Criteria of regularity for norm-sequences. II

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Abstract. The regularity property of a norm-sequence $\rho(n) = \|T^n\|$ $(n \in \mathbf{N})$ ensures that the operator T can be intertwined with an isometry V, which relation can be exploited to obtain a lot of information for T itself, as it was shown in [1] and [2]. In [3] general sufficient conditions of regularity were provided. In the present note a necessary and sufficient condition of regularity is given. Applying this criterion a non-regular norm-sequence ρ of positive radius is exhibited, settling the question, posed in [1] and [3], in the negative.

1. Introduction

A mapping $\rho : \mathbf{N}_0 = \mathbf{N} \cup \{0\} \to \mathbf{R}^+$ is called a *norm-sequence*, if $\rho(0) = 1$ and $\rho(m+n) \leq \rho(m)\rho(n)$ holds, for every $m, n \in \mathbf{N}_0$. These sequences are exactly those, which arise as $\rho(n) = \|T^n\|$ $(n \in \mathbf{N}_0)$ with a non-nilpotent Hilbert or Banach space operator T.

We say that the norm-sequence ρ is *regular*, if there exists a gauge function p adjusted to ρ, more precisely, if there exist a mapping $p : \mathbf{N}_0 \to \mathbf{R}^+$ and a positive number c satisfying the conditions

$$\rho(n) \leq p(n) \quad \text{for every} \quad n \in \mathbf{N}_0,$$

$$\lim_{N\to\infty} \sup_{m\in\mathbf{N}_0} \frac{1}{N} \sum_{n=m}^{m+N-1} \left| \frac{p(n+1)}{p(n)} - c \right| = 0 \tag{1}$$

and

$$\limsup_{N\to\infty} \sup_{m\in\mathbf{N}_0} \frac{1}{N} \sum_{n=m}^{m+N-1} \frac{\rho(n)}{p(n)} > 0. \tag{2}$$

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It is not difficult to verify that c coincides with the radius $r(\rho) := \lim_{n\to\infty} \rho(n)^{1/n} = \inf_{n\in\mathbf{N}} \rho(n)^{1/n}$ of ρ. Thus the regularity of ρ implies the positivity of the radius $r(\rho)$. (See [1] for details.)

General sufficient conditions of regularity were provided in [3] in terms of the existence of arbitrarily long monotone sections in the derived sequence $\{\rho(n)^{1/n}\}_{n\in\mathbf{N}}$. Our aim in this paper is to give a necessary and sufficient condition of regularity. This criterion enables us to exhibit a non-regular norm-sequence of positive radius, settling a question, posed in [1] and [3], in the negative.

2. Main results

Given a norm-sequence ρ of positive radius $r(\rho)$, $m \in \mathbf{N}_0$ and $k \in \mathbf{N}$, let us introduce the notation

$$M_\rho(m,k) := \max\{r(\rho)^{-n}\rho(n) : m \le n < m+k\},$$

and

$$c_\rho(m,k) := \sum_{n=m}^{m+k-1} \frac{r(\rho)^{-n}\rho(n)}{k\cdot M_\rho(m,k)}.$$

If no confusion can arise then we omit the index ρ and write for short $M(m,k)$ and $c(m,k)$. Furthermore, let

$$c(\rho) := \inf_{k\in\mathbf{N}} \sup_{m\in\mathbf{N}_0} c_\rho(m,k).$$

Note that if ρ is a norm-sequence of positive radius, then the norm- sequence ρ' defined by $\rho'(n) = r(\rho)^{-n}\rho(n)$ satisfies $r(\rho') = 1$ and $c_\rho(m,k) = c_{\rho'}(m,k)$ for all m,k. Thus $c(\rho') = c(\rho)$.

In this way it is possible to reduce problems concerning norm-sequences of positive radius to the case of norm-sequences of radius equal to 1. Formulas defining $M(m,k)$ and $c(m,k)$ become then particularly simple.

The following lemma summarizes basic properties of the numbers $c(m,k)$.

Lemma 1. Let ρ be a norm-sequence of positive radius. Then:
(i) $0 < c(m,k) \le 1$ for all m,k;
(ii) if $q \ge 2$ and $0 \le m_0 < m_1 < \cdots < m_q$ then

$$(m_q - m_0)\cdot c(m_0, m_q - m_0) \le \sum_{j=0}^{q-1}(m_{j+1} - m_j)\cdot c(m_j, m_{j+1} - m_j).$$

Proof. (i) is clear.

(ii) We may assume that $r(\rho) = 1$. We have

$$(m_q - m_0) \cdot c(m_0, m_q - m_0) = \sum_{n=m_0}^{m_q-1} \frac{\rho(n)}{M(m_0, m_q - m_0)}$$

$$\leq \sum_{j=0}^{q-1} \sum_{n=m_j}^{m_{j+1}-1} \frac{\rho(n)}{M(m_j, m_{j+1} - m_j)} = \sum_{j=0}^{q-1} (m_{j+1} - m_j) \cdot c(m_j, m_{j+1} - m_j).$$

Q. E. D.

Our main result is the following

Theorem. *The norm-sequence ρ is regular if and only if $r(\rho) > 0$ and $c(\rho) > 0$.*

This statement is an immediate consequence of the subsequent Propositions 1, 2 and [1, Proposition 1].

Corollary. *There exists a non-regular norm-sequence ρ of positive radius $r(\rho)$.*

Proof. The sequence ρ is defined in the following way. Any $n \in \mathbf{N}_0$ can be uniquely expressed as $n = \sum_{i=0}^{\infty} a_i 2^i$, where $a_i \in \{0, 1\}$ for every i, and $\sum_{i=0}^{\infty} a_i < \infty$. Then $\rho(n) := \prod_{i=0}^{\infty} (i+2)^{a_i}$.

We show that ρ is a norm-sequence. The inequality $\rho(m+n) \leq \rho(m)\rho(n)$ is clear if $0 \leq m, n < 2$. Let $k \in \mathbf{N}$ and suppose that $\rho(m+n) \leq \rho(m)\rho(n)$ for all $m, n < 2^k$. Let $m, n < 2^{k+1}, m = a \cdot 2^k + m', n = b \cdot 2^k + n'$, where $a, b \in \{0, 1\}$ and $m', n' < 2^k$. Set $c := 0$ if $m' + n' < 2^k$ and $c := 1$ if $m' + n' \geq 2^k$. It is easy to check the inequality $\rho(m+n) \leq \rho(m)\rho(n)$ for any possible choice of the triple $(a, b, c) \in \{0, 1\}^3$.

By induction on k we get that ρ is a norm-sequence.

The radius of ρ is

$$r(\rho) = \lim_{n\to\infty} \rho(n)^{1/n} = \lim_{k\to\infty} \rho(2^k)^{1/2^k} = \lim_{k\to\infty} (k+2)^{1/2^k} = 1.$$

Further, for any $k \in \mathbf{N}$, we have

$$\sum_{n=0}^{2^k-1} \rho(n) = \prod_{j=2}^{k+1} (j^0 + j^1) = \frac{(k+2)!}{2}$$

and

$$M(0, 2^k) = \rho(2^k - 1) = (k+1)!$$

so that

$$c(0, 2^k) = \frac{k+2}{2^{k+1}}.$$

Furthermore, for all $j \in \mathbf{N}$ and $0 \le n < 2^k$ we have $\rho(j \cdot 2^k + n) = \rho(j \cdot 2^k)\rho(n)$, so that $c(j \cdot 2^k, 2^k) = c(0, 2^k) = (k+2)2^{-(k+1)}$.

To show that $c(\rho) = 0$, fix $k \ge 1$. Let $m \in \mathbf{N}_0$ and $d \in \mathbf{N}, d > 2^{k+1}$. Find $s, t \in \mathbf{N}_0$ such that $s \cdot 2^k \le m < (s+1)2^k$ and $t \cdot 2^k \le m + d < (t+1)2^k$. By Lemma 1 we have

$$d \cdot c(m,d) \le ((s+1) \cdot 2^k - m) + \sum_{j=s+1}^{t-1} 2^k \cdot c(j2^k, 2^k) + (m + d - t \cdot 2^k) \le 2^{k+1} + (t-s-1)2^k \frac{k+2}{2^{k+1}}$$

so that

$$\sup_{m \in \mathbf{N}_0} c(m,d) \le \frac{2^{k+1}}{d} + \frac{k+2}{2^{k+1}}.$$

Hence $c(\rho) \le \frac{k+2}{2^{k+1}}$ and since k was arbitrary, we conclude that $c(\rho) = 0$.

In view of our Theorem the norm-sequence ρ can not be regular. Q. E. D.

3. Necessity

Proposition 1. Let ρ be a regular norm-sequence. Then $c(\rho) > 0$.

Proof. Suppose on the contrary that ρ is regular and $c(r) = 0$. We infer by [1, Proposition 1] that $r(\rho)$ is positive. Without loss of generality we can assume that $r(\rho) = 1$.

The regularity of ρ means that there exists a gauge function p adjusted to ρ, that is

$$\rho(n) \le p(n) \qquad \text{for every} \qquad n \in \mathbf{N}_0,$$

$$\lim_{N \to \infty} \sup_{m \in \mathbf{N}_0} \frac{\nu(m,N)}{N} = 0$$

and there is $\alpha > 0$ such that

$$\limsup_{N \to \infty} \sup_{m \in \mathbf{N}_0} \frac{\tau(m,N)}{N} > \alpha,$$

where we write for short

$$\nu(m,N) = \sum_{n=m}^{m+N-1} \left| \frac{p(n+1)}{p(n)} - 1 \right| \qquad \text{and} \qquad \tau(m,N) = \sum_{n=m}^{m+N-1} \frac{\rho(n)}{p(n)}.$$

Thus $\nu(m,N)$ characterizes the smoothness of p and $\tau(m,N)$ the distance of ρ and p.

Since $c(\rho) = 0$ there exists $k \in \mathbf{N}$ such that $c(m,k) < \alpha/4$ for each $m \in \mathbf{N}_0$.

Find $\varepsilon > 0$ such that $\varepsilon < \alpha/4$, $(1 - k\varepsilon)^k > 1/2$ and $(1 + k\varepsilon)^k < 2$.

Finally find $d \ge 4k(\varepsilon + 1)\alpha^{-1}$ and $m \in \mathbf{N}_0$ such that

$$\frac{\nu(m,d)}{d} < \varepsilon^2 \qquad \text{and} \qquad \frac{\tau(m,d)}{d} > \alpha.$$

Let $t \in \mathbf{N}_0$ satisfy $tk \le d < (t+1)k$.

Consider the partition

$$\{m, \dots, m+d-1\} = \bigcup_{j=0}^{t-1} \{m+jk, \dots, m+jk+k-1\} \cup \{m+tk, \dots, m+d-1\}.$$

Denote by H the set of all $j \in \{0, 1, \dots, t-1\}$ such that $\nu(m+jk, k) \ge k\varepsilon$. Clearly

$$d\varepsilon^2 > \nu(m, d) \ge \sum_{j \in H} \nu(m+jk, k) \ge k\varepsilon \cdot \operatorname{card} H$$

so that $\operatorname{card} H \le d\varepsilon/k \le \varepsilon(t+1)$.

Suppose now that $j \in \{0, \dots, t-1\} \setminus H$, that is, $\nu(m+jk, k) < k\varepsilon$. Then

$$2^{-1/k} < 1 - k\varepsilon < \frac{p(n+1)}{p(n)} < 1 + k\varepsilon < 2^{1/k} \qquad (m+jk \le n < m+jk+k)$$

so that

$$\frac{p(n)}{p(i)} > \frac{1}{2} \qquad (m+jk \le n, i < m+jk+k).$$

Thus

$$\begin{aligned}
\tau(m+jk, k) &\le 2 \cdot \sum_{n=m+jk}^{m+jk+k-1} \frac{\rho(n)}{\max\{p(m+jk), \dots, p(m+jk+k-1)\}} \\
&\le 2 \cdot \sum_{n=m+jk}^{m+jk+k-1} \frac{\rho(n)}{M(m+jk, k)} = 2k \cdot c(m+jk, k) < \frac{k\alpha}{2}.
\end{aligned}$$

Hence

$$\begin{aligned}
d\alpha < \tau(m, d) &= \sum_{j \in H} \tau(m+jk, k) + \sum_{j \notin H} \tau(m+jk, k) + \tau(m+tk, d-tk) \\
&\le k \cdot \operatorname{card} H + \frac{tk\alpha}{2} + k \le k\varepsilon(t+1) + k + \frac{d\alpha}{2} \\
&\le d\varepsilon + k(\varepsilon+1) + \frac{d\alpha}{2} \le \frac{d\alpha}{4} + \frac{d\alpha}{4} + \frac{d\alpha}{2} = d\alpha,
\end{aligned}$$

a contradiction. Q. E. D.

4. Sufficiency

We need the following lemma:

Lemma 2. Let ρ be a norm-sequence satisfying $r(\rho) = 1$ and $c(\rho) > 0$. Then for all $m_0 \in \mathbf{N}_0$ and $k \in \mathbf{N}$ there exists $m \ge m_0$ such that $c(m, k) \ge c(\rho)/2$.

Proof. Suppose on the contrary that $c(m,k) < c(\rho)/2$ for all $m \ge m_0$.

Choose $t \ge 4m_0/c(\rho)$ and set $k_1 = kt$. By the assumption there is $m_1 \in \mathbf{N}_0$ such that $c(m_1,k_1) > \frac{3}{4}c(\rho)$. By Lemma 1 we have

$$k_1 c(m_1,k_1) \le \sum_{j=0}^{t-1} k \cdot c(m_1 + jk, k).$$

Clearly $c(m_1 + jk, k) < c(\rho)/2$ for all $j \ge m_0$. Thus

$$\frac{3}{4}c(\rho) < c(m_1,k_1) \le \frac{1}{k_1}\Big(m_0 k + (t-m_0)k\frac{c(\rho)}{2}\Big) \le \frac{m_0}{t} + \frac{c(\rho)}{2} \le \frac{3}{4}c(\rho),$$

a contradiction. Q. E. D.

Proposition 2. Let ρ be a norm-sequence satisfying $r(\rho) > 0$ and $c(\rho) > 0$. Then ρ is regular.

Proof. Without loss of generality we can assume that $r(\rho) = 1$.

Find numbers $k_j \ge j \quad (j = 1,2,\ldots)$ such that $\rho(n) \le (1+1/j)^n$ for all $n \ge k_j$.

We construct inductively numbers $d_1, a_1, d_2, a_2, \ldots$ in the following way: set formally $d_0 = a_0 = 0$. Suppose that $j \ge 1$ and that the numbers $d_1, a_1, \ldots, d_{j-1}, a_{j-1}$ are already constructed. Choose $d_j \ge 4k_j/c(\rho)$. By the previous lemma we can find $a_j \ge \max\{4(a_{j-1}+d_{j-1}), 3d_j\}$ such that $c(a_j, d_j) \ge c(\rho)/2$.

Clearly the numbers $d_j, a_j \quad (j = 1,2,\ldots)$ constructed in this way satisfy $a_1 < a_1 + d_1 < a_2 < a_2 + d_2 < \cdots$.

Write for short $M_j = M(a_j, d_j)$. Define function p by

$$p(n) = \begin{cases} \max\{\rho(0),\ldots,\rho(a_1+d_1-1)\} & (n < a_1), \\ M_j & (j \ge 1, a_j \le n \le a_j + d_j - k_j), \\ M_j \cdot (1+3/j)^{n-(a_j+d_j-k_j)} & (j \ge 1, a_j + d_j - k_j < n < a_{j+1}). \end{cases}$$

We prove that p is the required gauge function adjusted to ρ.

Clearly $\rho(n) \le p(n)$ if $a_j \le n \le a_j + d_j - 1$ for some $j \ge 1$.

If $a_j + d_j \le n < a_{j+1}$ then

$$\rho(n) \le \rho(a_j + d_j - k_j)\cdot \rho\big(n - (a_j + d_j - k_j)\big) \le M_j \cdot \Big(1 + 1/j\Big)^{n-(a_j+d_j-k_j)} \le p(n).$$

Thus $\rho(n) \le p(n)$ for all $n \in \mathbf{N}_0$.

Further

$$\sum_{n=a_j}^{a_j+d_j-k_j-1} \frac{\rho(n)}{p(n)} = \sum_{n=a_j}^{a_j+d_j-k_j-1} \frac{\rho(n)}{M_j} \ge \sum_{n=a_j}^{a_j+d_j-1} \frac{\rho(n)}{M_j} - k_j$$
$$= d_j \cdot c(a_j, d_j) - k_j \ge \frac{d_j c(\rho)}{2} - k_j \ge \frac{d_j c(\rho)}{4}$$

so that

$$\frac{1}{d_j-k_j}\sum_{n=a_j}^{a_j+d_j-k_j-1}\frac{\rho(n)}{p(n)}\ge\frac{c(\rho)}{4}.$$

Since $d_j-k_j\to\infty$, the sequence p satisfies (2).

It remains to prove (1). Let $j\ge 2$ and $a_j\le n<a_j+d_j$. Then $n<a_j+d_j\le \frac{4}{3}a_j\le 2(a_j-(a_{j-1}+d_{j-1}))$ so that

$$\rho(n)\le\Big(1+1/j\Big)^n\le\Big(1+1/j\Big)^{2(a_j-(a_{j-1}+d_{j-1}))}\le\Big(1+3/j\Big)^{a_j-(a_{j-1}+d_{j-1})}.$$

Thus

$$p(a_j)=M_j\le\Big(1+3/j\Big)^{a_j-(a_{j-1}+d_{j-1})}\le p(a_j-1)$$

and

$$\Big|\frac{p(a_j)}{p(a_j-1)}-1\Big|\le 1\qquad(j\ge 2).$$

Obviously

$$\Big|\frac{p(n+1)}{p(n)}-1\Big|\le\frac{3}{j}\qquad(j\ge 1, a_j\le n<a_{j+1}-1).$$

Since the sequence (a_j) is "lacunary" $(a_{j+1}\ge 4a_j)$, it is easy to prove (1). Indeed, let $q\ge 1$, $N\ge 1$ and $m\ge 0$. Then the cardinality of the set of all n such that $m\le n<m+N$ and $\Big|\frac{p(n+1)}{p(n)}-1\Big|>\frac{3}{q}$ is at most $a_q+\log_4 N$ so that

$$\lim_{N\to\infty}\sup_{m\in\mathbf{N}_0}\frac{1}{N}\sum_{n=m}^{m+N-1}\Big|\frac{p(n+1)}{p(n)}-1\Big|\le\lim_{N\to\infty}\frac{3(a_q+\log_4 N)+3N/q}{N}=\frac{3}{q}.$$

Since q was arbitrary, we conclude that the above limit is equal to 0.

Thus ρ is regular. Q. E. D.

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