Dominant Taylor spectrum and invariant subspaces

C. Ambrozie and V. Müller

Abstract. Let $T = (T_1, \dots, T_n)$ be a von Neumann n-tuple of commuting Hilbert space operators of class C_{00} with dominating Taylor spectrum. Then T has a nontrivial joint invariant subspace. Stronger results can be obtained if T possesses a dilation.

I. Introduction

The Scott Brown technique has been a successful method of constructing invariant subspaces of Hilbert space contractions and, more generally, of polynomially bounded operators on Banach spaces. On the other hand, the situation for n-tuples of commuting operators is much more complicated. Even the first question - whether each von Neumann n-tuple of commuting Hilbert space operators with dominant Taylor spectrum has a nontrivial joint invariant subspace - has not been solved yet. Note that for $n = 1$ this was an early result [BCP] that started a long development of the technique.

The first results concerning joint invariant subspaces of commuting n-tuples of operators were obtained by Apostol [A], who studied the left Harte spectrum. The invariant subspaces for n-tuples possessing a dilation with dominant Harte spectrum were obtained by Eschmeier [E1], [E3] and Kosiek-Octavio [KO].

Invariant subspaces for von Neumann n-tuples of Hilbert spaces operators of class C_{00} with dominant essential Taylor spectrum were constructed by Albrecht and Chevreau [AC]. Didas [D2], following ideas of Eschmeier was able to use some points of the Taylor spectrum which are not in the essential Taylor spectrum, in particular all inner points of the Taylor spectrum.

The aim of this paper is to improve the results of Didas, Eschmeier and Albrecht-Chevreau and to show that all points of the Taylor spectrum can be used for the Scott Brown technique. In particular we show that every von Neumann n-tuple of Hilbert space operators with dominant Taylor spectrum satisfying condition C_{00} has a nontrivial joint invariant subspace. Stronger results are true if T is assumed to have a dilation.

II. Taylor spectrum and Scott Brown technique

Let $T = (T_1, \dots, T_n)$ be a commuting n-tuple of operators on a Hilbert space H. Let

$$0 \to \Lambda^0(H) \xrightarrow{\delta_T^0} \Lambda^1(H) \xrightarrow{\delta_T^1} \cdots \xrightarrow{\delta_T^{n-1}} \Lambda^n(H) \to 0$$

The research was supported by the grant No. 201/06/0128 of GA CR, by the Institutional Research Plan AV 0Z 10190503 and by the grant of Romanian government 2-CEx06-11-34.

Keywords and phrases: Taylor spectrum, Scott Brown technique, dominant spectrum, invariant subspaces.

2000 Mathematics Subject Classification: 47A15, 47A13, 47A10.

be the (cochain) Koszul complex of T. For $p = 0, 1, \dots, n$ let $H^p(T) = \operatorname{Ker} \delta_T^p / \operatorname{Im} \delta_T^{p-1}$.

In order to simplify the notations we consider the Koszul complex of T "globally". Let $\Lambda(H) = \bigoplus_{p=0}^n \Lambda^p(H)$. We can identify $\Lambda(H)$ with H^N where $N = 2^n$. The operators δ_T^p define naturally the operator $\delta_T : \Lambda(H) \to \Lambda(H)$ by $\delta_T(x_0 \oplus \cdots \oplus x_n) = \sum_{p=0}^{n-1} \delta_T^p x_p$. The Taylor spectrum $\sigma(T)$ is defined by

$$\sigma(T) = \{\lambda \in \mathbb{C}^n : \operatorname{Ker} \delta_{T-\lambda} \neq \operatorname{Im} \delta_{T-\lambda}\}$$

and the essential Taylor spectrum $\sigma_e(T)$ by

$$\sigma_e(T) = \{\lambda \in \mathbb{C}^n : \dim \operatorname{Ker} \delta_{T-\lambda} / \operatorname{Im} \delta_{T-\lambda} = \infty\}.$$

We start with the following simple lemma:

Lemma 1. *Let $0 < r < 1$, $k \in \mathbb{N}$, let X be a Banach space and let $v_0, \dots, v_{k-1} \in X$. Then there exists a polynomial function $p : \mathbb{C} \to X$ of degree $\le k-1$ satisfying*

$$p(re^{2\pi i m/k}) = v_m$$

for all $m = 0, \dots, k-1$ and

$$\max_{|\mu| \le 1} \|p(\mu)\| \le \Big(\frac{1+r}{r}\Big)^{k-1} \cdot \max\{\|v_m\| : 0 \le m \le k-1\}.$$

Proof. For $m = 0, 1, \dots, k-1$ set $\lambda_m = re^{2\pi i m/k}$. Let

$$p(\mu) = \sum_{m=0}^{k-1} \left(\prod_{j \neq m} \frac{\mu - \lambda_j}{\lambda_m - \lambda_j} \right) v_m.$$

Obviously $p(\lambda_m) = v_m$ for each $m = 0, \dots, k-1$.

For $\mu \in \mathbb{C}$ and $m \in \{0, 1, \dots, k-1\}$ we have

$$\prod_{j \neq m} (\mu - \lambda_j) = \frac{(\mu - \lambda_0) \cdots (\mu - \lambda_{k-1})}{\mu - \lambda_m} = \frac{\mu^k - r^k}{\mu - \lambda_m} = \frac{\mu^k - \lambda_m^k}{\mu - \lambda_m}$$
$$= \mu^{k-1} + \lambda_m \mu^{k-2} + \cdots + \lambda_m^{k-1}.$$

Thus $\left|\prod_{j \neq m} (\lambda_m - \lambda_j)\right| = |k\lambda_m^{k-1}| = kr^{k-1}$.

It follows that for any $\mu \in \mathbb{C}$ with $|\mu| \le 1$ we have the estimate

$$\Big|\prod_{j \neq m} \frac{\mu - \lambda_j}{\lambda_m - \lambda_j}\Big| \le \frac{(1+r)^{k-1}}{kr^{k-1}}$$

and

$$\|p(\mu)\| \le \Big(\frac{1+r}{r}\Big)^{k-1} \max\{\|v_m\| : 0 \le m \le k-1\}.$$

□

Lemma 2. *Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting operators on a Hilbert space H. Let $\varepsilon > 0$ and $g : D_\varepsilon \to \mathbb{C}^{n-1}$ be an analytic function on $D_\varepsilon := \{\lambda \in \mathbb{C} : |\lambda| < \varepsilon\}$. Let $k \geq 1$ and suppose that*

$$\{(\lambda^k g(\lambda), \lambda) \,:\, |\lambda| < \varepsilon\} \subset \sigma(T) \setminus \sigma_e(T).$$

Then the n-tuple $S := (T_1, \ldots T_{n-1}, T_n^k)$ satisfies

$$\dim \operatorname{Ker} \delta_S \,/\, \operatorname{Im} \delta_S \;\geq\; k.$$

Proof. Set $\phi(\lambda) := \lambda^k g(\lambda)$ and $c := \max_{|\lambda| \leq \varepsilon/2} \|g(\lambda)\|_1$, where $\|\cdot\|_1$ denotes the ℓ_1-norm in $\mathbb{C}^{n-1}$. Then

$$\|\phi(\lambda)\|_1 \leq c\,|\lambda|^k$$

for all λ with $|\lambda| < \varepsilon/2$. Let $0 < r < \min\{1, \varepsilon/2\}$. By Lemma 1, there exists a polynomial function $p_r : \mathbb{C} \to \mathbb{C}^{n-1}$ of degree $\leq k-1$ such that

$$p_r(re^{2\pi i m/k}) = \phi(r^k) - \phi(re^{2\pi i m/k})$$

for all $m = 0, 1, \ldots, k-1$ and

$$\max_{|\mu| \leq 1} \|p_r(\mu)\|_1 \leq \Big(\frac{1+r}{r}\Big)^{k-1} \cdot 2cr^k \leq 2^k cr.$$

For $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ write $z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1}$.
Let the function $f_r : \mathbb{C}^n \to \mathbb{C}^n$ be defined by

$$f_r(z) = (z' + p_r(z_n), z_n^k).$$

Set $w^{(r)} := f_r(\phi(r), r)$. Then

$$w^{(r)} = (\phi(r) + p_r(r), r^k) = (\phi(r^k), r^k).$$

Hence $w^{(r)} \in \sigma(T) \setminus \sigma_e(T)$ and $w^{(r)} \to 0$ as $r \to 0$.
For every $w \in \mathbb{C}^n$ with $w_n \neq 0$, the equation

$$f_r(z) = w$$

has exactly k solutions $z = (z', z_n)$. Indeed, if

$$z' + p_r(z_n) = w' \quad \text{and} \quad z_n^k = w_n,$$

then z_n can be any of the k roots of order k of w_n and $z' = w' - p_r(z_n)$. In particular, for $w^{(r)}$ $(= (\phi(r^k), r^k))$, there exist k vectors $z^{(0)}, \ldots, z^{(k-1)} \in \mathbb{C}^n$ satisfying $f_r(z^{(m)}) = w^{(r)}$ $\quad (m = 0, \ldots, k-1)$, namely $(z^{(m)})_n = re^{2\pi i m/k}$ and

$$(z^{(m)})' = w^{(r)\prime} - p_r((z^{(m)})_n) = \phi(r^k) - p_r(re^{2\pi i m/k}) = \phi(re^{2\pi i m/k}).$$

Hence $z^{(m)} \in \sigma(T) \setminus \sigma_e(T)$ for all m. Thus $\sum_{p=0}^{n} \dim H^p(T - z^{(m)}) \geq 1$. Furthermore, for every $p = 0, \ldots, n$ we have an isomorphism

$$H^p(f_r(T) - w^{(r)}) \equiv \bigoplus_{m=0}^{k-1} H^p(T - z_r^{(m)}),$$

see the proof of Theorem 10.3.13 from [EP].

Therefore

$$\sum_{p=0}^{n} \dim H^p(f_r(T) - w^{(r)}) = \sum_{p=0}^{n} \sum_{m=0}^{k-1} \dim H^p(T - z_r^{(m)}) \geq k.$$

For each polynomial mapping $q : \mathbb{C} \to \mathbb{C}^{n-1}$ of degree $\leq k-1$, $q(\mu) = \sum_{j=0}^{k-1} \alpha_j \mu^j$ with coefficients $\alpha_0, \ldots \alpha_{k-1} \in \mathbb{C}^{n-1}$ we have by the Cauchy formulas that $\|\alpha_j\|_1 \leq \max\{\|q(\mu)\|_1 : |\mu| \leq 1\}$. Thus

$$\|p_r(T_n)\|_1 \leq \max\{\|p_r(\mu)\|_1 : |\mu| \leq 1\} \cdot \sum_{j=0}^{k-1} \|T_n^j\| \leq 2^k cr \sum_{j=0}^{k-1} \|T_n^j\|.$$

Hence $\|p_r(T_n)\|_1 \to 0$ and $f_r(T) \to S$ as $r \to 0$.

Then, using the upper semicontinuity of the dimensions of the cohomology space $H^p(\,\cdot\,)$, we derive that

$$\dim \operatorname{Ker} \delta_S / \operatorname{Im} \delta_S = \sum_{p=0}^{n} \dim H^p(S) \geq \limsup_{r \to 0} \sum_{p=0}^{n} \dim H^p(f_r(T) - w^{(r)}) \geq k.$$

□

Lemma 3. *Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be a commuting n-tuple of operators. Let $r > 0$ and $f : \{\mu \in \mathbb{C} : |\mu| < r\} \to \mathbb{C}^{n-1}$ be analytic such that $f(0) = 0$. Suppose that $(f(\mu), \mu) \in \sigma(T) \setminus \sigma_e(T)$ for all $|\mu| < r$. Let $k \in \mathbb{N}$. Then there are orthonormal vectors $x_1, \ldots, x_k \in H^N$ such that $\langle p(T)x_j, x_j \rangle = p(0)$ for all $j = 1, \ldots, k$ and all polynomials p in n variables.*

Proof. Write $f(\mu) = p(\mu) + \mu^k g(\mu)$, where $p = (p_1, \ldots, p_{n-1})$ is an $(n-1)$-tuple of polynomials (of degree $\leq k-1$) and $g = (g_1, \ldots, g_{n-1})$ is an $(n-1)$-tuple of functions analytic in $\{\mu : |\mu| < r\}$. Clearly $p(0) = 0$.

Set $S = (T' - p(T_n), T_n)$ where $T' = (T_1, \ldots, T_{n-1})$. Thus $S = h(T)$ for h defined by $h(w', w_n) = (w' - p(w_n), w_n)$. Hence $h(f(\mu), \mu) = (f(\mu) - p(\mu), \mu)$ for $|\mu| < r$. It follows, by the spectral mapping theorem, that $\{(\mu^k g(\mu), \mu) : \mu \in \mathbb{C}, |\mu| < r\} \subset \sigma(S)$.

Suppose on the contrary that there is a μ, $|\mu| < r$ and $(\mu^k g(\mu), \mu) \in \sigma_e(S)$. By the spectral mapping property for the essential Taylor spectrum, there is a $z \in \sigma_e(T)$ such that $(\mu^k g(\mu), \mu) = h(z) = (z' - p(z_n), z_n)$. Thus $z_n = \mu$ and $z' = \mu^k g(\mu) + p(\mu) = f(\mu)$, a contradiction with the assumption that $(f(\mu), \mu) \notin \sigma_e(T)$. Hence

$$\{(\mu^k g(\mu), \mu) : \mu \in \mathbb{C}, |\mu| < r\} \subset \sigma(S) \setminus \sigma_e(S).$$

By Lemma 2, $\dim \operatorname{Ker} \delta_{(S',S_n^k)}/\operatorname{Im} \delta_{(S',S_n^k)} \geq k$. It is well known that S acts in the quotient space $\operatorname{Ker} \delta_{(S',S_n^k)}/\operatorname{Im} \delta_{(S',S_n^k)}$ as the n-tuple $(0,\ldots,0,N)$ where N is a nilpotent operator. By [D2, Lemma 3.2], there are orthonormal vectors $x_1,\ldots,x_k \in H^N$ such that $\langle p(S)x_j, x_j\rangle = p(0)$ for $j = 1,\ldots,k$ and all polynomials p. This means that $\|x_j\| = 1$ and $\langle S^\alpha x_j, x_j\rangle = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 1$.

We prove that $\langle T^\alpha x_j, x_j\rangle = 0$ for $\alpha \in \mathbb{Z}_+^n, \alpha \neq 0$.

Write $\alpha = (\alpha', \alpha_n)$ where $\alpha' \in \mathbb{Z}_+^{n-1}$ and $\alpha_n \in \mathbb{Z}_+$. We show that $\langle T'^{\alpha'} T_n^{\alpha_n} x_j, x_j\rangle = 0$ by induction on $|\alpha'|$.

If $\alpha' = 0$ and $\alpha_n \neq 0$ then $\langle T_n^{\alpha_n} x_j, x_j\rangle = \langle S_n^{\alpha_n} x_j, x_j\rangle = 0$.

Let $\alpha' \in \mathbb{Z}_+^{n-1}$, $\alpha' \neq 0$, $\alpha_n \in \mathbb{Z}_+$ and suppose that $\langle T'^{\beta'} T_n^{\beta_n} x_j, x_j\rangle = 0$ for all $\beta' \in \mathbb{Z}_+^n$, $\beta_n \in \mathbb{Z}_+$ with $|\beta'| < |\alpha'|$, $|\beta'| + \beta_n \neq 0$. We have

$$\begin{aligned}
0 &= \langle S^\alpha x_j, x_j\rangle \\
&= \big\langle (T_1 - p_1(T_n))^{\alpha_1}(T_2 - p_2(T_n))^{\alpha_2} \cdots (T_{n-1} - p_{n-1}(T_n))^{\alpha_{n-1}} T_n^{\alpha_n} x_j, x_j \big\rangle \\
&= \langle T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{n-1}^{\alpha_{n-1}} T_n^{\alpha_n} x_j, x_j\rangle + \Big\langle \sum_\beta c_\beta T'^{\beta'} T_n^{\beta_n} x_j, x_j \Big\rangle,
\end{aligned}$$

where all terms in the last sum satisfy $|\beta'| < |\alpha'|$. Since $p(0) = 0$, by the induction assumption we have $\big\langle T_1^{\alpha_1} T_2^{\alpha_2} \cdots T_{n-1}^{\alpha_{n-1}} T_n^{\alpha_n} x_j, x_j \big\rangle = 0$.

Hence $\langle p(T)x_j, x_j\rangle = p(0)$ for all $j = 1,\ldots,k$ and all polynomials p. □

Lemma 4. [D2, Lemma 3.3] *Let H be a separable Hilbert space and let $A \subset H$ be a subset which, for each natural number $k \geq 1$, contains an orthonormal system of length k. Then A contains a weak zero sequence of unit vectors.*

Notation Let $\sigma_r(T)$ be the set of those points $z \in \sigma(T) \setminus \sigma_e(T)$ for which there exists a one-dimensional complex-analytic submanifold M of $\mathbb{C}^n$ with $z \in M$ such that $M \subset \sigma(T)$.

Corollary 5. *Let $T \in B(H)^n$ be a commuting n-tuple of operators on a complex Hilbert space H. Then for every $\lambda \in \sigma_r(T)$ there exists a sequence $(x_k)_{k\geq 1}$ of unit vectors $x_k \in H^N$ such that $x_k \to 0$ weakly as $k \to \infty$ and $\langle p(T)x_k, x_k\rangle = p(\lambda)$ for all k and all polynomials p.*

Proof. Without loss of generality we may assume that $\lambda = 0$. By a permutation of variables we can use Lemma 3 and Lemma 4. □

Lemma 6. *Let T be a commuting n-tuple of operators and let λ be an accumulation point of $\sigma(T) \setminus \sigma_e(T)$. Then $\lambda \in \overline{\sigma_r(T)}$.*

Proof. As it is known, the set $\sigma(T) \setminus \sigma_e(T)$ is analytic [L1], [L2], see also [M]. Then, locally around $\lambda \in \sigma(T) \setminus \sigma_e(T)$, say on a small open ball B centered at λ, the set $\sigma(T) \setminus \sigma_e(T)$ is a finite union of irreducible varieties V_j for $j = \overline{1,k}$ with the property $V_i \not\subset \bigcup_{j\neq i} V_j$ [GR, Theorem II.E.15].

We can suppose, for B sufficiently small, that $\sigma_e(T) \cap B = \emptyset$ and that $\lambda \in V_j$ for all j. Therefore we have the equality

$$\sigma(T) \cap B = \bigcup_{j=1}^k V_j.$$

There is at least one j_0 such that V_{j_0} has dimension ≥ 1, for otherwise $\sigma(T) \cap B$ is a discrete set and λ is an isolated point of $\sigma(T)$, see [GR, Lemma III.C.12].

Moreover, V_{j_0} has a dense subset $M \subset V_{j_0}$ consisting of regular points [GR, III.A.10, III.C.3]. By definition, $M \subset \sigma_r(T)$. Thus $\lambda \in \overline{\sigma_r(T)}$. $\square$

III. Invariant subspaces

In the following we assume that G is either the unit polydisc

$$\mathbb{D}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| < 1 \quad (i = 1, \ldots, n)\}$$

or the unit ball

$$B_n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

However, most of the results remain true for more general domains in $\mathbb{C}^n$.

Let $\mathcal{P}(G)$ be the normed space of all complex polynomials in n variables with the norm $\|p\|_G = \sup\{|p(z)| : z \in G\}$.

Let $H^\infty(G)$ be the space of all bounded analytic functions $f = f(z)$ on G, endowed with the sup norm $\|f\|_G := \sup_{z \in G} |f(z)|$. As it is known, $H^\infty(G)$ is a dual space. Clearly $\mathcal{P}(G)$ is a subspace of $H^\infty(G)$. Thus $\mathcal{P}(G)$ inherits the w^*-topology from $H^\infty(G)$.

For any $\lambda \in G$, let $\mathcal{E}_\lambda \in \mathcal{P}(G)^*$ denote the functional of evaluation at the point λ, namely

$$\mathcal{E}_\lambda(p) := p(\lambda)$$

for all $p \in \mathcal{P}(G)$. It is well known that $\mathcal{E}_\lambda$ is w^*-continuous.

Let $T = (T_1, \ldots, T_n)$ be an n-tuple of mutually commuting bounded linear operators on a complex Hilbert space H. One calls T *von Neumann* (over G) if

$$\|p(T)\| \leq \|p\|_G$$

for all $p \in \mathcal{P}(G)$.

Let T be a von Neumann n-tuple of operators on a Hilbert space. For any $x, y \in H$, let $x \otimes y \in \mathcal{P}(G)^*$ denote the functional defined by

$$(x \otimes y)\,(p) := \langle p(T)x, y\rangle$$

for all $p \in \mathcal{P}(G)$. Clearly $x \otimes y$ is continuous and $\|x \otimes y\| \leq \|x\| \cdot \|y\|$.

Set $N = 2^n$ and let H^N be the direct sum of N copies of H, endowed with the norm

$$\|x\| := \Big(\sum_{j=1}^{N} \|x_j\|^2\Big)^{1/2}$$

for $x = (x_j)_{j=1}^N \in H^N$.

For any $x=(x_j)_{j=1}^N$ and $y=(y_j)_{j=1}^N$ in H^N, let $x\,\square\, y\in\mathcal{P}(G)^*$ denote the functional defined by

$$x\,\square\, y := \sum_{j=1}^N x_j\otimes y_j,$$

that is,

$$(x\,\square\, y)\,(p) := \sum_{j=1}^N \langle p(T)x_j, y_j\rangle$$

for all $p\in\mathcal{P}(G)$.

Lemma 7. *Let $T=(T_1,\dots,T_n)$ be a von Neumann n-tuple of operators on a Hilbert space H. Let λ be a non-isolated point in $\sigma(T)\cap G$. Then there exists a sequence (x_k) of unit vectors in H^N such that $x_k\to 0$ weakly and $\|x_k\,\square\, x_k-\mathcal{E}_\lambda\|\to 0$.*

Proof. The statement was proved in [AC, Lemma 1.4] for points of $\sigma_e(T)$ (note that the proof works as well for the norm closure $\overline{\mathcal{P}(G)}$, which also has Gleason's property).

For points $\lambda\in\sigma_r(T)$ the statement was proved in Corollary 5.

Let $\lambda\in(\sigma(T)\setminus\sigma_e(T))\cap G$. By Lemma 6, there exists a sequence λ_k of points of $\sigma_r(T)$ such that $\lambda_k\to\lambda$. Note that $\|\mathcal{E}_{\lambda_k}-\mathcal{E}_\lambda\|\to 0$. Indeed, let $r=\operatorname{dist}\{\lambda,\partial G\}$ and $|\lambda_k-\lambda|<r/2$. Let $p\in\mathcal{P}(G)$, $\|p\|_G=1$. By the Cauchy formula we have

$$|p(\lambda)-p(\lambda_k)|\le|\lambda-\lambda_k|\cdot\max\{|p'(\mu)|:|\mu-\lambda|<r/2\}\le|\lambda-\lambda_k|\cdot\frac{2}{r}.$$

Thus

$$\|\mathcal{E}_{\lambda_k}-\mathcal{E}_\lambda\|=\sup\{|p(\lambda_k)-p(\lambda)|:p\in\mathcal{P}(G),\|p\|_G=1\}\le|\lambda_k-\lambda|\cdot\frac{2}{r}\to 0$$

as $\lambda_k\to\lambda$.

Note also that for any double sequence $(x_j^{(k)})$ of unit vectors in H^N such that $(x_j^{(k)})\to 0$ weakly as $j\to\infty$ and $x_j^{(k)}\,\square\, x_j^{(k)}\to\mathcal{E}_{\lambda_k}$ for each k there exist $j_1<j_2<\cdots$ such that $x_{j_k}^{(k)}\to 0$ weakly as $k\to\infty$ and $x_{j_k}^{(k)}\,\square\, x_{j_k}^{(k)}\to\mathcal{E}_\lambda$. □

Lemma 8. (see [AC]) *Let $T=(T_1,\dots,T_n)$ be a von Neumann n-tuple of operators on a Hilbert space H. Let $\varepsilon>0$. Let $\lambda\in G$ and $x=(x_j)_{j=1}^N\in H^N$ with $\|x\|=1$ and $\lambda\in G$ such that $\|x\,\square\, x-\mathcal{E}_\lambda\|<\varepsilon$. Then there exists $j\in\{1,\dots,N\}$ such that $\|x_j\otimes x_j-\mathcal{E}_\lambda\|<1-2^{-n}+\varepsilon$.*

Proof. We have $\sum_{i=1}^N\|x_i\|^2=\|x\|^2=1$ Fix an index i such that

$$\|x_i\|^2\ge\frac{1}{N}.$$

Then for every $p\in\mathcal{P}(G)$ with $\|p\|\le 1$, we have the estimates

$$|\langle p(T)x_i,x_i\rangle-p(\lambda)|\le\left|\langle p(T)x_i,x_i\rangle-\sum_{j=1}^N\langle p(T)x_j,x_j\rangle\right|+\left|\sum_{j=1}^N\langle p(T)x_j,x_j\rangle-p(\lambda)\right|\le$$

$$\sum_{j:\, j\neq i} |\langle p(T)x_j, x_j\rangle| + \varepsilon \leq \sum_{j:\, j\neq i} \|p(T)\| \cdot \|x_j\|^2 + \varepsilon \leq \sum_{j=1}^{N} \|x_j\|^2 - \|x_i\|^2 + \varepsilon$$

$$\leq \|x\|^2 - \|x_i\|^2 + \varepsilon \leq 1 - \frac{1}{N} + \varepsilon.$$

□

Let $T = (T_1, \ldots, T_n)$ be a von Neumann n-tuple of operators on a Hilbert space H. We say that the n-tuple T is of class $C_{0.}$ if the polynomial functional calculus $p \mapsto p(T)$ from $\mathcal{P}(G)$ to $\mathcal{L}(H)$ is sequentially w^* - SOT continuous. Equivalently, $p_k(T) \to 0$ in the strong operator topology for each Montel sequence (p_k) of polynomials (i.e., $\sup \|p_k\| < \infty$ and $p_k(z) \to 0$ for all $z \in G$).

We say that T is of class $C_{.0}$ if $T^* = (T_1^*, \ldots, T_n^*)$ is of class $C_{0.}$. We say that T is of class C_{00} if it is both of class $C_{0.}$ and $C_{.0}$.

Note that in the case of the unit polydisc, the $C_{0.}$ property reduces to the well known condition $T_j^k \to 0$ in the strong operator topology for all $j = 1, \ldots, n$, see [A, Proposition 1.8].

It is well known that if T is either of class $C_{0.}$ or $C_{.0}$ then the functionals $x \otimes y$ are w^*-continuous for all vectors $x, y \in H$.

Let (x_k) be a sequence of vectors in H weakly converging to zero. It is well known that if T is of class $C_{0.}$ then $a \otimes x_k \to 0$ for all $a \in H$. If T is of class $C_{.0}$ then $x_k \otimes b \to 0$ for all $b \in H$.

We say that a set $A \subset \overline{G}$ is *dominating* in G if $\|f\|_G = \sup\{|f(z)| : z \in A \cap G\}$ for all $f \in H^\infty(G)$.

Theorem 9. *Let G be either the unit ball or the unit polydisc. Let $T = (T_1, \ldots, T_n)$ be a von Neumann n-tuple (over G) of operators on a Hilbert space H. Suppose that T is of class C_{00} and that the Taylor spectrum $\sigma(T)$ is dominating in G. Then T has a nontrivial common invariant subspace.*

Moreover, if the accumulation points in $\sigma(T)$ are dominating in G, then T is reflexive.

Proof. If there is an isolated point in $\sigma(T)$ then the Taylor functional calculus gives the existence of a common invariant subspace.

Thus we may suppose that the accumulation points in $\sigma(T)$ are dominating in G. Let $\theta = 1 - 2^{-n}$. Denote by $\mathcal{L}(\theta)$ the set of all w^*-continuous functionals φ on $\mathcal{P}(G)$ with the property that for each $\varepsilon > 0$ and finite families $a_1, \ldots, a_s, b_1, \ldots, b_s \in H$ there are $x, y \in H$ such that:
(1) $\|x\| \leq 1$, $\|y\| \leq 1$;
(2) $\|\varphi - x \otimes y\| < \theta + \varepsilon$;
(3) $\|a_i \otimes y\| < \varepsilon$ and $\|x \otimes b_i\| < \varepsilon$ for all $i = 1, \ldots, s$.

It is well known that the set $\mathcal{L}(\theta)$ is closed and absolutely convex, see [E2] or [AC]. By Lemmas 7 and 8, $\mathcal{L}(\theta)$ contains the closed absolutely convex hull of the set $\{\mathcal{E}_\lambda : \lambda \text{ accumulation point of } \sigma(T) \cap G\}$. It is well known that this set is equal to $\{\varphi \in \mathcal{P}(G) : \|\varphi\| = 1, \varphi \text{ is } w^*\text{continuous}\}$. By [AC, Proposition 0.1], see also [BFP], T has property $A_{\aleph_0}$, and hence it is reflexive. □

Remark 10. Theorem 9 remains true for all domains $G \subset \mathbb{C}^n$ satisfying the Gleason property for the algebra $\overline{\mathcal{P}(G)}^{w^*}$, i.e., for each $\lambda \in G$ there is a constant $c_\lambda > 0$ such that every function $f \in \overline{\mathcal{P}(G)}^{w^*}$ with $f(\lambda) = 0$ can be written as $f = \sum_{j=1}^n (z_j - \lambda_j) f_j$ for some $f_j \in \overline{\mathcal{P}(G)}^{w^*}$ with $\|f_j\|_G \le c_\lambda \quad (j = 1, \dots, n)$. For details see [AC].

It is well known that the condition C_{00} can be sometimes omitted if we assume that the n-tuple T has a dilation.

We consider the ball case.

We say that an n-tuple $T = (T_1, \dots, T_n)$ has a *spherical dilation* if there are a larger Hilbert space $K \supset H$ and an n-tuple $N = (N_1, \dots, N_n)$ of commuting normal operators on K such that $\sigma(N) \subset \partial G$ and $T^\alpha = P_H N^\alpha | H$ for all $\alpha \in \mathbb{Z}_+^n$, where P_H denotes the orthogonal projection onto H.

Note that an n-tuple possessing a spherical dilation is automatically von Neumann over B_n.

Theorem 11. *Let $T = (T_1, \dots, T_n)$ be an n-tuple of commuting operators on a Hilbert space H. Suppose that T possesses a spherical dilation and that the Taylor spectrum $\sigma(T)$ is dominating in B_n. Then T has a joint invariant subspace.*

Moreover, if T is of class $C_{\cdot 0}$ over B_n, and the set of all accumulation points of $\sigma(T)$ is dominating in B_n, then T is reflexive.

Proof. Suppose first that T is of class $C_{\cdot 0}$ and the set of all accumulation points of $\sigma(T)$ is dominating in B_n.

Let $\theta = 1 - 2^{-n}$. Denote by $\mathcal{E}_\theta^r(T)$ the set of all w^*-continuous functionals $\varphi \in \mathcal{P}(G)^*$ for which there are sequences $(x_k), (y_k) \subset H$ such that
(1) $\|x_k\|, \|y_k\| \le 1$ for all k;
(2) $\limsup_{k\to\infty} \|\varphi - x_k \otimes y_k\| \le \theta$;
(3) $x_k \otimes z \to 0$ for all $z \in H$.

Since the set of non-isolated points of $\sigma(T)$ is dominating in B_n, by Lemmas 7 and 8 we have

$$\begin{aligned}\overline{\Gamma}\mathcal{E}_\theta^r(T) &\supset \overline{\Gamma}(\{\mathcal{E}_\lambda : \lambda \text{ accumulation point of } \sigma(T) \cap G\}) \\ &= \{\varphi \in \mathcal{P}(G)^* : \|\varphi\| \le 1 \text{ and } \varphi \text{ is } w^*\text{-continuous}\},\end{aligned}$$

where $\overline{\Gamma}$ denotes the closed absolutely convex hull.

By [D1, Corollary 4.4.3], this implies property $A_{1,\aleph_0}$. So T is reflexive.

Let T be a general n-tuple with dominant Taylor spectrum and possessing a spherical dilation. If there is an isolated point in $\sigma(T)$ then the Taylor functional calculus provides a nontrivial joint invariant subspace. Thus we may assume that there are no isolated points in $\sigma(T)$.

If T is neither of class $C_{0\cdot}$ nor $C_{\cdot 0}$ then T has a joint invariant subspace by [E1, Theorem 2.3].

If T is of class $C_{\cdot 0}$ then the statement was proved above. If T is of class $C_{0\cdot}$ then we can use the same result for the n-tuple $T^* = (T_1^*, \dots, T_n^*)$. □

Remark 12. Theorem 11 remains true for all strictly pseudoconvex domains, see [D1, Corollary 4.4.3 and Theorem 3.1.1].

References

[AC] E. Albrecht, B. Chevreau, *Invariant subspaces for certain representations of $H^\infty(G)$*, in: Functional Analysis, editors Bierstedt, Pietsch, Ruess, Vogt, Marcel Dekker, New York, 1994, pp. 293–305.

[A] C. Apostol, *Functional calculus and invariant subspaces*, J. Operator Theory 4 (1980), 159–190.

[BFP] H. Bercovici, C. Foiaş, C. Pearcy, *Dual Algebras with Applications to Invariant Subspaces and Dilation Theory*, CBMS Regional Conf. Ser. in Math. 56, Amer. Math. Soc., Providence, RI, 1985.

[BCP] S. Brown, B. Chevreau, C. Pearcy, *Contractions with rich spectrum have invariant subspaces*, J. Operator Theory, 1 (1979), 123–136.

[D1] M. Didas, *Dual algebras generated by von Neumann tuples over strictly pseudo-convex sets*, Dissertationes Math. (Rozprawy Mat.) 425, Warszawa 2004.

[D2] M. Didas, *Invariant subspaces for commuting pairs with normal boundary dilation and dominating Taylor spectrum*, J. Operator Theory 54 (2005), 169–187.

[E1] J. Eschmeier, *Invariant subspaces for spherical contractions*, Proc. London Math. Soc. 75 (1996), 157–176.

[E2] J. Eschmeier, *C_{00}-Representations of $H^\infty(G)$ with dominating Harte spectrum.* In Banach Algebras '97 Proceedings of the 13th international conference on Banach algebras (Eds. E. Albrecht, M. Mathie), Walter de Gruyter, Berlin, 135–151.

[E3] J. Eschmeier, *Invariant subspaces for commuting contractions*, J. Operator Theory 45 (2001), 413–443.

[EP] J. Eschmeier, M. Putinar, *Spectral decompositions and analytic sheaves.* LMS Monographs Series, Clarendon Press, Oxford 1996.

GR] R.C. Gunning, H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1965.

[KO] M. Kosiek, A. Octavio, *On common invariant subspaces for commuting contractions with rich spectrum*, Indiana Univ. Math. J. 53 (2004), 823–844.

[L1] R. Levi, *Notes on the Taylor joint spectrum of commuting operators*, in: Spectral Theory, Banach Center Publications, vol. 8, PWN, Warsaw 1982, 321–322.

[L2] R. Levi, *Cohomological invariants for essentially commuting systems of operators* (Russian), Funktsional Anal. i Prilozhen. 17 (1983), 79–80.

[M] V. Müller, *On the punctured neighbourhood theorem*, J. Operator Theory 43 (2000), 83–95.

Mathematical Institute AV ČR
Zitna 25, 115 67 Praha 1
Czech Republic

ambrozie@math.cas.cz
muller@math.cas.cz