_k(z) \widetilde{\psi_k}(w)$$

where the series converges uniformly and absolutely on each compact subset of $D \times \widetilde{D}$.

Proof. Let $L \subset D$ be a compact set and d_L the (finite) supremum of the continuous function $\|C_z\|^2 = C(z, \overline{z})$ on L. Then for each $z \in L$,

$$\sum_k |\psi_k(z)|^2 = \Big\| \sum_k \overline{\psi_k(z)} \cdot \psi_k \Big\|^2 \geq d_L^{-1} \cdot \Big| \Big(\sum_k \overline{\psi_k(z)} \psi_k \Big)(z) \Big|^2 = d_L^{-1} \cdot \Big(\sum_k |\psi_k(z)|^2 \Big)^2.$$

Thus $\sum |\psi_k(z)|^2 \le d_L$. Hence the sum $h_z = \sum_k \overline{\psi_k(z)}\psi_k$ is convergent for each $z \in D$. For $h \in \mathcal{H}$ we have

$$\langle h, h_z\rangle = \Big\langle h, \sum_k \overline{\psi_k(z)}\psi_k\Big\rangle = \sum_k \psi_k(z)\langle h, \psi_k\rangle = \Big(\sum_k \langle h, \psi_k\rangle\psi_k\Big)(z) = h(z) = \langle h, C_z\rangle$$

so $h_z = C_z$. In particular, for $w \in \widetilde{D}$ we have

$$C(z,w) = \overline{C_z(\overline{w})} = \overline{h_z(\overline{w})} = \overline{\sum_k \overline{\psi_k(z)}\psi_k(\overline{w})} = \sum_k \psi_k(z)\widetilde{\psi_k}(w).$$

It is easy to see that the convergence is uniform and absolute on each compact subset of $D \times \widetilde{D}$. $\square$

Let $\mathcal{H}$ be a D-space and H a Hilbert space. Denote by $\mathcal{H} \otimes H$ the (completed) Hilbertian tensor product. Elements of $\mathcal{H}\otimes H$ can be viewed upon as H-valued functions analytic on D.

Consider the multiplication operators M_{z_j} on $\mathcal{H} \otimes H$ defined by

$$M_{z_j} := Z_j \otimes I_H \qquad (j = 1, \dots, n)$$

and write

$$M_z := (M_{z_1}, \dots, M_{z_n}).$$

The basic prototype of a D-space $\mathcal{H}$ is the Hardy space H^2 on the unit disc, see Example 1 at the end of this paper. In this case $C(z,w) = (1 - zw)^{-1}$ and M_z^* is a backward shift of infinite multiplicity. Theorem 0 can thus be restated as saying that if an operator T on H satisfies $\frac{1}{C}(T,T^*) = I - TT^* \ge 0$ and (a) $\sigma(T) \subset D$ or (b) $T^{*n} \to 0$, then T^* is unitarily equivalent to the restriction of M_z^* to an invariant subspace.

We are going to study commuting n-tuples $T = (T_1, \dots, T_n)$ of operators acting on H for which M_z, for a general D-space $\mathcal{H}$, serves as a model in the manner just described. We start with n-tuples having the spectrum contained in D (part I). In this case $h(T)$ is defined for any $h \in \mathcal{H}$. Then we deal, under slightly stronger assumptions on $\mathcal{H}$, with n-tuples whose spectrum need not lie in D (part II).

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I. Let H be a Hilbert space. For $A \in B(H)$, denote by L_A and R_A the left (right) multiplication operators by A on $B(H)$, i.e., $L_A X := AX$ and $R_A X := XA$ ($X \in B(H)$).

Let $S = (S_1, \dots, S_n)$ be a commutative n-tuple of operators on a Banach space X. Denote by $\sigma(S)$ the Taylor spectrum and by $\sigma_s(S)$ the split-spectrum of S, see [E],[CF]. These two spectra coincide for Hilbert space operators.

Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators on H. Set $T^* := (T_1^*, \dots, T_n^*)$. Write $L_T = (L_{T_1}, \dots, L_{T_n})$, $R_{T^*} = (R_{T_1^*}, \dots, R_{T_n^*})$ and $M_T = (L_T, R_{T^*})$. Clearly M_T is a commuting $2n$–tuple of operators acting on the Banach space $B(H)$.

Further

$$\sigma_s(L_T) = \sigma(L_T) = \sigma(T), \quad \sigma_s(R_{T^*}) = \sigma(R_{T^*}) = \sigma(T^*)$$

and

$$\sigma_s(M_T) \subset \sigma_s(L_T) \times \sigma_s(R_{T^*}) = \sigma(T) \times \sigma(T^*).$$

Let f, g be analytic in a neighbourhood of $\sigma(T)$ and h be analytic in a neighbourhood of $\sigma(T^*) = \widetilde{\sigma(T)}$. Then we have the equalities

$$L_{f(T)} = f(L_T), \quad R_{h(T^*)} = h(R_{T^*}) \tag{1}$$

and

$$g(T)^* = \widetilde{g}(T^*), \tag{2}$$

as a consequence of the uniqueness of the functional calculus [Pu].

Let f be a function analytic on a neighbourhood of $\sigma(M_T)$. Define $f(T, T^*) \in B(H)$ by

$$f(T, T^*) := f(M_T)(I). \tag{3}$$

If $f(z, w) := z^\alpha w^\beta$ for some multiindices $\alpha, \beta \in \mathbf{Z}_+^n$, then $f(T, T^*) = T^\alpha T^{*\beta}$. This differs slightly from the hereditary calculus of [A1] (giving in this case rather $f(T, T^*) := T^{*\beta} T^\alpha$). Clearly the mapping $f \mapsto f(T, T^*)$ is linear. The multiplicativity is replaced by the property stated in Lemma 3 below, cf. [A1], see also [At].

Lemma 3. *Let T be a tuple on a Hilbert space. Let $f = f(z)$, $g = g(z, w)$ and $h = h(w)$ be analytic in neighbourhoods of $\sigma(T)$, $\sigma(M_T)$ and $\sigma(T^*)$, respectively. Set $F(z, w) := f(z) g(z, w) h(w)$. Then*

$$F(T, T^*) = f(T)\, g(T, T^*)\, h(T^*).$$

Proof. We use the equalities (2), as well as the multiplicativity of the functional calculus for M_T. We have

$$f(T) g(T, T^*) h(T^*) = f(T)\Big(g(M_T)(I)\Big) h(T^*) =$$

$$= R_{h(T^*)} L_{f(T)} \Big(g(L_T, R_{T^*})(I)\Big) = \Big(R_{h(T^*)} L_{f(T)} g(L_T, R_{T^*})\Big)(I) =$$

$$= \Big(h(R_{T^*}) f(L_T) g(L_T, R_{T^*})\Big)(I) = \Big((hfg)(L_T, R_{T^*})\Big)(I) =$$

$$= (fgh)(T, T^*) = F(T, T^*). \qquad \square$$

For a function f analytic on a neighbourhood U of $\sigma_s(M_T)$ we can express $f(T, T^*)$ by means of the Martinelli kernel:

$$f(T, T^*) = \int_{\partial G} f(u, v) k_{M_T}(u, v)(I).$$

Here G is an open domain with smooth boundary such that $\sigma_s(M_T) \subset G$ and $\overline{G} \subset U$. The exact form of the Martinelli kernel k_{M_T} can be found in [KM], for Hilbert space operators see [V1], [V2]. For our purpose it is sufficient to know that it is a differential form of degree $2n-1$ in $d\bar{u}_1,\ldots,d\bar{u}_n,d\bar{v}_1,\ldots,d\bar{v}_n$ and of the maximal degree $2n$ in $du_1,\ldots,du_n,dv_1,\ldots,dv_n$, whose coefficients are smooth $B(B(H))$-valued functions defined on the complement of the split-spectrum.

Thus the coefficients of the form $k_{M_T}(u,v)(I)$ are smooth $B(H)$-valued functions. Similarly, for $h,g\in H$ we can consider $\langle k_{M_T}(u,v))(I)h,g\rangle$ to be a differential form of total degree $4n-1$ whose coefficient are C^∞- scalar valued functions.

It is easy to see that the functional calculus defined by (3) is continuous in the following sense: if W is an open neighbourhood of $\sigma(M_T)$ and f,f_l $(l=1,2,\ldots)$ functions analytic on W such that $f_l\to f$ uniformly on compact subsets of W, then $f_l(T,T^*)\to f(T,T^*)$.

Let now $D\subset \mathbf{C}^n$ be a domain and $\mathcal{H}$ a D-space. Let T be an n-tuple of operators on a Hilbert space H such that $\sigma(T)\subset D$.

Define the linear map $C_T:H\to\mathcal{H}\otimes H$ by

$$C_T h := \int_{\partial\Delta} C_{\overline{w}}\otimes k_{T^*}(w)h \qquad (h\in H), \tag{4}$$

where Δ is a bounded open domain with smooth boundary such that $\sigma(T^*)\subset\Delta$ and $\overline{\Delta}\subset\widetilde{D}$. This definition is motivated by the formal identities

$$C_T(z)=C(z,T^*)=\int_{\partial\Delta} C(z,w)k_{T^*}(w)=\int_{\partial\Delta} C_{\overline{w}}(z)k_{T^*}(w).$$

Lemma 4. *We have*

$$\langle C_T h\,,\, f\otimes h'\rangle=\langle h\,,\, f(T)h'\rangle$$

for all $h,\ h'\in H$ and $f\in\mathcal{H}$. In particular, C_T does not depend on the choice of Δ. Moreover, $C_T:H\to\mathcal{H}\otimes H$ is a bounded operator.

Proof. The integral makes sense, because $\partial\Delta$ is compact, the map $z\mapsto C_z$ is continuous on D and k_{T^*} is smooth on $\partial\Delta$. Since the integral commutes with bounded linear maps, we obtain the following equalities.

$$\langle C_T h, f\otimes h'\rangle=\left\langle\int_{\partial\Delta} C_{\overline{w}}\otimes(k_{T^*}(w)h)\,,\, f\otimes h'\right\rangle=$$

$$=\int_{\partial\Delta}\left\langle C_{\overline{w}}\otimes(k_{T^*}(w)h)\,,\, f\otimes h'\right\rangle=\int_{\partial\Delta}\langle C_{\overline{w}}\,,\, f\rangle\left\langle k_{T^*}(w)h\,,\, h'\right\rangle=$$

$$=\int_{\partial\Delta}\overline{f(\overline{w})}\langle k_{T^*}(w)h\,,\, h'\rangle=\int_{\partial\Delta}\left\langle\widetilde{f}(w)k_{T^*}(w)h\,,\, h'\right\rangle=$$

$$=\left\langle\int_{\partial\Delta}\widetilde{f}(w)k_{T^*}(w)h\,,\, h'\right\rangle=\langle\widetilde{f}(T^*)h\,,\, h'\rangle=\langle h, f(T)h'\rangle.$$

For the last equality, see (2).

Since finite linear combinations of elementary tensor products $f\otimes h'$ are dense in $\mathcal{H}\otimes H$, the definition of C_T does not depend on the choice of Δ.

It follows easily from (4) that C_T is a bounded operator. $\square$

Let $\mathcal{H}$ be a D-space. Let T be an n-tuple of operators on H such that $\sigma(T) \subset D$ and $\frac{1}{C}(T,T^*) \geq 0$. Set $D_T = \frac{1}{C}(T,T^*)^{1/2}$ and define the mapping $V : H \to \mathcal{H} \otimes H$ by the equality

$$V := \big(I_{\mathcal{H}} \otimes D_T\big) C_T. \tag{5}$$

Note that D_T plays the rôle of the defect operator of the Sz.-Nagy–Foiaş theory.

By Lemma 4, it is easy to see that $V^* : \mathcal{H} \otimes H \to H$ is defined by the formula $V^*(f \otimes h) = f(T) D_T h$ ($f \in \mathcal{H}, h \in H$). This formula can be used equivalently to define V.

Theorem 5. *Let $\mathcal{H}$ be a D-space. Let T be an n-tuple of operators on H such that $\sigma(T) \subset D$ and $\frac{1}{C}(T,T^*) \geq 0$. Then the mapping $V : H \to \mathcal{H} \otimes H$ defined by* (5) *satisfies the equalities*

$$VT_j^* = M_{z_j}^* V \qquad (j = 1, \ldots, n).$$

Proof. It suffices to verify the equalities on elementary tensor products. Let h, $h' \in H$ and $f \in \mathcal{H}$ be arbitrary. By Lemma 4, we have

$$\big\langle VT_j^* h\,,\, f \otimes h' \big\rangle = \big\langle h, T_j V^*(f \otimes h' \big\rangle = \big\langle h, T_j f(T) D_T h' \big\rangle.$$

On the other hand, by Lemma 4 again,

$$\big\langle M_{z_j}^* V h\,,\, f \otimes h' \big\rangle = \big\langle h, V^*(Z_j f \otimes h')\big\rangle = \big\langle h, (z_j f)(T) D_T h' \big\rangle = \big\langle h, T_j f(T) D_T h' \big\rangle.$$

Thus $VT_j^* = M_{z_j}^* V$. $\square$

Theorem 6. *Let $\mathcal{H}$ be a D-space. Let T be an n-tuple of commuting operators on a Hilbert space H such that $\sigma(T) \subset D$ and $\frac{1}{C}(T,T^*) \geq 0$. Then the mapping $V : H \to \mathcal{H} \otimes H$ defined by* (5) *is an isometry.*

Proof. Let $\{\psi_k\}$ be an orthonormal basis in $\mathcal{H}$ and $\{h_j\}$ an orthonormal basis in H. Then $\{\psi_k \otimes h_j\}_{k,j}$ is an orthonormal basis in $\mathcal{H} \otimes H$. Let $h \in H$. We have

$$\begin{aligned}
\|Vh\|^2 &= \sum_{k,j} |\langle Vh, \psi_k \otimes h_j \rangle|^2 = \sum_{k,j} |\langle h, V^*(\psi_k \otimes h_j)\rangle|^2 = \sum_{k,j} |\langle h, \psi_k(T) D_T h_j \rangle|^2 = \\
&= \sum_{k,j} |\langle D_T \psi_k(T)^* h, h_j \rangle|^2 = \sum_k \|D_T \psi_k(T)^* h\|^2 \\
&= \sum_k \langle \psi_k(T) D_T^2 \psi_k(T)^* h, h \rangle = \sum_k \langle F_k(T,T^*) h, h \rangle
\end{aligned}$$

where $F_k(z,w) = \dfrac{\psi_k(z)\widetilde{\psi_k(w)}}{C(z,w)}$. By Lemma 2, $\sum_k F_k = 1$ uniformly on a neighbourhood of $\sigma(T) \times \sigma(T^*)$ so $\sum_k F_k(T,T^*) = I$. Thus $\|Vh\|^2 = \|h\|^2$ and V is an isometry. $\square$

Corollary 7. *Let $\mathcal{H}$ be a D-space. Let T be an n-tuple of operators on a Hilbert space H satisfying $\sigma(T) \subset D$ and $\frac{1}{C}(T,T^*) \geq 0$. Then T^* is unitarily equivalent to a restriction of M_z^* to an invariant subspace.*

In many interesting cases D is bounded and $\mathcal{H}$ is isometrically embedded into a space $L^2(\phi)$, where ϕ is a finite nonnegative Borel measure on $\overline{D}$, see the examples below. Then $Z = (Z_j)_j$ is a subnormal n-tuple. Namely it extends to the (bounded and normal) n-tuple M of multiplications by the coordinate functions on $L^2(\phi)$. Note that $\sigma(M) = \operatorname{supp}\phi \subset \overline{D}$.

Corollary 8. *Let $D \subset \mathbf{C}^n$ be a bounded domain, let $\mathcal{H}$ be a D-space and suppose that $\mathcal{H}$ is a subspace of $L^2(\phi)$ where ϕ is a nonnegative finite Borel measure with $\operatorname{supp}\phi \subset \overline{D}$. Let T be an n-tuple of operators such that $\sigma(T) \subset D$ and $\frac{1}{C}(T, T^*) \geq 0$. Then T has a normal dilation N. More precisely, there are a Hilbert space $K \supset H$ and a commuting n-tuple N of normal operators on K such that $\sigma(N) \subset \overline{D}$ and*

$$p(T) = P_H p(N)|_H$$

for all polynomials p in n variables.

Proof. Let M be the n-tuple of multiplications by the variables on $L^2(\phi)$. Set $N := M \otimes I_H$ componentwise. Then N is a normal tuple on $L^2(\phi) \otimes H$, and $\sigma(N) = \sigma(M) \subset \overline{D}$. Now $\mathcal{H} \otimes H \subset L^2(\phi) \otimes H$ and

$$M_z = Z \otimes I = (M \otimes I)|_{\mathcal{H} \otimes H} = N|_{\mathcal{H} \otimes H}.$$

By Corollary 7, we can assume that $H \subset \mathcal{H} \otimes H$ and $T^* = M_z^*|H$. Hence

$$T^\alpha = P_H M_z^\alpha |H = P_H N^\alpha |H$$

for all $\alpha \in \mathbf{Z}_+^n$. $\square$

Corollary 9. (von Neumann inequality) *Let $\mathcal{H}$ be a D-space satisfying the conditions of the preceding corollary. Let T be a commuting n-tuple of operators on a Hilbert space H such that $\sigma(T) \subset D$ and $\frac{1}{C}(T, T^*) \geq 0$. Then*

$$\|p(T)\| \leq \sup_{z \in D} |p(z)|$$

for all polynomials p. Moreover, if D is polynomially convex, then the above von Neumann inequality is true for all functions analytic on D.

* * *

II. From now on we are going to study models for n-tuples of operators which need not satisfy $\sigma(T) \subset D$. We need stronger assumptions on $\mathcal{H}$. Nevertheless, the assumptions will be satisfied in most of the interesting cases.

Let $\mathcal{H}$ be a D-space such that:

- *$\mathcal{H}$ contains the constant functions (hence, also all polynomials) and the polynomials are dense in $\mathcal{H}$,*
- *$\frac{1}{C}$ is a polynomial.*

The monomials z^α ($\alpha \in \mathbf{Z}_+^n$) are then a (non-orthogonal) basis for $\mathcal{H}$. Arranging them in some order, by the Gram-Schmidt orthogonalization we can find (and fix from now on) an orthonormal basis $\{\psi_k\}$ consisting of polynomials and such that, conversely, any polynomial is a finite linear combination of ψ_k. (The latter property will be needed in the proof of Lemma 11 below).

For $m \geq 0$ set $f_m(z,w) = \sum_{k=m}^{\infty} \psi_k(z) \frac{1}{C}(z,w) \widetilde{\psi_k}(w)$. By Lemma 2, the series converges and $f_0(z,w) = 1$. Note that $f_m(z,w) = 1 - \sum_{k=1}^{m-1} \psi_k(z) \frac{1}{C}(z,w) \widetilde{\psi_k}(w)$ is a polynomial for each m. In particular, $f_m(T,T^*)$ makes sense for any operator tuple T.

Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators satisfying $\frac{1}{C}(T,T^*) \geq 0$ and $\sup_m \|f_m(T,T^*)\| < \infty$. As above set $D_T = \frac{1}{C}(T,T^*)^{1/2}$. Define $V : H \to \mathcal{H} \otimes H$ by

$$Vh = \sum_k \psi_k \otimes D_T \psi_k(T)^* h. \tag{6}$$

We show first that this definition is correct (i.e. that the right-hand side converges). Moreover, V is bounded and this definition of V coincides with the definition (5) above.

Proposition 10. *Let $D, \mathcal{H}, T$ be as above. Let $h \in H$. Then $I = f_0(T,T^*) \geq f_1(T,T^*) \geq f_2(T,T^*) \geq \ldots$ and we have $\|Vh\|^2 = \|h\|^2 - \lim_m \langle f_m(T,T^*)h, h \rangle$.*

Proof. For any $k \in \mathbf{N}$ and $h \in H$,

$$\Big\langle \psi_k(T) \frac{1}{C}(T,T^*) \widetilde{\psi_k}(T^*)h, h \Big\rangle = \Big\langle \frac{1}{C}(T,T^*) \psi_k(T)^* h, \psi_k(T)^* h \Big\rangle \geq 0.$$

Thus $f_m(T,T^*) \geq f_{m+1}(T,T^*)$ for each m. Hence the limit $\lim_{m\to\infty} \langle f_m(T,T^*)h, h \rangle$ exists for each $h \in H$. Further for any j, m $(j < m)$

$$\Big\| \sum_{k=j}^{m-1} \psi_k \otimes D_T \psi_k(T)^* h \Big\|^2 = \sum_{k=j}^{m-1} \|D_T \psi_k(T)^* h\|^2$$
$$= \sum_{k=j}^{m-1} \langle \psi_k(T) \frac{1}{C}(T,T^*) \psi_k(T)^* h, h \rangle = \langle (f_j - f_m)(T,T^*)h, h \rangle$$

and it follows that the partial sums of the right-hand side of (6) form a Cauchy sequence. Thus Vh is well-defined, and letting $j = 0$ and $m \to \infty$ in the last calculation we see that

$$\|Vh\|^2 = \lim_m \langle (f_0 - f_m)(T,T^*)h, h \rangle$$
$$= \lim_m \langle (1 - f_m)(T,T^*)h, h \rangle = \|h\|^2 - \lim_m \langle f_m(T,T^*)h, h \rangle. \qquad \square$$

The condition $\|f_m(T,T^*)\| < \infty$ is really necessary. The simplest example is when $\mathcal{H}$ is the Bergman space on the unit disc (see Example 2 in the end of this paper) and $T = 2I$; then even $\langle f_m(T,T^*)h, h \rangle \to -\infty$ for each $h \in \mathcal{H}$ so V is not defined.

Lemma 11. *For all $g \in H$ and any polynomial $f \in \mathcal{H}$,*

$$V^*(f \otimes g) = f(T) D_T g.$$

Proof. In view of our choice of the basis (namely, the property that any polynomial be a finite linear combination of the ψ_j) it suffices to verify this for $f = \psi_j$. But for any $h \in H$,

$$\langle h, V^*(\psi_j \otimes g)\rangle = \langle Vh, \psi_j \otimes g\rangle = \Big\langle \sum_k \psi_k \otimes D_T\psi_k(T)^* h, \psi_j \otimes g\Big\rangle$$
$$= \langle D_T\psi_j(T)^* h, g\rangle = \langle h, \psi_j(T)D_T g\rangle. \qquad \square$$

Thus the definition of V coincides with (5).

Proposition 12. *Let $\mathcal{H}$ be a D-space. Suppose that $\frac{1}{C}$ is a polynomial and the polynomials are (contained and) dense in $\mathcal{H}$. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators satisfying $\frac{1}{C}(T, T^*) \geq 0$ and $\lim_m \langle f_m(T, T^*)h, h\rangle = 0$ for each $h \in H$. Then T^* is unitarily equivalent to the restriction of M_z^* to an invariant subspace.*

Proof. Let V be the operator defined by (6). By Proposition 10, V is an isometry. As in the proof of Theorem 5 one can prove easily (using Lemma 11 instead of Lemma 4) that $VT_j^* = M_{z_j}^* V$ $(j = 1, \ldots, n)$, so that T^* is unitarily equivalent to a restriction of M_z^* to an invariant subspace. $\square$

The above conditions on T are also necessary.

Proposition 13. *Assume that C^{-1} is a polynomial and $\mathcal{H}$ contains the constant functions. Then*

$$\frac{1}{C}(Z, Z^*) = \|1\|_{\mathcal{H}}^2 P,$$

where P is the orthogonal projection onto the constant functions in $\mathcal{H}$.

Proof. Using the equalities $Z_j^* C_z = \overline{z_j} C_z$ it is easy to prove that

$$\big\langle p(Z, Z^*)C_\lambda \,,\, C_\mu\big\rangle = p(\mu, \overline{\lambda})C(\mu, \overline{\lambda}) \tag{7}$$

for each polynomial p. Thus we have

$$\Big\langle \frac{1}{C}(Z, Z^*)C_\lambda, C_\mu\Big\rangle = 1.$$

Let $x := \sum_{r=1}^m \alpha_r C_{\lambda_r}$ and $y := \sum_{s=1}^k \beta_s C_{\lambda_s}$ be finite linear combinations of the functions C_{λ_r}. Then

$$\Big\langle \frac{1}{C}(Z, Z^*)x, y\Big\rangle = \sum_{r,s} \alpha_r \overline{\beta_s}.$$

On the other hand,

$$\langle \|1\|_{\mathcal{H}}^2 Px, y\rangle = \langle x, 1\rangle\langle 1, y\rangle = \sum_{r,s} \alpha_r \overline{\beta}_s \overline{1(\lambda_r)} 1(\lambda_s) = \sum_{r,s} \alpha_r \overline{\beta}_s,$$

and the result follows. $\square$

Lemma 14. *Let $\mathcal{H}$ be a D-space such that $\frac{1}{C}$ is a polynomial and the polynomials are dense in $\mathcal{H}$. Let f_m be defined as above. Then $f_m(Z, Z^*)$ is the orthogonal projection onto $\vee\{\psi_k, k \geq m\}$. In particular, $f_m(Z, Z^*) \geq 0$ and $\lim_{m\to\infty} f_m(Z, Z^*)h = 0$ for each $h \in \mathcal{H}$.*

Proof. In view of the last proposition,

$$(f_0 - f_m)(Z, Z^*)h = \sum_{k=0}^{m-1} \psi_k(Z)\frac{1}{C}(Z, Z^*)\widetilde{\psi_k}(Z^*)h = \sum_{k=0}^{m-1} \psi_k(Z)\big(\langle \widetilde{\psi_k}(Z^*)h, 1\rangle \cdot 1\big)$$

$$= \sum_{k=0}^{m-1} \langle h, \psi_k(Z)1\rangle \psi_k(Z)1 = \sum_{k=0}^{m-1} \langle h, \psi_k\rangle\, \psi_k.$$

As $f_0 = 1$, we thus get

$$f_m(Z, Z^*)h = \sum_{k\geq m} \langle h, \psi_k\rangle\, \psi_k,$$

as claimed. $\square$

Corollary 15. *Let $\mathcal{H}$ be a D-space such that the polynomials are dense in $\mathcal{H}$ and $\frac{1}{C}$ is a polynomial. Let T be a commuting n-tuple of operators on a Hilbert space H. The following statements are equivalent:*
(i) *T^* is unitarily equivalent to the restriction of M_z^* to an invariant subspace;*
(ii) *$\frac{1}{C}(T, T^*) \geq 0$ and $\lim_m f_m(T, T^*)h = 0$ ($h \in H$).*

Proof. If $\frac{1}{C}(T, T^*) \geq 0$ and $f_m(T, T^*) \to 0$ in the strong operator topology, then (i) holds by Proposition 12.

To prove the implication (i) $\Rightarrow$ (ii), take $h \in H$ and let $p = \sum_{\alpha,\beta\in\mathbf{Z}_+^n} d_{\alpha,\beta} u^\alpha v^\beta$ be a polynomial in u, v. Then

$$\langle p(T, T^*)h, h\rangle = \sum_{\alpha,\beta} d_{\alpha,\beta}\langle T^\alpha T^{*\beta} h, h\rangle = \sum_{\alpha,\beta} d_{\alpha,\beta}\langle T^{*\beta} h, T^{*\alpha} h\rangle =$$

$$= \sum_{\alpha,\beta} d_{\alpha,\beta}\langle VT^{*\beta} h, VT^{*\alpha} h\rangle = \sum_{\alpha,\beta} d_{\alpha,\beta}\langle M_z^{*\beta} Vh, M_z^{*\alpha} Vh\rangle = \langle p(M_z, M_z^*)Vh, Vh\rangle.$$

Thus $p(T, T^*) = V^* p(M_z, M_z^*)V$. In particular,

$$\Big\langle \frac{1}{C}(T, T^*)h, h\Big\rangle = \Big\langle \frac{1}{C}(M_z, M_z^*)Vh, Vh\Big\rangle.$$

Now the right hand side is nonnegative by Proposition 13.

Since f_m are also polynomials,

$$f_m(T, T^*) = V^* f_m(M_z, M_z^*)V \to 0$$

in the strong operator topology, by Lemma 14. $\square$

We also get an analogue of Corollaries 8 and 9.

Corollary 16. (dilation and a von Neumann inequality) *Let $\mathcal{H}$ be a D-space on a bounded domain $D \subset \mathbf{C}^n$. Suppose that $\mathcal{H}$ is the closure of the polynomials in $L^2(\phi)$ where ϕ is a finite nonnegative Borel measure with $\operatorname{supp}\phi \subset \overline{D}$, and that $\frac{1}{C}(z,w)$ is a polynomial. Let T be an n-tuple of operators on a Hilbert space H such that $\frac{1}{C}(T,T^*) \geq 0$ and $f_m(T,T^*) \to 0$ in the strong operator topology. Then the multiplications $N = (N_1,\ldots,N_n)$ by the coordinate functions on $L^2(\phi)\otimes H$ are a normal dilation for T, and for any polynomial p*

$$\|p(T)\| \leq \sup_{z\in D} |p(z)|.$$

Moreover, if D is polynomially convex, then $\sigma(T) \subset \overline{D}$. To see this, denote by σ_π the approximate point spectrum. Clearly $\sigma_\pi(M_z) \subset \sigma_\pi(N) = \sigma(N) \subset \overline{D}$, so, by [SZ], $\sigma(M_z) \subset \overline{D}$. Further $\sigma_\pi(T^*) \subset \sigma_\pi(M_z^*) \subset \widetilde{\overline{D}}$ so $\sigma(T^*) \subset \widetilde{\overline{D}}$ and $\sigma(T) \subset \overline{D}$.

* * *

Remarks. (i) Let $\mathcal{H}_i$ be D_i-spaces with reproducing kernels C_i, where $D_i \subset \mathbf{C}^{n_i}$ $(i = 1,\ldots,l)$. Set $D = \prod_{i=1}^l D_i$ and let $\mathcal{H} = \otimes_{i=1}^l \mathcal{H}_i$ be the completed Hilbertian tensor product. Then $\mathcal{H}$ is a D-space with reproducing kernel $C = \prod_{i=1}^l C_i$.

(ii) Let $\mathcal{H}$ be a D-space with the reproducing kernel C, $D \subset \mathbf{C}^n$, let $D' \subset \mathbf{C}^{n'}$ be a domain and $f : D' \to D$ an analytic embedding. Set $\mathcal{H}' = \{h \circ f : h \in \mathcal{H}\}$. Let $\mathcal{H}_0 = \{h \in \mathcal{H} : h \circ f \equiv 0\}$. Clearly $\mathcal{H}_0$ is a closed subspace of $\mathcal{H}$. Define a linear mapping $\Phi : \mathcal{H}/\mathcal{H}_0 \to \mathcal{H}'$ by $\Phi(h + \mathcal{H}_0) = h \circ f$. Define on $\mathcal{H}'$ a norm by $\|h \circ f\|_{\mathcal{H}'} = \|h + \mathcal{H}_0\|_{\mathcal{H}/\mathcal{H}_0}$ so that Φ becomes a unitary operator.

It is easy to verify that $\mathcal{H}'$ is a D'-space with the reproducing kernel $C' = C \circ (f, \widetilde{f})$.

In particular, if f is a biholomorphism, then $\mathcal{H}_0 = \{0\}$ and $\mathcal{H}'$ is unitarily equivalent to $\mathcal{H}$.

Further, let L be a linear subspace of $\mathbf{C}^n$ and set $D' = C \cap L$. Consider the inclusion $f : D' \to D$. Then $\mathcal{H}'$ defined as above consists of all restrictions $\{h|D' : h \in \mathcal{H}\}$ and the reproducing kernel $C' = C|(D' \times \widetilde{D'})$.

(iii) Let $D_i \subset \mathbf{C}^n$ $(i = 1,\ldots,l)$ be domains, let $\mathcal{H}_i$ be D_i-spaces with reproducing kernels C_i. For $D = \bigcap_{i=1}^l D_i$ we can define a D-space as follows: Set $D' = \prod D_i$ and let $\mathcal{H}' = \otimes \mathcal{H}_i$ be the D'-space defined in (i). Then $D = \cap D_i$ can be identified (by means of a biholomorphism) with the "diagonal" $\{(z_i,\ldots,z_l) : z_1 = z_2 = \ldots = z_l\}$. Thus we can define a D space as in (ii).

Examples. (1) As has already been pointed out, the basic example is the Hardy space H^2 over the unit disc $D \subset \mathbf{C}$. In this case $C(z,w) = (1-zw)^{-1} = \sum_{j=0}^\infty z^j w^j$, $\mathcal{H} \otimes H$ is the set of all analytic H-valued functions on D satisfying

$$\|f\|^2 := \sup_{r\leq 1} \frac{1}{2\pi} \int \|f(r(e^{it})\|^2 dt < \infty,$$

$C_T h = \sum_j z^j T^{*j} h$ and $Vh = \sum_j z^j (I - TT^*)^{1/2} T^{*j} h$. The monomials $\psi_k(z) = z^k$ are an orthonormal basis for H^2, and with this choice $f_m(T,T^*) = T^m T^{*m}$. Thus we recover the well-known fact that $V : H \to \mathcal{H} \otimes H$ is defined for any contraction T and is an isometry if and only if $T^{*n} h \to 0$ for all $h \in H$.

(2) Let $D \subset \mathbf{C}$ be the unit disc and let $\mathcal{H}$ be the generalized Bergman space of all analytic functions on D satisfying

$$\|f\|^2 := \frac{k-1}{\pi} \int_D (1-r^2)^{k-2} |f(re^{it})|^2 dm < \infty$$

where m is the area Lebesgue measure and k is an integer ≥ 2. In this case $C(z,w) = (1-zw)^{-k}$, so $\frac{1}{C}(T,T^*) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} T^j T^{*j}$. The contractions satisfying $\frac{1}{C}(T,T^*) \geq 0$ are called k-hypercontractions in [A2].

For $k=2$ we get the classical Bergman space. Choosing the standard orthonormal basis $\psi_j(z) = \sqrt{j+1}\, z^j$, we have

$$f_m(T,T^*) = (m+1)T^m T^{*m} - mT^{m+1}T^{*m+1}.$$

Thus the adjoint of any operator T satisfying $1 - 2TT^* + T^2T^{*2} \geq 0$ and $f_m(T,T^*) \to 0$ in the strong operator topology is unitarily equivalent to the restriction of M_z^* to an invariant subspace. By [A2] (see also [M]) this is equivalent to the condition that T is a contraction satisfying $1 - 2TT^* + T^2T^{*2} \geq 0$ and $T^{*m} \to 0$ strongly. It would be amusing to have a direct proof of this result.

(3) Let $D \subset \mathbf{C}^n$ be the unit ball and $\mathcal{H} = H^2(D)$ the Hardy space on D. Then $\mathcal{H} \subset L^2(\phi)$ where ϕ is the unique rotation invariant Borel measure on ∂D normalized by $\phi(\partial D) = 1$. In this case $C(z,w) = (1 - z_1w_1 - \ldots - z_nw_n)^{-n}$.

More generally, for any $k > n$, the k-Bergman space consists of functions analytic on D that belong to $L^2((1-|z|^2)^{k-n-1}m)$ where m is the Lebesgue measure on D. The reproducing kernel is, up to a constant factor, $C(z,w) = (1 - z_1w_1 - \ldots - z_nw_n)^{-k}$ (for $k = n+1$ we get the classical Bergman space).

The models in these cases (under the assumption that $I - T_1^*T_1 - \cdots T_n^*T_n \geq 0$) were expressed in [MV] in terms of weighted shifts.

(4) Let $D \subset \mathbf{C}^n$ be the unit polydisc. Let $\mathcal{H}$ be the Hardy space over D (= the completion of the space of all polynomials in $L^2(\phi)$ where ϕ is the normalized Lebesgue measure on the Shilov boundary $\partial_0 D = \{z = (z_1, \ldots, z_n) : |z_1| = \cdots = |z_n| = 1\}$). Now $C(z,w) = \prod_{i=1}^n (1 - z_iw_i)^{-1}$, see Remark (i). Under the assumption $\|T_i\| \leq 1$ $(i = 1, \ldots, n)$ the existence of the regular dilation of the n-tuple $T = (T_1, \ldots, T_n)$ is equivalent to Brehmer's condition

$$\sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} T^{*\alpha} T^\alpha \geq 0$$

for all $\beta \leq e = (1, \ldots, 1)$.

Our inequality $\frac{1}{C}(T,T^*) \geq 0$ is the condition of the maximal degree for T^*. Choosing the standard orthonormal basis of monomials, the condition $f_m(T,T^*) \to 0$ is readily seen to be equivalent to

$$\sum_j T_j^m T_j^{*m} - \sum_{j<k} T_j^m T_k^m T_j^{*m} T_k^{*m} + \sum_{j<k<l} T_j^m T_k^m T_l^m T_j^{*m} T_k^{*m} T_l^{*m} - \ldots \to 0$$

(or, phrased another way, $(1 - \frac{1}{C})(T^m, T^{*m}) \to 0$). Models for operators corresponding to the Bergman spaces over the polydisc were studied in [CV].

(5) For domains $D = \{z \in \mathbf{C}^n : \sum_j c_{ij}|z_j|^2 < 1, i = 1, \ldots, l\}$ where $c_{ij} \geq 0$, the corresponding Bergman spaces and models for operators were studied in [V3]. Note that $D = \cap_i D_i$ where $D_i = \{z \in \mathbf{C}^n : \sum_j c_{ij}|z_j|^2 < 1\}$ and each D_i is biholomorphic to the unit ball. Thus our theory can be applied to the corresponding D-spaces, at least for tuples T with $\sigma(T) \subset D$, see Remark (iii).

(6) Domains $\{z \in \mathbf{C}^n : P(|z_1|^2, \ldots, |z_n|^2) < 1\}$ where P is a polynomial with nonnegative coefficients were studied in [Po].

(7) The unit disc in $\mathbf{C}$ and the unit ball in $\mathbf{C}^n$ are special cases of Cartan domains; for complete description and basic properties of the Cartan domains see [H], [U]. An example of a Cartan domain is

$$D = \left\{ (z_1, \ldots, z_4) \in \mathbf{C}^4 : \| \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \| < 1 \right\}$$

where $\| \cdot \|$ is the operator norm on the 2-dimensional Hilbert space $\mathbf{C}^2$.

Let $D \subset \mathbf{C}^n$ be a Cartan domain. Let ϕ denote the unique probability measure on the Shilov boundary $\partial_0 D$ of D that is invariant under the group $GL(D)$ of linear automorphisms of D. The Hardy space $H^2(\partial_0 D)$ consisting of all analytic functions $f : D \to \mathbf{C}$ such that $\sup_{r \to 1} \int_{\partial_0 D} |f(rw)|^2 \, d\phi(w) < \infty\}$ is isometrically embedded into the space $L^2(\partial_0 D, \phi)$. In this case $C(z, w) = \Delta(z, \overline{w})^{-n/r}$ $(z,\ w \in D)$, where r is the rank and Δ is the Jordan triple determinant of D. Then $\mathcal{H} := H^2(\partial_0 D)$ is a D-space, and our theory applies. Moreover, the polynomials are always dense in $\mathcal{H}$ and C^{-1} is a polynomial if n/r is an integer.

(8) More generally, for any λ in the Wallach set $W = W(D)$ of a Cartan domain D the function $C(z, \overline{u}) = \Delta(z, u)^{-\lambda}$ is positive–definite on $D \times D$. Thus C is the reproducing kernel of a Hilbert space $H^2_\lambda(D)$ of analytic functions on D.

Moreover, if λ is in the continuous part W_c of the Wallach set, the space $\mathcal{H} := H^2_\lambda(D)$ is a D-space and our theory applies as well. In this case also the polynomials are dense in $\mathcal{H}$, and $\frac{1}{C}$ is a polynomial if and only if λ is an integer.

In particular, if $\lambda > g - 1$, where $g \in \mathbf{Z}_+$ denotes the genus of D, then $H^2_\lambda(D)$ consists of all analytic functions in $L^2(D, \phi_\lambda)$. Here the measure ϕ_λ is absolutely continuous with respect to the Lebesgue measure dz on D. More precisely, $d\phi_\lambda(z) = c(\lambda)\, \Delta(z, z)^{\lambda - g} dz$ where $c(\lambda) > 0$ is the constant making $\phi_\lambda(D) = 1$. Then $\mathcal{H} := H^2_\lambda(D)$ is a D-space, called the λ-Bergman space. For $\lambda := g$ one obtains the Bergman space of D.

Note also that in general the continuous part of the Wallach set is strictly larger than $(g-1, +\infty)$. For instance if $\lambda := n/r$ $(\leq g-1)$, then we obtain a D-space $H^2_{n/r}(D)$ that coincides isometrically with the Hardy space $H^2(\partial_0 D)$ discussed above.

(9) Let D be a bounded symmetric domain. Then there exist Cartan domains D_i and a biholomorphic map $f : D \to \prod_i D_i$ (the Harish-Chandra realization). For any $\lambda_i \in W_c(D_i)$ we consider the D_i-space $\mathcal{H}_i := \mathcal{H}^2_{\lambda_i}(D_i)$. Let $C_i(z, w) = \Delta_i(z, \overline{w})^{-\lambda_i}$ denote the corresponding reproducing kernel.

Let $\lambda := (\lambda_i)_i$ be an arbitrary collection of values λ_i as above. The constructions of Remarks (i) and (ii) give a D-space $\mathcal{H}$ with reproducing kernel $C = (\prod C_i) \circ (f, \widetilde{f})$.

(10) Let $\mathcal{H}$ be a regular model atom of Agler [A1]. Then clearly $\mathcal{H}$ satisfies conditions (a) – (c) of Definition 1 so that $\mathcal{H}$ is a D-space (where D is the unit disc in $\mathbf{C}$). Thus the model described in this paper generalizes the results of Agler.

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