EXTENSIONS OF TOPOLOGICAL ALGEBRAS.

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ABSTRACT. We prove that, in the class of commutative topological algebras with separately continuous multiplication, an element is permanently singular if and only if it is a topological divisor of zero. This extends the result given by R. Arens [**1**] for the Banach algebra case. We also give sufficient conditions for non-removability of ideals in commutative topological algebras with jointly continuous multiplication.

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Introduction. By a topological algebra we mean a topological vector space with a jointly continuous multiplication making of it a complex algebra. The topology of a topological algebra A can be given by a system $\mathcal{U}$ of zero-neighbourhoods satisfying the following properties:

(i) For every $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W + W \subset V$.

(ii) For every $V \in \mathcal{U}$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$, $\alpha V \subset V$.

(iii) Every $V \in \mathcal{U}$ is absorbent.

(iv) For every $V \in \mathcal{U}$, there exists $W \in \mathcal{U}$ such that $W \cdot W \subset V$.

Every algebra in this paper will be a commutative complex algebra with unit element denoted usually by e.

A locally convex algebra is a topological algebra with a system of convex zero-neighbourhoods. The topology of a locally convex algebra A can be given by a directed system of seminorms $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$ (in this case, (iv) above can be written as follows: for every $\alpha \in \mathcal{D}$ there exists $\beta \in \mathcal{D}$ such that $|xy|_\alpha \leq |x|_\beta |y|_\beta$ for all $x, y \in A$).

Let A and B be topological algebras with units e_A and e_B, respectively. We say that B is an extension of A if there exists a unit preserving, injective algebra homomorphism $f : A \to B$ such that A is topologically isomorphic to its image $f(A)$. In this case, we identify A with $f(A)$ and simply write $A \subset B$.

Let A be a topological algebra and $I \subset A$ an ideal. We say that I is removable if there exists an extension $B \supset A$ such that I is not contained in any proper ideal of B. It is easy to see that this condition is equivalent to the existence of a finite number of elements $x_1, \ldots, x_k \in I$ and $y_1, \ldots, y_k \in B$ such that $x_1y_1 + \cdots + x_ky_k = e$. An ideal which is not removable will be called non-removable. The notion of non-removable ideal was introduced by R. Arens [**2**]. Non-removable ideals in commutative Banach algebras have been studied, e.g., in [**2**], [**6**], [**4**] and [**5**], and in topological algebras in [**8**], [**9**] and [**10**].

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§**1.** The aim of this section is to give a sufficient condition for an ideal in a topological algebra to be non-removable. This condition will be shown to be more general than the one given in [**10**]. However, it seems that there is no simple necessary and sufficient condition characterizing non-removability. Our result will be reformulated also for permanently singular elements.

Theorem 1. *Let A be a commutative topological algebra with unit e, and $\mathcal{U}(A)$ a system of zero-neighbourhoods defining the topology of A and satisfying* (i)–(iv). *Let $I \subset A$ be an ideal such that*

(1) *For every finite subset* $\{x_1, \cdots, x_k\} \subset I$
$\exists\ V \in \mathcal{U}(A),\ \forall\ W \in \mathcal{U}(A),\ \exists\ n \geq 1,\ \forall r > 0,\ \exists\ u \in A \setminus V$
such that $ux_i^n \in rW \qquad (i = 1, \ldots, k)$

then I is non-removable.

Proof. Suppose, on the contrary, that there exists an extension $B \supset A$, and elements $x_1, \ldots, x_k \in I$; $y_1, \ldots, y_k \in B$ such that $x_1y_1 + \cdots + x_ky_k = e$. Let $\mathcal{U}(B)$ be a system of zero-neighbourhoods for the topology of B. Let $V \in \mathcal{U}(A)$ be the neighbourhood given by condition (1). Take $V', W' \in \mathcal{U}(B)$ such that $V' \cap A \subset V$ and $\underbrace{W'W' + \cdots + W'W'}_{k \text{ times}} \subset V'$, and $W \in \mathcal{U}(A)$ satisfying $W \subset W' \cap A$. Let n be the integer from condition (1) (for V and W) and $m = k(n-1) + 1$. Then we have

$$e = e^m = \left(\sum_{i=1}^{k} x_i y_i\right)^m = \sum_{i_1 + \cdots + i_k = m} \frac{m!}{i_1! \cdots i_k!} (x_1y_1)^{i_1} \cdots (x_ky_k)^{i_k}.$$

In every term of this sum at least one exponent $i_j \geq n$, so that, for some $v_i \in B$, we may write

$$e = \sum_{i=1}^{k} x_i^n v_i.$$

Take $s > 0$ such that $v_i \in sW'$ for $i = 1, \ldots, k$, let $r = s^{-1}$ and take $u \in A \setminus V$ given by condition (1). Then

$$u \in A \setminus V \subset B \setminus V'$$

but, on the other hand

$$ux_i^n v_i = (ux_i^n)v_i \in rW \cdot sW' \subset W' \cdot W' \qquad (i = 1, \ldots, k)$$

and therefore

$$u = ue = \sum_{i=1}^{k} ux_i^n v_i \in \underbrace{W'W + \cdots + W'W'}_{k \text{ times}} \subset V',$$

a contradiction.

Remark 1. For a locally convex algebra A, with the topology given by a system of seminorms $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$ condition (1) can be reformulated as follows:

(1') For every finite subset $\{x_1, \ldots, x_k\} \subset I$

$$\exists\ \alpha \in \mathcal{D},\ \forall\ \beta \in \mathcal{D},\ \exists\ n \geq 1 \text{ such that } \quad \inf\left\{\sum_{i=1}^{k} |ux_i^n|_\beta \ : \ u \in A,\ |u|_\alpha = 1\right\} = 0.$$

Therefore, if I is an ideal in A satisfying (1') then it is non-removable.

Remark 2. In [**10**, Prop.2.18] was given the following sufficient condition for the non-removability of an ideal I in a topological algebra A with a system of zero-neighbourhoods $\mathcal{U}$:

(2) I is contained in an ideal $J = I_1 + I_s(A)$ where:
I_1 consists locally of joint topological divisors of zero, *i.e.,*
for every finite $\{y_1, \ldots, y_r\} \subset I_1$ there exists a net $\{u_\gamma\}_\gamma \subset A$
such that $u_\gamma \not\to 0$ but $u_\gamma y_i \to 0$, for $i = 1, \ldots, r$.
$I_s(A)$ is the set of all elements of A with small powers:
$z \in A$ is said to have small powers if for every zero-neighbourhood V
there exists an integer $n \geq 1$ such that $\lambda z^n \in V$ for all $\lambda \in \mathbb{C}$.

Proposition. *Let A be a topological algebra and $I \subset A$ an ideal satisfying* (2)*, then I satisfies* (1).

Proof. Let $\mathcal{U}$ be a system of zero-neighbourhoods in A satisfying (i)–(iv). To see that I satisfies condition (1), take $x_1, \ldots, x_k \in I$. Then, since I satisfies condition (2), we can find $y_1, \ldots, y_k \in I_1$ and $z_1, \ldots, z_k \in I_s(A)$ such that $x_i = y_i + z_i$ for $i = 1, \ldots, k$. It is easy to see that the y_i's and the z_i's satisfy the following conditions:

(a) $\exists\ V \in \mathcal{U},\ \forall\ W \in \mathcal{U},\ \exists\ u \in A \setminus V$ such that $uy_i \in W$ for $i = 1, \ldots, k$.

(b) $\forall\ U \in \mathcal{U},\ \exists\ n \geq 1$ such that $z_i^n \in \bigcap_{r>0} rU$ for $i = 1, \ldots, k$.

Let $V \in \mathcal{U}$ be given by (a), and for $W \in \mathcal{U}$ arbitrary take $U \in \mathcal{U}$ such that $UU + UU \subset W$. Let $n \geq 1$ be the integer from (b), then we can write:

$$x_i^n = (y_i + z_i)^n = z_i^n + y_i \left[\sum_{j=1}^{n} \binom{n}{j} y_i^{j-1} z_i^{n-j}\right] = z_i^n + y_i v_i \qquad (i = 1, \ldots, k)$$

for some $v_1, \ldots, v_k \in A$. Fix $r > 0$ and let $s > 0$ be such that $v_i \in sU$ for $i = 1, \ldots, k$, then by using (a) we can find $u \in A \setminus V$ such that

$$uy_i \in rs^{-1}U \qquad (i = 1, \ldots, k).$$

Therefore, we can write $ux_i^n = uz_i^n + (uy_i)v_i$ where, for some $t > 0$,

$$uz_i^n \in (tU)\left(\bigcap_{r'>0} r'U\right) \subset \bigcap_{r'>0} r'UU \subset rUU$$

and, on the other hand,

$$(uy_i)v_i \in rs^{-1}U \cdot sU \subset rUU.$$

Hence $ux_i^n \in rUU + rUU \subset rW$, for $i = 1, \ldots, k$, which proves that I satisfies (1).

An element x of a topological algebra A is called permanently singular if x is singular in every extension $B \supset A$. Clearly, $x \in A$ is permanently singular if and only if the ideal xA generated by x is non-removable. Therefore Theorem 1 yields the following

Corollary. *Let A be a commutative topological algebra with unit e, and a system of zero-neighbourhoods $\mathcal{U}$ satisfying* (i)–(iv). *Suppose $x \in A$ satisfies the following condition*

(3) $\exists\, V \in \mathcal{U},\ \forall\, W \in \mathcal{U},\ \exists\, n \geq 1,$ *such that* $(A \setminus V)x^n \cap rW \neq \emptyset$ *for every* $r > 0$,

then x is permanently singular.

Remark 3. The previous corollary for locally convex algebras has been proved in [**8**, Prop. 2]. If A is a locally convex algebra, and $\{|\cdot|_\alpha : \alpha \in \mathcal{D}\}$ is the corresponding system of seminorms, condition (3) may be written as follows:

(3′) $\exists\, \alpha \in \mathcal{D},\ \forall \beta \in \mathcal{D},\ \exists\, n \geq 1,$ such that $\inf\{|zx^n|_\beta : z \in A\ \ |z|_\alpha = 1\} = 0.$

We construct now an example showing that condition (1) is more general than (2) even in the case of simply generated ideals in locally convex algebras.

Example. Let A be the algebra of all polynomials with complex coefficients in the variable x, endowed with the topology given by the system of seminorms $|\cdot|_k$, $k = 1, 2, \ldots$ defined by:

$$\left|\sum_{i=0}^{\infty} \alpha_i x^i\right|_k = \sum_{i=0}^{\infty} c_{ki}|\alpha_i| \qquad (k = 1, 2, \ldots)$$

(actually, all sums are finite) where c_{ki}, $(k = 1, 2, \ldots;\ i = 0, 1, 2, \ldots)$ are positive numbers satisfying:

$$
\begin{aligned}
&(\alpha) \quad c_{k,0} = 1\\
&(\beta) \quad c_{k,i+j} \leq c_{k+1,i}c_{k+1,j}\\
&(\gamma) \quad c_{k+1,i} \geq c_{k,i}\\
&(\delta) \quad c_{k+1,i+1} \geq c_{k,i}\\
&(\varepsilon) \quad \inf\left\{\frac{c_{k,k+j}}{c_{1,j}} : j = 0, 1, \ldots\right\} = 0.
\end{aligned}
$$

Conditions (β) and (γ) imply that A is a locally convex algebra. It is clear, since all $c_{ik} > 0$, that $|\sum \alpha_i x^i|_k > 0$ for every non-zero polynomial $\sum \alpha_i x^i$ and every index $k = 1, 2, \ldots$, this means that there are no elements with small powers in A. Condition (δ) imply $|ax|_{k+1} \geq |a|_k$ for all $a \in A$ and $k = 1, 2, \ldots$, *i.e.* x is not a topological divisor of zero in A. Therefore, the ideal xA does not satisfy (2).

On the other hand, x satisfies (3'): take $\alpha = 1$, and for arbitrary seminorm $|\cdot|_k$ put $n = k$, then, by condition (ε):

$$\inf\{|ux^k|_k \,:\, u \in A,\ |u|_1 = 1\} \leq \inf_{j\geq 0} \left\{ \frac{|x^{k+j}|_k}{|x^j|_1} \right\} = \inf_{j\geq 0} \left\{ \frac{c_{k,k+j}}{c_{1,j}} \right\} = 0.$$

It remains to show that it is possible to find numbers c_{ki} satisfying $(\alpha) - (\varepsilon)$. To see this, assume we can construct sets $M_k \subset \{0, 1, 2, \ldots\}$, $k = 1, 2, \ldots$, satisfying $0 \in M_k$ and

$$\begin{aligned}
&M_{k+1} + M_{k+1} \subset M_k,\\
&M_{k+1} - 1 \subset M_k,\\
&M_{k+1} \subset M_k \text{ and}\\
&\forall\ k \geq 1,\ \forall\ n \geq k,\ \exists\ m \text{ such that } m, m+1, \ldots, m+n \notin M_1, \text{ and } m+n+k \in M_k.
\end{aligned}$$

Now take

$$c_{k,i} = 2^{i - \max\{j \leq i,\ j \in M_k\}} \qquad (i = 0, 1, \ldots, \quad k = 1, 2, \ldots).$$

It is a matter of routine to check that the above properties of the sets M_k imply $(\alpha)-(\delta)$ for c_{ki}. To prove (ε) consider the infimum over those $j = n + m$, where m is the index existing for given k and $n \geq k$, *i.e.*,

$$\inf \left\{ \frac{c_{k,k+j}}{c_{1,j}},\ j \geq 0 \right\} \leq \inf \left\{ \frac{c_{k,n+m+k}}{c_{1,n+m}},\ n \geq k \right\} \leq \inf \left\{ \frac{1}{2^n},\ n \geq k \right\} = 0.$$

The sets M_k can be constructed as follows: put

$$N_k^{(0)} = \left\{ 2^{2^i},\ i \geq k \right\} \qquad (k = 1, 2, \ldots),$$

$$N_k^{(r)} = N_{k+1}^{(r-1)} \cup \left(N_{k+1}^{(r-1)} - 1 \right) \cup \bigcup_{s=0}^{r-1} \left(N_{k+1}^{(r-1)} + N_{k+1}^{(s)} \right) \quad (k = 1, 2, \ldots, \quad r = 1, 2, \ldots)$$

and now take

$$M_k = \bigcup_{r=0}^{\infty} N_k^{(r)} \cup \{0\} \qquad (k = 1, 2, \ldots).$$

Clearly $M_{k+1} \subset M_k$ since $N_{k+1}^{(r-1)} \subset N_k^{(r)}$ for $r = 0, 1, \ldots$. The properties $M_{k+1} - 1 \subset M_k$ and $M_{k+1} + M_{k+1} \subset M_k$ can be checked analogously.

Finally, fix k and $n \geq k$, $n \geq 2$ and put $m = 2^{2^n} - 2n$. Then $2^{2^n} \in M_n$, $2^{2^n} - 1 \in M_{n-1}$ and by induction $m + n + k = 2^{2^n} - (n - k) \in M_k$. It remains to prove that $m, m+1, \ldots, m+n \notin M_1$. First note that for $j = 1, 2, \ldots$, $r = 0, 1, 2, \ldots$ we have:

$$\min N_j^{(r)} = \min N_{j+1}^{(r-1)} - 1 = \cdots = \min N_{j+r}^{(0)} - r = 2^{2^{j+r}} - r.$$

Further, the open interval $\left(2^{2^{n-1}}, 2^{2^n}\right)$ and $N_j^{(0)}$ are disjoint: $N_j^{(0)} \cap \left(2^{2^{n-1}}, 2^{2^n}\right) = \emptyset$, and it is easy to prove, by induction on r, that, as a matter of fact, we have:

$$N_j^{(r)} \cap \left(2^r 2^{2^{n-1}}, 2^{2^n} - r\right) = \emptyset$$

for every r and j as before. Therefore,

$$\begin{aligned} M_1 \cap \left[2^{2^n} - 2n, 2^{2^n} - n\right] &= \bigcup_{r=0}^{n-1} \left\{N_1^{(r)} \cap \left[2^{2^n} - 2n, 2^{2^n} - n\right]\right\} \subset \\ &\subset \bigcup_{r=0}^{n-1} \left\{N_1^{(r)} \cap \left(2^r 2^{2^{n-1}}, 2^{2^n} - r\right)\right\} = \emptyset. \end{aligned}$$

Hence $m = 2^{2^n} - 2n$, $m+1$, $\ldots$, $m+n = 2^{2^n} - n \notin M_1$.

§**2.** In this section we deal with algebras having multiplication only separately continuous. These algebras have been also called topological algebras by some authors (see e.g. [**7**]). To avoid misunderstanding, these algebras will be called s-algebras in this paper.

In terms of zero-neighbourhoods, the difference is that for an s-algebra A we assume (i)–(iii) plus the following (iv') which is weaker than (iv):

(iv') For every $V \in \mathcal{U}$ and $x \in A$, there exists $W \in \mathcal{U}$ such that $xW \subset V$.

An element x of an s-algebra A is said to be a topological divisor of zero if there exists a net $\{u_\alpha\}_\alpha \in A$ such that $u_\alpha \not\to 0$ but $u_\alpha x \to 0$. Clearly, x is not a topological divisor of zero if and only if the mapping $f_x(a) = xa$ is a homeomorphism from A onto xA. The notion of s-extension is defined analogously to the notion of extension for topological algebras. Let A be an s-algebra and $x \in A$ be a topological divisor of zero, then x is singular in any s-extension $B \supset A$. If this were not the case, we could find an s-extension $B \supset A$ and $y \in B$ such that $xy = e$. But for $(u_\alpha)_\alpha$, the net in A such that $u_\alpha \not\to 0$ and $u_\alpha x \to 0$ we would have $u_\alpha = u_\alpha e = (u_\alpha x)y \to 0$ (by the separate continuity of multiplication in B), a contradiction.

The purpose of this section is to prove the converse of the statement above. This will mean that in the class of s-algebras there exists a simple characterization of permanently singular elements, similar to the one that holds for Banach algebras (recall that if A is a Banach s-algebra, then A is a Banach algebra by the Banach–Steinhaus theorem).

Let A be an s-algebra with unit e and $\mathcal{U}$ a system of zero-neighbourhoods in A satisfying (i)–(iii) and (iv'). Let $A[x]$ be the algebra of all polynomials with coefficients from A in one variable x. We define a topology in $A[x]$ in the following way: Let $\tilde{V} = (V_i)_{i=0}^\infty$ be a sequence from $\mathcal{U}$ and define

$$N_{\tilde{V}} = \left\{\sum_{i=0}^n a_i x^i \in A[x] \,:\, a_i \in V_i,\ i = 0, 1, \ldots\right\}.$$

Let $\mathcal{V}$ be the set of all $N_{\tilde{V}}$ obtained from all sequences V. It is easy to see that $\mathcal{V}$ satisfies (i), (ii), (iii) and (iv'), therefore $A[x]$ is an s-algebra (if we identify $A[x]$ with the countable direct sum of copies of A by means of

$$\sum_{i=0}^{n} a_i x^i \in A[x] \to (a_0, a_1, \ldots, a_n, 0, \ldots) \in \bigoplus_{m=0}^{\infty} A$$

the topology defined above is precisely the direct sum topology). By identifying elements of A with constant polynomials we see that $A[x]$ is an s-extension of A. Moreover, if A is a locally convex s-algebra, then $A[x]$ is also locally convex.

Let A be an s-algebra and $I \subset A$ a closed ideal. Then A/I is again an s-algebra. To see this we only need to prove (iv'): let $a+I \in A/I$ and let $V+I$ be a zero-neighbourhood in A/I. Take W such that $aW \subset V$, then

$$(a+I)(W+I) \subset aW + aI + IW + I^2 \subset V + I.$$

Theorem 2. *Let A be an s-algebra with unit e and $u \in A$. Then u is invertible in some s-extension $B \supset A$ if and only if u is not a topological divisor of zero in A.*

Proof. One implication was proved above. Conversely, assume that u is not a topological divisor of zero in A, *i.e.* that $a \mapsto au$ is a homeomorphism from A onto uA. This implies that for every $V \in \mathcal{U}$ there exists $V' \in \mathcal{U}$ such that $V' \cap uA = uV$. Consider the s-algebra $A[x]$ and let I be the ideal generated by $e - ux$, $I = (e - ux)A[x]$. We prove firstly that I is closed in $A[x]$: Let $(p_\alpha)_\alpha$ be a net of elements from I,

$$p_\alpha = (e - ux)\sum_{i=0}^{\infty} b_i^{(\alpha)} x^i = b_0^{(\alpha)} + \sum_{i=0}^{\infty} (b_i^{(\alpha)} - u b_{i-1}^{(\alpha)}) x^i$$

(where only a finite number of coefficients $b_i^{(\alpha)}$ are non-zero for every α) and suppose that $p_\alpha \to p = \sum_{i=0}^{n} a_i x^i$ in the topology of $A[x]$. Then, coordinate-wise, we have:

$$\begin{aligned}
&b_0^{(\alpha)} \to a_0, \\
&b_i^{(\alpha)} - u b_{i-1}^{(\alpha)} \to a_i \qquad \text{for } i = 1, \ldots, n, \\
&b_i^{(\alpha)} - u b_{i-1}^{(\alpha)} \to 0 \qquad \text{for } i > n.
\end{aligned}$$

Since $b_0^{(\alpha)} u \to a_0 u$, we have $b_1^{(\alpha)} \to a_1 + a_0 u$, and inductively:

$$\begin{aligned}
&b_i^{(\alpha)} \to c_i := a_i + a_{i-1}u + \cdots + a_0 u^i \qquad \text{for } i = 0, \ldots, n, \\
&b_i^{(\alpha)} \to a_n u^{i-n} + a_{n-1} u^{i-n+1} + \cdots + a_0 u^i = c_n u^{i-n} \qquad \text{for } i > n
\end{aligned}$$

where $c_n = a_n + a_{n-1}u + \cdots + a_0 u^n$. Suppose $c_n \neq 0$ and let $V_0 \in \mathcal{U}$ such that $c_n \notin V_0$. Let $W_0 \in \mathcal{U}$ such that $W_0 + W_0 \subset V_0$. Construct $V_i, W_i \in \mathcal{U}$ such that:

$$V_{i+1} \cap uA \subset uW_i \qquad \text{and} \qquad W_{i+1} + W_{i+1} \subset V_{i+1} \qquad \text{for } i = 0, 1, 2, \ldots$$

and consider the zero-neighbourhood N_W in $A[x]$ given by the sequence

$$(\underbrace{W_0, \dots, W_0}_{n \text{ times}}, W_0, W_1, W_2, \dots).$$

Since $p_\alpha \to p$ and $b_n^{(\alpha)} \to c_n$, there exists an index α such that

$$b_{n+i}^{(\alpha)} - ub_{n+i-1}^{(\alpha)} \in W_i \qquad (i = 1, 2, \dots)$$

and also

$$b_n^{(\alpha)} - c_n \in W_0.$$

This implies $b_n^{(\alpha)} \notin W_0$ since $c_n \notin V_0$. We prove now, by induction, that $b_{n+i}^{(\alpha)} \notin W_i$ for $i = 0, 1, \dots$: suppose $b_{n+i}^{(\alpha)} \notin W_i$, then $ub_{n+i}^{(\alpha)} \notin uW_i$, and so $ub_{n+i}^{(\alpha)} \notin V_{i+1}$,. Write $ub_{n+i}^{(\alpha)} = \left(-b_{n+i+1}^{(\alpha)} + ub_{n+i}^{(\alpha)}\right) + b_{n+i+1}^{(\alpha)}$ to deduce that $b_{n+i+1}^{(\alpha)} \notin W_{i+1}$. Therefore, we have that $b_{n+i}^{(\alpha)} \notin W_i$ for $i = 0, 1, \dots$ which implies $b_{n+i}^{(\alpha)} \neq 0$ for all $i \geq 0$ and this contradicts the fact that $\sum b_i^{(\alpha)} x^i$ is a polynomial and, consequently, has only a finite number of non-zero coefficients. We have proved that $c_n = 0$ and therefore p, the limit of p_α, can be written as:

$$p = \sum_{i=0}^{n} a_i x^i = (e - ux) \sum_{i=0}^{n-1} c_i x^i \in I = (e - ux)A.$$

Now, let $q : A[x] \to A[x]/I$ be the canonical homomorphism and let $g : A \to A[x]$ be the natural embedding. Denote by $f = q \circ g$. Since $e - ux \in I$, we have $(u + I)(x + I) = e + I$, hence $f(u)$ is invertible in $A[x]/I$. Finally, we must check that $A[x]/I$ is an s-extension of A. Clearly f is a continuous algebra homomorphism. To prove that f is 1-1 and $f(A)$ is topologically isomorphic to A, it suffices to prove that for all $V \in \mathcal{U}$ there exists a sequence $\tilde{W} = \{W_i\}_{i=0}^{\infty}$, $W_i \in \mathcal{U}$ $(i = 0, 1, \dots)$ such that $f(a) \in N_{\tilde{W}}$ implies $a \in V$.

Let $V \in \mathcal{U}$. We can find $W_0 \in \mathcal{U}$ such that $W_0 + W_0 \subset V$ (hence $W_0 \subset V$). Choose $V_1 \in \mathcal{U}$ such that $ua \in V$ implies $a \in W_0$, and take $W_1 \in \mathcal{U}$ such that $W_1 + W_1 \subset V_1$. Define inductively neighbourhoods $V_i, W_i \in \mathcal{U}$ such that

$$ua \in V_{i+1} \text{ implies } a \in W_i,$$
$$W_{i+1} + W_{i+1} \subset V_{i+1}, \text{ (hence } W_{i+1} \subset V_{i+1}).$$

Let $N_{\tilde{W}} \in \mathcal{V}$ be the zero-neighbourhood in $A[x]$ corresponding to the sequence $\tilde{W} = (W_i)_{i=0}^{\infty}$.

Let $a \in A$ satisfy $f(a) \in N_{\tilde{W}} + I$. This means that $a - p \in N_{\tilde{W}}$ for some $p = (e - xu) \sum_{i=0}^{n} b_i x^i \in I$ (we identify A with the constant polynomials $g(A) \subset A[x]$). We have

$$a - p = (a - b_0) + x(ub_0 - b_1) + x^2(ub_1 - b_2) + \dots + x^{n+1}(ub_n).$$

Since $ub_n \in W_{n+1} \subset V_{n+1}$ we have $b_n \in W_n$. Furthemore $ub_{n-1} = (ub_{n-1} - b_n) + b_n \in W_n + W_n \subset V_n$, so that $b_{n-1} \in W_{n-1}$. We continue in the same way and obtain $b_i \in W_i$ for $i = n - 1, \dots, 1, 0$. Finally, since $b_0 \in W_0, a = (a - b_0) + b_0 \in W_0 + W_0 \subset V$.

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