Norms on unitizations of Banach algebras revisited [1]

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Abstract. Let A be an algebra without unit. If $\| \ \|$ is a complete regular norm on A it is known that among the regular extensions of $\| \ \|$ to the unitization of A there exists a minimal (operator extension) and maximal (ℓ_1-extension) which are known to be equivalent. We shall show that the best upper bound for the ratio of these two extensions is exactly 3. This improves the results represented by A.K. Gaur and Z.V. Kovářík and later by T.W. Palmer.

Let A be an algebra and let $\| \ \|$ be a submultiplicative norm on A. So if $\| \ \|$ is complete, then $(A, \| \ \|)$ is a Banach algebra. We shall say that $\| \ \|$ is regular if

$$\|a\| = \sup\{\|ax\|, \|xa\| : x \in A, \|x\| = 1\}$$

for all $a \in A$. A norm $\| \ \|$ is called weakly regular if there exists a constant $M > 0$ such that

$$\|a\| \leq M \cdot \sup\{\|ax\|, \|xa\| : x \in A, \|x\| = 1\}$$

for all $a \in A$. If an algebra A has an identity element (denoted by e) for which $\|e\| = 1$, then $\| \ \|$ is automatically regular.

Let $(A, \| \ \|)$ be a non-unital regular Banach algebra i.e. the norm $\| \ \|$ is regular and let $A^+ = \{a + \lambda e : a \in A, \lambda \in \mathbb{C}\}$ be the unitization of A with the unit element denoted by e. The norm of A can be extended to A^+ in many ways, however, all extensions $| \ |$ satisfying $|e| = 1$ i.e. regular extensions are equivalent.

Denote by $\| \cdot \|_1$ the ℓ_1-norm on A^+,

$$\|a + \lambda e\|_1 = \|a\| + |\lambda|,$$

and by $\| \cdot \|_{op}$ the operator norm

$$\|a + \lambda e\|_{op} = \sup\{\|ax + \lambda x\|, \|xa + \lambda x\| : x \in A, \|x\| = 1\}.$$

By [1], both of the norms above on A^+ are extensions of $\| \ \|$ and the ℓ_1-norm is the maximal and the operator norm the minimal among all regular extensions of the original norm on A. It was shown in [2] that

$$\|a + \lambda e\|_1 \leq 6 \exp(1) \|a + \lambda e\|_{op} \tag{1}$$

[1] The second author was partially supported by the grant No. 201/03/0041 of GAČR
1991 *Mathematics Subject Classification.* Primary 46H05.
Key words and phrases. regular norm, unitization.

for all non-unital regular Banach algebras A and all $a \in A$ and $\lambda \in \mathbb{C}$. It was noted by T.W. Palmer in his review [3] on the paper [2] that the universal constant $6\exp(1)$ in (1) can be improved to $1 + 2\exp(1)$. The same constant has been used also in [4], (Proposition 1.1.13.).

The aim of this note is to show that in fact

$$\|a + \lambda e\|_1 \leq 3\|a + \lambda e\|_{op}$$

for all non-unital regular Banach algebras A and all $a \in A$ and $\lambda \in \mathbb{C}$. Moreover, the constant 3 is the best possible, so that the situation is similar to the case of C^*-algebras and hermitian elements, see [2].

Similar results hold for weakly regular algebras.

Theorem 1. Let A be a non-unital weakly regular Banach algebra, let $\|a\| \leq M \cdot \sup\{\|ax\|, \|xa\| : x \in A, \|x\| = 1\}$ for all $a \in A$. Then

$$\|a + \lambda e\|_1 \leq (2M + 1)\|a + \lambda e\|_{op}$$

for all $a \in A$ and $\lambda \in \mathbb{C}$.

Proof. Without loss of generality we may assume that $\|a + \lambda e\|_1 = 1$.

We distinguish two cases:

A. Let $|\lambda| \leq \frac{1}{2M+1}$. Then $\|a\| = 1 - |\lambda| \geq 1 - \frac{1}{2M+1} = \frac{2M}{2M+1}$, and so

$$\|a\|_{op} = \sup\{\|ax\|, \|xa\| : x \in A, \|x\| = 1\} \geq \frac{2}{2M + 1}.$$

Hence

$$\|a + \lambda e\|_{op} \geq \|a\|_{op} - |\lambda| \geq \frac{2}{2M + 1} - \frac{1}{2M + 1} = \frac{1}{2M + 1}$$

and $\|a + \lambda e\|_1 \leq (2M + 1)\|a + \lambda e\|_{op}$.

B. Let $|\lambda| > \frac{1}{2M+1}$. Clearly A^+ is complete with the $\|\cdot\|_{op}$-norm, and consequently $(A^+, \|\cdot\|_{op})$ is a Banach algebra. Note that A is a proper two-sided ideal in A^+, and so $\|b - e\|_{op} \geq 1$ for each $b \in A$. In particular, $\|\frac{-a}{\lambda} - e\|_{op} \geq 1$, and so $\|a + \lambda e\|_{op} \geq |\lambda|$. Hence

$$\|a + \lambda e\|_{op} \geq |\lambda| > \frac{1}{2M + 1} = \frac{1}{2M + 1}\|a + \lambda e\|_1.$$

Corollary 2. Let A be a non-unital regular Banach algebra. Then

$$\|a + \lambda e\|_1 \leq 3 \cdot \|a + \lambda e\|_{op}$$

for all $a \in A$ and $\lambda \in \mathbb{C}$.

The constant 3 is the best possible, cf. [2]. For the sake of convenience we give a simple example here.

Example 3. Let A be the algebra of all continuous functions $f : [0, \infty) \to \mathbb{C}$ such that $\lim_{t\to\infty} f(t) = 0$. Consider the sup-norm on A defined by $\|f\| = \sup\{|f(t)| : t \geq 0\}$. Clearly A is a non-unital Banach algebra and

$$\sup\{\|fg\| : g \in A, \|g\| = 1\} \geq \left\| f \cdot \frac{f}{\|f\|} \right\| = \|f\|$$

for all $f \neq 0$. Thus A is regular.

Consider the function f defined by

$$f(t) = \begin{cases} 2/3 & (0 \leq t \leq 1), \\ 4/3 - 2t/3 & (1 \leq t \leq 2), \\ 0 & (t \geq 2). \end{cases}$$

Then $\|f - 1/3\|_1 = \|f\| + 1/3 = 1$ and $\|f - 1/3\|_{op} = \sup\{|f(t) - 1/3| : t \geq 0\} = 1/3$. Hence $\|f - 1/3\|_1 = 3 \cdot \|f - 1/3\|_{op}$.

Remark 4. The assumption that the algebra is non-unital is essential. Suppose that A is a unital Banach algebra with the unit e_1. Then A is automatically regular. Suppose that we add to A a new unit e_2, i.e., the multiplication in the unitization A^+ is defined by $(a + \lambda e_2)(a' + \lambda' e_2) = aa' + \lambda a' + \lambda' a + \lambda\lambda' e_2$. In particular, $e_1 e_2 = e_2 e_1 = e_1$.

Clearly $\|e_1 - e_2\|_1 = 2$ and

$$\|e_1 - e_2\|_{op} = \sup\{\|(e_1 x - e_2 x\|, \|xe_1 - xe_2\| : x \in A, \|x\| = 1\} = 0.$$

Hence the seminorm $\|\cdot\|_{op}$ defined on A^+ is not equivalent to the norm $\|\cdot\|_1$.

Remark 5. The assumption that the algebra is complete is also essential (cf. [3]). The point is that it is possible to have a non-unital normed algebra whose completion is unital. For example, let B be the ℓ_1-algebra over the free unital semigroup with countably many generators $x_1, x_2, \dots$ satisfying $\|x_n\| = \frac{1}{n}$. Let A_0 be the (non-closed) subalgebra of B generated by the elements $e + x_n$, where e is the unit element in B. Then $e \in \overline{A_0}$. It is a matter of routine to verify that A_0 is regular but the norms $\|\cdot\|_1$ and $\|\cdot\|_{op}$ on the unitization A^+ are not equivalent.

The statement remains true for normed algebras if we assume that the algebra possesses no bounded uniform approximate unit (left or right).

Corollary 6. Let A be a regular normed algebra. Let ε be a positive number such that

$$\sup\{\|ax - x\|, \|xa - x\| : x \in A, \|x\| = 1\} \geq \varepsilon$$

for all $a \in A$ with $\|a\| = 1$. Then

$$\|a + \lambda e\|_1 \leq 3\|a + \lambda e\|_{op}$$

for all $a \in A$ and $\lambda \in \mathbb{C}$.

Remark 7. Sometimes (see [1], [2]) the regular norms are defined as those satisfying $\|a\| = \sup\{\|ax\| : x \in X, \|x\| = 1\}$ for all a. All results of this paper remain true for this definition without any change.

In this paper we used the "symmetrized" definition of regular norms used e.g. in [4].

References

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