Dvoretzky's type result for operators on Banach spaces

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Abstract. Let $\lambda_1,\dots,\lambda_n$ be elements of the essential approximate point spectrum of a bounded Banach space operator. Then there are corresponding approximate eigenvectors $x_1,\dots,x_n$ such that the norm on the subspace generated by them is almost symmetric.

The result can be used in the Scott Brown technique for Banach space operators. Another application is for the local behaviour of operators.

By the Dvoretzky theorem, each infinite dimensional Banach space contains "nice" finite dimensional subspaces. More precisely,

Theorem 0. Let $k\in \mathbf{N}$ and $\varepsilon>0$. Then there exists $n\in\mathbf{N}$ such that each Banach space X with $\dim X\ge n$ contains an ε-Hilbert subspace Y with $\dim Y=k$.

By saying that Y is an ε-Hilbert space we mean that there are a Hilbert space H and an invertible operator $S:Y\to H$ such that $\|S\|\cdot\|S^{-1}\|<1+\varepsilon$.

The aim of this paper is to study an analogous question for operators on Banach spaces.

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All Banach spaces are considered to be complex. Denote by $\mathcal{L}(X)$ the set of all bounded operators on a Banach space X. Let $\dim X=\infty$ and $T\in\mathcal{L}(X)$. Clearly the question whether there are finite dimensional subspaces $Y\subset X$ such that the restriction $T|Y$ behaves "nicely" (in the sense of Theorem 0) is closely connected with approximate eigenvalues (= elements of the approximate point spectrum $\sigma_\pi(T)$).

Even more appropriate notion in many situations is the essential approximate point spectrum

$$\sigma_{\pi e}(T)=\{\lambda\in\mathbf{C}: T-\lambda \text{ is not upper semi-Fredholm}\}.$$

Equivalently, $\lambda\in\sigma_{\pi e}(T)$ if and only if

$$\inf\bigl\{\|(T-\lambda)x\|: x\in M, \|x\|=1\bigr\}=0$$

for each subspace $M\subset X$ with $\operatorname{codim} M<\infty$.

It is well-known that $\sigma_{\pi e}(T)$ is a nonempty compact subset of $\mathbf{C}$. Further $\sigma_{\pi e}(T)$ contains the topological boundary of the essential spectrum $\sigma_e(T)$.

The following technical lemma will be used frequently:

Lemma 1. ([M1], Lemma 1) Let X be an infinite dimensional Banach space, $F\subset X$ a finite dimensional subspace and $\varepsilon>0$. Then there exists a subspace $M\subset X$ of a finite codimension such that

$$\|f+m\|\ge(1-\varepsilon)\max\{\|f\|,\|m\|/2\}$$

for all $f\in F$ and $m\in M$.

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Corollary 2. Let $T \in \mathcal{L}(X)$, $\lambda \in \sigma_{\pi e}(T)$, $k \in \mathbf{N}$ and $\varepsilon > 0$. Let M be a subspace of X of a finite codimension. Then there exists an ε-Hilbert subspace $Y \subset M$ with $\dim Y = k$ such that

$$\|(T-\lambda)y\| \le \varepsilon \cdot \|y\| \qquad (y \in Y).$$

Proof. By the Dvoretzky theorem, there is $n \in \mathbf{N}$ such that each n-dimensional Banach space contains a k-dimensional ε-Hilbert subspace.

Using Lemma 1 inductively we can find vectors $x_1, \dots, x_n \in M$ of norm one such that $\|(T-\lambda)x_i\| < \frac{\varepsilon}{3n} \quad (i = 1, \dots, n)$ and

$$\Big\|\sum_{i=1}^{n} \alpha_i x_i\Big\| \ge \frac{1}{3} \max_{1 \le i \le n} |\alpha_i|$$

for all $\alpha_1, \dots, \alpha_n \in \mathbf{C}$.

Find a k-dimensional ε-Hilbert subspace $Y \subset \vee\{x_1, \dots, x_n\}$. Let $y = \sum_{i=1}^{n} \alpha_i x_i \in Y$. Then $\|y\| \ge \frac{1}{3} \max_i |\alpha_i|$ so that

$$\|(T-\lambda)y\| \le \sum_{i=1}^{n} |\alpha_i| \cdot \|(T-\lambda)x_i\| \le n \cdot \max_i |\alpha_i| \cdot \frac{\varepsilon}{3n} \le \varepsilon \cdot \|y\|.$$

Much more interesting problem is to find approximate eigenvectors $x_1, \dots, x_k$ corresponding to distinct elements $\lambda_1, \dots, \lambda_k \in \sigma_{\pi e}(T)$. In general we cannot expect to find $x_1, \dots, x_k$ such that the subspace generated by them is ε-Hilbert. Indeed, consider the operator $T = \lambda_1 I \oplus \lambda_2 I$ on the space $X \oplus X$ (the ℓ_1 direct sum). If x_1, x_2 are approximate eigenvectors corresponding to λ_1, λ_2, respectively, then the norm on $\vee\{x_1, x_2\}$ is "almost ℓ_1" rather than "almost Hilbert".

A variation of this example shows that one cannot even expect to have an "almost ℓ_p-norm" (for some p) on the space $\vee\{x_1, \dots, x_k\}$.

On the other hand, it is always possible to find $x_1, \dots, x_k$ such that the norm on $\vee\{x_1, \dots, x_k\}$ is "almost symmetric", i.e., $\left\|\sum \alpha_i x_i\right\| \doteq \left\|\sum \alpha_i' x_i\right\|$ whenever $|\alpha_i| = |\alpha_i'| \quad (i = 1, \dots, k)$.

The proof will be based on a powerful combinatorial principle — the Ramsey theorem. The author wishes to thank to K. John for drawing his attention to applications of the Ramsey theorem in the functional analysis (see [DJT], p. 298) and to J. Nešetřil for detailed information about the Ramsey theory.

Let A be a set and $n > 0$. Denote by $P_n(A)$ the family of all subsets of A of cardinality n. The classical Ramsey theorem says that, given $n > 0$, $r > 0$ and $m > 0$, there exists $u > 0$ with the property that if A is a set with $\operatorname{card} A \ge u$ and $P_n(A)$ is r-coloured (i.e., there is a partition of $P_n(A)$ into r disjoint parts) then there is a set $B \subset A$ with $\operatorname{card} B = m$ such that $P_n(B)$ is monochromatic (i.e., it is contained in one of the parts).

We need the following generalization of the Ramsey theorem, see [GRS], Theorem 5.1.5.

Theorem 3. Let $k > 0$, $n > 0$, $r > 0$ and $m > 0$. Then there exists $u > 0$ with the following property: if $A_1, \ldots, A_k$ are sets with $\operatorname{card} A_i \geq u \quad (1 \leq i \leq k)$ and $P_n(A_1) \times \cdots \times P_n(A_k)$ is r-coloured, then there are sets $B_i \subset A_i$ with $\operatorname{card} B_i = m \quad (1 \leq i \leq k)$ such that $P_n(B_1) \times \cdots \times P_n(B_k)$ is monochromatic.

Remark 4. The classical Ramsey theorem has also an infinite version: if $\operatorname{card} A = \infty$, $n, r \in \mathbf{N}$ and $P_n(A)$ is r-coloured, then there is an infinite subset $B \subset A$ such that $P_n(B)$ is monochromatic.

Such an infinite version of Theorem 3 is not true. Let $k = 2$, $n = 1$ and $A_1 = A_2 = \mathbf{N}$ so that $P_1(A_1) \times P_1(A_2)$ is the complete bipartite graph. Consider the 2-colouring

$$\{(i, j) : i < j\} \cup \{(i, j) : i \geq j\}.$$

It is easy to see that there is no infinite monochromatic complete bipartite subgraph (on the other hand, there are monochromatic bipartite subgraphs of any finite size by Theorem 3).

Lemma 5. Let $k, m, n \in \mathbf{N}$, $\varepsilon > 0$. Then there exists $u \in \mathbf{N}$ with the following property: if X is a Banach space and $M_1, \ldots, M_k$ its subspaces with $\dim M_i \geq u \quad (1 \leq i \leq k)$, then there are vectors $x_{i,t} \in M_i \quad (1 \leq i \leq k, 1 \leq t \leq m)$ of norm one such that

$$(1 - \varepsilon) \sum_{t=1}^{m} |\beta_t|^2 \leq \Big\| \sum_{t=1}^{m} \beta_t x_{it} \Big\|^2 \leq (1 + \varepsilon) \sum_{t=1}^{m} |\beta_t|^2 \qquad (i = 1, \ldots, k, \beta_t \in \mathbf{C}),$$

and

$$\Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} x_{i,t_{il}} \Big\| - \Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} x_{i,t'_{il}} \Big\| < \varepsilon$$

for all $\beta_{il} \in \mathbf{C}$, $|\beta_{il}| \leq 1$ and $1 \leq t_{i,1} < \cdots < t_{i,n} \leq m, 1 \leq t'_{i,1} < \cdots < t'_{i,n} \leq m$.

Proof. Let u and v be positive integers large enough (the precise value of u and v will be clear from the proof; in fact $u >> v >> m$).

Let $M_1, \ldots, M_k \subset X$ and $\dim M_i \geq u \quad (1 \leq i \leq k)$. If u is large enough, then by the Dvoretzky theorem there are ε-Hilbert subspaces $M'_i \subset M_i$ with $\dim M'_i = v \quad (1 \leq i \leq k)$. If we choose an "orthonormal basis" $y_{i1}, \ldots, y_{i,v}$ in M'_i, then $\|y_{ij}\| = 1$ for all i, j and

$$(1 - \varepsilon) \sum_{j=1}^{v} |\beta_j|^2 \leq \Big\| \sum_{j=1}^{v} \beta_j y_{ij} \Big\|^2 \leq (1 + \varepsilon) \sum_{j=1}^{v} |\beta_j|^2 \qquad (\beta_j \in \mathbf{C}).$$

Let F be a finite $\frac{\varepsilon}{3nk}$-net in the unit ball of $\mathbf{C}^{nk}$ with the ℓ_∞-norm.

Let $\langle 0, kn \rangle = G_1 \cup \cdots \cup G_s$ where the sets G_j are pairwise disjoint and $\operatorname{diam} G_j < \frac{\varepsilon}{3}$ for all j. Consider the following colouring of $\prod_{i=1}^{k} P_n(\{1, \ldots, v\})$: if $j_{i,l}, j'_{i,l} \in \mathbf{N} \quad (1 \leq i \leq k, 1 \leq l \leq n)$ and $1 \leq j_{i,1} < j_{i,2} < \cdots < j_{i,n} \leq v$, $1 \leq j'_{i,1} < j'_{i,2} < \cdots < j'_{i,n} \leq v$, then $\prod_{i=1}^{k} \{j_{i,1}, \ldots, j_{i,n}\}$ and $\prod_{i=1}^{k} \{j'_{i,1}, \ldots, j'_{i,n}\}$ are of the same colour if and only if

$$\Big\| \sum_{1 \leq i \leq k} \sum_{1 \leq l \leq n} \beta_{il} y_{i,j_{i,l}} \Big\| \in G_r \iff \Big\| \sum_{1 \leq i \leq k} \sum_{1 \leq l \leq n} \beta_{il} y_{i,j'_{i,l}} \Big\| \in G_r$$

for all $(\beta_{il}) \in F$ and $1 \le r \le s$.

If v is large enough, then by Theorem 3 there are subsets $L_i \subset \{1, \dots, v\}$ such that $\operatorname{card} L_i = m \quad (i = 1, \dots, k)$ and

$$\Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} y_{i,j_{il}} \Big\| - \Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} y_{i,j'_{il}} \Big\| < \frac{\varepsilon}{3}$$

for all $(\beta_{il}) \in F, j_{i,l}, j'_{i,l} \in L_i, j_{i,1} < \dots < j_{i,n}, j'_{i,1} < \dots < j'_{i,n}$. Since F is an $\frac{\varepsilon}{3nk}$-net in the unit ball of $\mathbf{C}^{nk}$ and $\|y_{ij}\| = 1$, it is easy to show that

$$\Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} y_{i,j_{il}} \Big\| - \Big\| \sum_{i=1}^{k} \sum_{l=1}^{n} \beta_{il} y_{i,j'_{il}} \Big\| < \varepsilon \tag{1}$$

for all $\beta_{il} \in \mathbf{C}, |\beta_{il}| \le 1, j_{i,l}, j'_{i,l} \in L_i, j_{i,1} < \dots < j_{i,n}, j'_{i,1} < \dots < j'_{i,n}$. Let $L_i = \{j_{i1}, \dots, j_{im}\} \quad (1 \le i \le k, j_{i1} < \ldots < j_{im})$. For $x_{it} = y_{i,j_{it}}$, (1) gives the statement of Lemma 5.

Theorem 6. Let $k \in \mathbf{N}$, $\varepsilon > 0$. Then there exists $u \in \mathbf{N}$ with the following property: if X is a Banach space and $M_1, \dots, M_k$ its subspaces with $\dim M_i \ge u \quad (1 \le i \le k)$, then there are vectors $x_i \in M_i \quad (1 \le i \le k)$ of norm one such that

$$\Big\| \sum_{i=1}^{k} \alpha_i x_i \Big\| - \Big\| \sum_{i=1}^{k} \alpha'_i x_i \Big\| < \varepsilon \qquad (\alpha_i, \alpha'_i \in \mathbf{C}, |\alpha'_i| = |\alpha_i| \le 1).$$

Proof. Choose $n \ge \frac{2k\pi}{\varepsilon}$ and $m \ge \frac{256k^2 n}{\varepsilon^2}$. Suppose that $M_1, \dots, M_k$ are sufficiently large subspaces of X. By the previous lemma there are vectors $y_{ij} \in M_i \quad (1 \le i \le k, 1 \le j \le mn + n)$ of norm one such that

$$\frac{1}{2} \sum_{j=1}^{mn+n} |\beta_j|^2 \le \Big\| \sum_{j=1}^{mn+n} \beta_j y_{ij} \Big\|^2 \le 2 \sum_{j=1}^{mn+n} |\beta_j|^2 \qquad (\beta_j \in \mathbf{C}),$$

and

$$\Big\| \sum_{i=1}^{k} \sum_{l=1}^{mn} \beta_{il} y_{i,j_{il}} \Big\| - \Big\| \sum_{i=1}^{k} \sum_{l=1}^{mn} \beta_{il} y_{i,j'_{il}} \Big\| < \frac{\varepsilon}{4}$$

for all $\beta_{il} \in \mathbf{C}$, $|\beta_{il}| \le 1$, $1 \le i \le k, 1 \le j_{i,1} < \dots < j_{i,mn} \le mn + n, 1 \le j'_{i,1} < \dots < j'_{i,mn} \le mn+n$. Let φ be the primitive n-th root of 1 so that the set $\{\varphi, \varphi^2, \dots, \varphi^{n-1}, 1\}$ is an $\frac{\varepsilon}{2k}$-net in the unit circle.

Set $x_i = a_i^{-1} \cdot \sum_{j=1}^{mn} \varphi^j y_{ij}$ where $a_i = \| \sum_{j=1}^{mn} \varphi^j y_{ij} \|$. Clearly $\|x_i\| = 1$ and $a_i \ge \frac{\sqrt{mn}}{2}$.

Let $\alpha_i, \alpha'_i \in \mathbf{C}, |\alpha'_i| = |\alpha_i| \le 1 \quad (1 \le i \le k)$. There are exponents $l_1, \dots, l_k \in \{1, \dots, n\}$ such that $|\alpha'_i - \varphi^{l_i} \alpha_i| < \frac{\varepsilon}{2k}$ so that

$$\Big\| \sum_{i=1}^{k} \alpha'_i x_i \Big\| \ge \Big\| \sum_{i=1}^{k} \varphi^{l_i} \alpha_i x_i \Big\| - \varepsilon/2.$$

Thus it is sufficient to show that

$$\Big\|\sum \varphi^{l_i}\alpha_i x_i\Big\| \ge \Big\|\sum \alpha_i x_i\Big\| - \varepsilon/2.$$

We have

$$\begin{aligned}
\Big\|\sum_{i=1}^k \varphi^{l_i}\alpha_i x_i\Big\| &= \Big\|\sum_{i=1}^k \frac{\alpha_i}{a_i}\sum_{j=1}^{mn}\varphi^{j+l_i}y_{i,j}\Big\| \\
&= \Big\|\sum_{i=1}^k \frac{\alpha_i}{a_i}\Big(\sum_{j=1}^{n-l_i}\varphi^{j+l_i}y_{i,j} + \sum_{j=n-l_i+1}^{mn+n-l_i}\varphi^{j+l_i}y_{i,j} - \sum_{j=mn+1}^{mn+n-l_i}\varphi^{j+l_i}y_{i,j}\Big)\Big\| \\
&\ge \Big\|\sum_{i=1}^k \frac{\alpha_i}{a_i}\sum_{j=n-l_i+1}^{mn+n-l_i}\varphi^{j+l_i}y_{i,j}\Big\| - \sum_{i=1}^k \frac{2n}{a_i} \\
&\ge \Big\|\sum_{i=1}^k \frac{\alpha_i}{a_i}\sum_{j=1}^{mn}\varphi^{j+n}y_{i,j}\Big\| - \frac{\varepsilon}{4} - \frac{4kn}{\sqrt{mn}} \ge \Big\|\sum_{i=1}^k \alpha_i x_i\Big\| - \frac{\varepsilon}{2}.
\end{aligned}$$

Corollary 7. Let $T \in \mathcal{L}(X)$, $\lambda_1, \dots, \lambda_k \in \sigma_{\pi e}(T)$, $\varepsilon > 0$, and let $M \subset X$ be a subspace of a finite codimension. Then there exist vectors $x_1, \dots, x_k \in M$ of norm one such that

$$\begin{aligned}
&\|(T-\lambda_i)x_i\| < \varepsilon \qquad (i = 1, \dots, k), \\
&\Big\|\sum_i \alpha_i x_i\Big\| \ge \frac{1}{3}\max_i |\alpha_i| \qquad (\alpha_i \in \mathbf{C}), \\
&\Big\|\sum_{i=1}^k \alpha_i' x_i\Big\| - \Big\|\sum_{i=1}^k \alpha_i x_i\Big\| < \varepsilon \qquad (\alpha_i, \alpha_i' \in \mathbf{C}, |\alpha_i'| = |\alpha_i| \le 1).
\end{aligned}$$

Proof. Use Corollary 2 and Lemma 1 inductively to construct subspaces $M_1, \dots, M_k \subset M$ large enough with the properties

$$\|(T-\lambda_i)y\| < \varepsilon\|y\| \qquad (y \in M_i, y \ne 0)$$

and

$$\Big\|\sum_{i=1}^k y_i\Big\| \ge \frac{1}{3}\max_i \|y_i\| \qquad (y_i \in M_i, i = 1, \dots, k).$$

Corollary 7 now follows immediately from Theorem 6.

Remark 8. By the same methods it is easy to prove also the following variant of the previous result:

Let $T \in \mathcal{L}(X)$, $\lambda_1, \dots, \lambda_k \in \sigma_{\pi e}(T)$, $s \in \mathbf{N}$, $\varepsilon > 0$. Then there are vectors $x_{i,j} \in X \quad (1 \le i \le k, 1 \le j \le s)$ of norm one such that

$$\|(T-\lambda_i)x_{i,j}\| < \varepsilon \qquad (1 \le i \le k, 1 \le j \le s),$$

$$(1-\varepsilon)\sum|\beta_r|^2 \le \Big\|\sum_r \beta_r x_{i,r}\Big\|^2 \le (1+\varepsilon)\sum|\beta_r|^2 \qquad (1\le i\le k, \beta_r\in \mathbf{C}),$$

$$\Big\|\sum_{i,r}\alpha_{i,r}x_{i,r}\Big\| \ge \frac{1}{3}\max_i \Big\|\sum_r \alpha_{i,r}x_{i,r}\Big\| \qquad (\alpha_{i,r}\in \mathbf{C})$$

and

$$\Big\|\sum_{i=1}^k\sum_{r=1}^s \alpha_{i,r}x_{i,r}\Big\| - \Big\|\sum_{i=1}^k\sum_{r=1}^s \alpha'_{i,r}x_{i,r}\Big\| < \varepsilon \qquad (|\alpha'_{i,r}| = |\alpha_{i,r}|\le 1).$$

Situations as in Corollary 7 appear naturally if the Scott Brown technique is applied to Banach space operators.

Let T be a polynomially bounded operator on a Banach space X. Let D be the open unit disc in the complex plane, and let $\mathcal{P}$ be the normed space of all polynomials with the sup-norm on D. Denote by $\mathcal{Q}$ the space of all bounded functionals on $\mathcal{P}$.

Important examples of elements of $\mathcal{Q}$ are the evaluations $\mathcal{E}_\lambda : p\mapsto p(\lambda)$ where $\lambda\in D$, and functionals $x\otimes x^* : p\mapsto \langle p(T)x, x^*\rangle$ where $x\in X$ and $x^*\in X^*$.

A typical problem in the Scott Brown technique is to approximate a finite linear combination $\sum_i \alpha_i\mathcal{E}_{\lambda_i}$, where $\lambda_i\in\sigma_{\pi e}(T)$ and $\alpha_i\in\mathbf{C}, \sum_i|\alpha_i| = 1$, by $x\otimes x^*$ for some $x\in X$ and $x^*\in X^*$, see [E], Corollary 1.11 or [EP], Corollary 5.2. The problem is relatively easy for Hilbert space operators. In the Banach space context this can be done by using the Zenger theorem [BD], page 20. Since the Zenger theorem is formulated only for positive coefficients α_i, it is necessary to consider the decompositions of α_i's into the real and imaginary, as well as into the positive and negative, parts. Corollary 7 enables to get the approximation in a more aesthetic way.

Theorem 9. Let T be a polynomially bounded operator on a Banach space X, let $\lambda_1,\dots,\lambda_k\in\sigma_{\pi e}(T)$, $\varepsilon>0$, let M, F be subspaces of X with $\dim F<\infty$ and $\operatorname{codim} M<\infty$. Let $\alpha_1,\dots,\alpha_k\in\mathbf{C}$ satisfy $\sum_i|\alpha_i|=1$. Then there exist $x\in M$ and $x^*\in F^\perp$ such that $\|x\|=1$, $\|x^*\|\le 2$ and $\|x\otimes x^* - \sum_i\alpha_i\mathcal{E}_{\lambda_i}\|_{\mathcal{Q}}<\varepsilon$.

Proof. Without loss of generality we can assume that all α_i's are nonzero. Let $\delta>0, \delta<1/2$. Let $K=\sup\{\|p(T)\| : \|p\|\le 1\}$.

By Lemma 1 there is a subspace $M'\subset X$ of a finite codimension such that

$$\|f+m'\|\ge \frac{1-\delta}{2}\|m'\| \qquad (f\in F, m'\in M').$$

By Corollary 7 there are vectors $u_1,\dots,u_k\in M\cap M'$ of norm one such that

$$\|(T-\lambda_i)u_i\|<\delta \qquad (i=1,\dots,k),$$

$$\Big\|\sum_{i=1}^k\beta_i u_i\Big\| \ge \frac{1}{3}\max_i|\beta_i| \qquad (\beta_i\in\mathbf{C}),$$

$$\Big\|\sum_{i=1}^k\beta'_i u_i\Big\| - \Big\|\sum_{i=1}^k\beta_i u_i\Big\| < \delta \qquad (\beta_i,\beta'_i\in\mathbf{C}, |\beta'_i|=|\beta_i|\le 1).$$

By the Zenger theorem there are $u^* \in X^*$ and a linear combination $u = \sum_i \mu_i u_i$ such that $\|u\| = 1 = \|u^*\|$ and $\langle \mu_i u_i, u^* \rangle = |\alpha_i| \quad (i = 1, \dots, k)$. Clearly $|\mu_i| \le 3$ for all i.

By the Hahn-Banach theorem there is a functional $v^* \in F^\perp$ such that $\langle \mu_i u_i, v^* \rangle = |\alpha_i| = \langle \mu_i u_i, u^* \rangle \quad (i = 1, \dots, k)$ and

$$
\begin{aligned}
\|v^*\| &= \sup\bigl\{ |\langle a + f, v^* \rangle| : a \in \vee_i u_i, f \in F, \|a + f\| \le 1 \bigr\} \\
&= \sup\bigl\{ |\langle a, u^* \rangle| : a \in \vee_i u_i, \operatorname{dist}\{a, F\} \le 1 \bigr\} \\
&\le \sup\bigl\{ \|a\| : a \in \vee_i u_i, \operatorname{dist}\{a, F\} \le 1 \bigr\} \le \frac{2}{1-\delta} \le 4.
\end{aligned}
$$

Set $v = \sum_{i=1}^k \mu_i \frac{\alpha_i}{|\alpha_i|} u_i$. Then $\|v\| - \|u\| < 3\delta$. Set further $x = \frac{v}{\|v\|}$ and $x^* = \frac{v^*}{\max\{1, \|v^*\|/2\}}$. Thus $\|x\| = 1$ and $\|x^*\| \le 2$. Further $\|x - v\| \le 3\delta$ and $\|x^* - v^*\| \le \frac{2}{1-\delta} - 2 = \frac{2\delta}{1-\delta} \le 4\delta$. We have

$$
\begin{aligned}
&\Bigl\| x \otimes x^* - \sum_{i=1}^k \alpha_i \mathcal{E}_{\lambda_i} \Bigr\| \le \|x \otimes (x^* - v^*)\| + \|(x - v) \otimes v^*\| + \Bigl\| v \otimes v^* - \sum_{i=1}^k \alpha_i \mathcal{E}_{\lambda_i} \Bigr\| \\
&\le K\|x^* - v^*\| + K\|x - v\| \cdot \|v^*\| + \sup\Bigl\{ \Bigl| \langle p(T)v, v^* \rangle - \sum_{i=1}^k \alpha_i p(\lambda_i) \Bigr| : \|p\| \le 1 \Bigr\} \\
&\le 16K\delta + \sup\Bigl\{ \Bigl| \sum_{i=1}^k \langle \mu_i \frac{\alpha_i}{|\alpha_i|} \bigl(p(T) - p(\lambda_i)\bigr) u_i, v^* \rangle \Bigr| : \|p\| \le 1 \Bigr\}.
\end{aligned}
$$

For $\|p\| \le 1$ and $1 \le i \le k$ set $q_i(z) = \frac{p(z) - p(\lambda_i)}{z - \lambda_i}$. Then $\|q_i\| \le \frac{2}{1 - |\lambda_i|}$ and

$$
\begin{aligned}
&\Bigl\| x \otimes x^* - \sum_{i=1}^k \alpha_i \mathcal{E}_{\lambda_i} \Bigr\| \le 16K\delta + \sup\Bigl\{ \Bigl| \sum_{i=1}^k \langle \mu_i \frac{\alpha_i}{|\alpha_i|} q_i(T)(T - \lambda_i) u_i, v^* \rangle \Bigr| : \|p\| \le 1 \Bigr\} \\
&\le 16K\delta + \sum_{i=1}^k 3K \cdot \frac{8\delta}{1 - |\lambda_i|}.
\end{aligned}
$$

For δ small enough we get $\Bigl\| x \otimes x^* - \sum_{i=1}^k \alpha_i \mathcal{E}_{\lambda_i} \Bigr\| < \varepsilon$.

Another application of Corollary 7 concerns the local behaviour of operators, see [M1], [M3]. We need the following lemma [F].

Lemma 10. Let $K \subset \mathbf{C}$ be a nonempty compact set and $k \in \mathbf{N}$. Then there exist elements $\lambda_0, \dots, \lambda_k \in K$ such that

$$
\max_{z \in K} |p(z)| \le \sum_{j=0}^k |p(\lambda_j)|
$$

for all polynomials p of degree $\deg p \le k$.

Proof. The statement is clear if $\operatorname{card} K \le k$.

Suppose that $\operatorname{card} K \geq k+1$. For $x_0, \dots, x_k \in K$ denote by $V(x_0, \dots, x_k) = \det(x_i^j)_{i,j=0}^k = \prod_{i<j}(x_j - x_i)$ the Wandermond determinant. Let $\lambda_0, \dots, \lambda_k \in K$ satisfy

$$|V(\lambda_0, \dots, \lambda_k)| = \max_{x_0, \dots, x_k \in K} |V(x_0, \dots, x_k)|.$$

Then $V(\lambda_0, \dots, \lambda_k) \neq 0$. For $j = 0, 1, \dots, k$ set

$$L_j(z) = V(\lambda_0, \dots, \lambda_{j-1}, z, \lambda_{j+1}, \dots, \lambda_k)/V(\lambda_0, \dots, \lambda_k).$$

Then $|L_j(z)| \leq 1 \quad (j = 0, 1, \dots, k, z \in K)$ and $L_j(\lambda_i) = \delta_{ij}$ (the Kronecker symbol). Thus the polynomials $L_0, \dots, L_k$ are linearly independent and form a basis in the space of all polynomials of degree $\leq k$.

Let p be a polynomial of degree $\leq k$. Then $p(z) = \sum_{j=0}^k p(\lambda_j) L_j(z)$ so that $\max_{z \in K} |p(z)| \leq \sum_{j=0}^k |p(\lambda_j)|$.

The following theorem was proved for Hilbert space operators in [M1], Lemma 4; for Banach spaces it improves [M1], Lemma 3.

Theorem 11. Let $T \in \mathcal{L}(X)$, $k \in \mathbf{N}$, $\varepsilon > 0$, let $M \subset X$ be a subspace of a finite codimension. Suppose that $\operatorname{card} \sigma_{\pi e}(T) \geq k+1$. Then there exists $x \in M$ of norm one such that

$$\|p(T)x\| \geq \frac{1-\varepsilon}{k+1} r_e(p(T))$$

for all polynomials p of degree $\leq k$; here r_e denotes the essential spectral radius.

Proof. Write $K = \sigma_{\pi e}(T)$. For a polynomial $p(z) = \sum_{j=0}^k \beta_j z^j$ write $|p| = \sum_{j=0}^k |\beta_j|$. Since $\operatorname{card} K \geq k+1$, $p|K \neq 0$ for all non-zero polynomials p of degree $\leq k$. Thus there exists a positive constant c such that $\|p\|_K \geq c \cdot |p|$ for all polynomials of degree $\leq k$.

By Lemma 10 there are elements $\lambda_0, \dots, \lambda_k \in K$ such that $\|p\|_K \leq \sum_{i=0}^k |p(\lambda_i)|$ for all polymomials p of degree $\leq k$.

By Corollary 7 we can find vectors $x_0, x_1, \dots, x_k \in M$ of norm one such that

$$\|(T - \lambda_i)x_i\| < \frac{\varepsilon c}{2(k+1)^3 \cdot \max\{1, \|T\|^k\}},$$

$$\|\sum_i \alpha_i x_i\| \geq \frac{1}{3} \max_i |\alpha_i| \qquad (\alpha_i \in \mathbf{C}),$$

$$\|\sum_i \alpha_i x_i\| - \|\sum_i \alpha_i' x_i\| < \frac{\varepsilon}{6(k+1)} \qquad (|\alpha_i'| = |\alpha_i| \leq 1.)$$

Let $p(z) = \sum_{j=0}^k \beta_j z^j$ be a polynomial. We have

$$\|(p(T) - p(\lambda_i))x_i\| = \Big\|\sum_j \beta_j (T^j - \lambda_i^j)x_i\Big\|$$

$$\leq \sum_{j=0}^k |\beta_j| \cdot \|T^{j-1} + \lambda_i T^{j-2} + \dots + \lambda_i^{j-1}\| \cdot \|(T - \lambda_i)x_i\|$$

$$\leq \sum_{j=0}^k |\beta_j| \cdot k\|T\|^{j-1} \cdot \|(T - \lambda_i)x_i\| \leq \frac{\varepsilon c}{2(k+1)^2} \cdot |p| \leq \frac{\varepsilon \|p\|_K}{2(k+1)^2}.$$

By the Zenger theorem there are a linear combination $x = \sum_{i=0}^k \mu_i x_i$ and $x^* \in X^*$ such that $\|x^*\| = 1 = \|x\|$ and $\langle \mu_i x_i, x^* \rangle = \frac{1}{k+1} \quad (i = 0, \dots, k)$. Clearly $|\mu_i| \leq 3 \quad (i = 0, \dots, k)$.

Let p be a polynomial of degree $\leq k$, $\|p\|_K = 1$. Then

$$\begin{aligned}
\|p(T)x\| &= \Big\|\sum_{i=0}^k p(T)\mu_i x_i\Big\| \geq \Big\|\sum_{i=0}^k p(\lambda_i)\mu_i x_i\Big\| - \Big\|\sum_{i=0}^k \big(p(T) - p(\lambda_i)\big)\mu_i x_i\Big\| \\
&\geq \Big\|\sum_{i=0}^k |p(\lambda_i)|\mu_i x_i\Big\| - \frac{\varepsilon}{2(k+1)} - \frac{\varepsilon}{2(k+1)} \geq \Big|\Big\langle \sum_{i=0}^k |p(\lambda_i)| \cdot \mu_i x_i, x^* \Big\rangle\Big| - \frac{\varepsilon}{k+1} \\
&\geq \frac{1}{k+1}\sum_{i=0}^k |p(\lambda_i)| - \frac{\varepsilon}{k+1} \geq \frac{1-\varepsilon}{k+1}.
\end{aligned}$$

Thus $\|p(T)x\| \geq \frac{1-\varepsilon}{k+1}\|p\|_K$ for all polynomials p of degree $\leq k$. By the spectral mapping property for the essential spectrum we have

$$\begin{aligned}
\|p\|_K &= \sup\{|p(z)| : z \in \sigma_{\pi e}(T)\} = \sup\{|p(z)| : z \in \sigma_e(T)\} \\
&= \sup\{|w| : w \in \sigma_e(p(T))\} = r_e(p(T)).
\end{aligned}$$

Example 12. The previous result is the best possible.

Let $X_0, \dots, X_k$ be copies of ℓ_1, $X = \oplus_{j=0}^k X_j$ (the ℓ_1 direct sum), let $\varphi = \exp(\frac{2\pi i}{k+1})$ and $T = \oplus_j \varphi^j I_{H_j}$. Then $K = \sigma_{\pi e}(T) = \{\varphi, \varphi^2, \dots, \varphi^{k-1}, 1\}$. We show that for each $x \in X$ of norm one there is a polynomial p of degree $\leq k$ such that $\|p(T)x\| \leq \frac{1}{k+1}$.

Let $x = \oplus_j x_j \in X$ and $\|x\| = \sum_{j=0}^k \|x_j\| = 1$. Then there is j such that $\|x_j\| \leq \frac{1}{k+1}$; without loss of generality we can assume that $\|x_k\| \leq \frac{1}{k+1}$. Consider the polynomial $p(z) = \frac{1}{k+1}(z^k + z^{k-1} + \dots + z + 1) = \frac{z^{k+1}-1}{z-1} \cdot \frac{1}{k+1} \quad (z \neq 1)$. Then $p(\varphi^j) = 0 \quad (j = 1, \dots, k)$ and $\|p\|_K = |p(1)| = 1$. We have $\|p(T)x\| = \|x_k\| \leq \frac{1}{k+1}$.

Using standard techniques (see [M1], Theorem 5 and [M2], Theorem 8) it is possible to obtain similar estimates for all polynomials.

Corollary 13. Let $T \in \mathcal{L}(X)$, let $x \in X$ and $\varepsilon > 0$. Then there exist $y \in X$ and a positive constant c (depending only on ε) such that $\|y - x\| < \varepsilon$ and

$$\|p(T)y\| \geq \frac{c \cdot r_e(p(T))}{(1 + \deg p)^{1+\varepsilon}}$$

for each polynomial p.

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