Invariant subspaces for polynomially bounded operators

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Abstract

Let T be a polynomially bounded operator on a Banach space X whose spectrum contains the unit circle. Then T^* has a nontrivial invariant subspace. In particular, if X is reflexive, then T itself has a nontrivial invariant subspace. This generalizes the well-known result of Brown, Chevreau, and Pearcy for Hilbert space contractions.

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1 Introduction

An operator T acting on a complex Banach space X is called polynomially bounded if there is a constant k such that $\|p(T)\| \le k \cdot \|p\|$ for all polynomials p, where $\|p\| = \sup\{|p(z)| : |z| \le 1\}$. The smallest constant k with this property is called the polynomial bound of T. By the von Neumann inequality, every Hilbert space contraction is polynomially bounded with constant $k = 1$.

An early application of the Scott Brown technique gave the existence of invariant subspaces for Hilbert space contractions with dominant spectrum [BCP1]. Further results for Hilbert and Banach space operators were given in [S], [Pr], [AC]. All these results assumed that there are many points of the spectrum in the open unit disc.

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Further development culminated by the well-known "second generation" result of Brown, Chevreau and Pearcy [BCP2] that each Hilbert space contraction with spectrum containing the unit circle has a nontrivial invariant subspace. This result is much stronger than the corresponding one for operators with dominant spectrum and the proof is much more complicated since it is not possible to use directly the information provided by the points of the spectrum in the unit circle. The proof used essentially the properties of the Sz.-Nagy Foiaş functional model for Hilbert space contractions, and so there is no direct way of generalizing it.

The aim of this paper is to give a new approach and to generalize the Brown, Chevreau, Pearcy result for Banach space operators.

The main result can be formulated as follows:

Theorem A. *Let T be a polynomially bounded operator acting on a Banach space X whose spectrum contains the unit circle. Then T^* has a nontrivial invariant subspace.*

In particular, if X is reflexive then T has also a nontrivial invariant subspace.

The result is new even for Hilbert space operators. Note that there are polynomially bounded operators on a Hilbert space that are not similar to a contraction by [P].

Note also that the situation is not symmetrical for nonreflexive Banach spaces. If $M \subset X$ is a nontrivial subspace invariant with respect to an operator $T \in B(X)$ then $M^\perp$ is a nontrivial invariant subspace of $T^* \in B(X^*)$.

Conversely, if $M' \subset X^*$ is a nontrivial subspace invariant with respect to T^* then ${}^\perp M'$ is an invariant subspace of T but it can be trivial (if M' is w^*-dense).

Since the proof of the main theorem A is rather technical we indicate briefly the plan of the proof in this section.

Without loss of generality it is possible to assume in Theorem A that T is of class $C_{0\cdot}$, i.e., $\|T^n x\| \to 0$ for all $x \in X$. It is sufficient to show the following Theorem B.

Theorem B. *Let T be a polynomially bounded operator whose spectrum contains the unit circle. Suppose that $\|T^n x\| \to 0$ for all $x \in X$. Then T has* *a nontrivial invariant subspace.*

The reduction of Theorem A to Theorem B is rather standard (at least for Hilbert space operators). It will be shown in the last section. The greatest part of the paper will be devoted to the proof of Theorem B.

Note that in B the invariant subspace is constructed for the operator T (not for T^* like in Theorem A). Thus Theorem B is not a consequence of Theorem A, and so it is rather the second main result of the paper.

The basic idea of the Scott Brown technique is to find vectors $x \in X$ and $x^* \in X^*$ such that

$$\langle T^n x, x^* \rangle = \begin{cases} 0 & (n \geq 1) \\ 1 & (n = 0). \end{cases} \tag{1}$$

If x and x^* satisfy (1) then $\bigvee\{T^n x : n \geq 1\}$ is a closed subspace invariant with respect to T which is different from X.

Let $\mathcal{P}$ denote the normed space of all complex polynomials with the supremum norm on the unit disc. Consider the dual space $\mathcal{P}^*$ with the usual dual norm. It is well known that (1) can be reformulated equivalently by

$$x \otimes x^* = \mathcal{E}_0 \tag{2}$$

where $x \otimes x^*$ is the functional on $\mathcal{P}$ defined by $p \mapsto \langle p(T)x, x^* \rangle$ and $\mathcal{E}_0$ is the evaluation functional $p \mapsto p(0)$ at the origin.

Since $T^n x \to 0$ for all x, all the functionals of the form $x \otimes x^*$ (and in fact all functional that will be of our interest) are w^*-continuous, i.e., they are of the form $M_f : p \mapsto \int_0^{2\pi} f(e^{it})p(e^{it})dt$ for some function $f \in L^1$ on the unit circle. Similarly, all the evaluation functionals $\mathcal{E}_\lambda : p \mapsto p(\lambda)$ for $|\lambda| < 1$ are of this form since $\mathcal{E}_\lambda = M_{P_\lambda}$ where P_λ is the Poisson kernel at λ. In particular, $\mathcal{E}_0 = M_1$ where 1 is the constant function on the unit circle.

The standard way of solving (2) is to find first an approximate solution and then a sequence of better and better solutions; the exact solution of (2) will be obtained as a limit of these approximate solutions.

The starting point of the proof is the result of Apostol that each polynomially bounded operator whose spectrum contains the unit circle has either a nontrivial invariant subspace or there is a large set Λ in the open unit disc consisting of "almost eigenvalues". Usually this is formulated that the set Λ is dominant, i.e., $\sup\{|f(z)| : z \in \Lambda\} = \|f\|$ for all $f \in H^\infty$. We use the full strength of the Apostol theorem that in fact almost all points of the unit circle are radial limits of points of Λ. Sets with this property will be called Apostol sets. Clearly each Apostol set is dominant but we do not use this property; in fact we avoid the use of H^∞ functions almost completely and speak only about polynomials.

It is easy to check that if $\lambda \in \Lambda$, x is a corresponding "almost eigenvector" and $x^* \in X^*$ is arbitrary, then

$$x \otimes x^* \approx \langle x, x^* \rangle \cdot \mathcal{E}_\lambda$$

and so $x \otimes x^*$ is approximately equal (in the sense of the norm in $\mathcal{P}^*$) to a scalar multiple of the evaluation functional $\mathcal{E}_\lambda$.

It is a technical fact that the constant function 1 can be approximated by a finite linear combination with positive coefficients of Poisson kernels P_λ with the numbers λ in a given Apostol set Λ. More precisely, there are elements $\lambda_i \in \Lambda$ and positive numbers α_i such that

$$\Big\|1 - \sum_i \alpha_i P_{\lambda_i}\Big\|_1 < c \tag{3}$$

where $c < 1$ is a universal constant; here $\|\cdot\|_1$ denotes the usual normalized L^1 norm on the unit circle.

Let x_i be the corresponding almost eigenvectors, i.e., $\|x_i\| = 1$ and $Tx_i \approx \lambda_i x_i$ for all i. An approximate solution of (2) will be found by the Zenger theorem, see Theorem 3.1. Applying this to the vectors x_i and numbers α_i we can find a functional $x^* \in X^*$ and a linear combination $x = \sum_i \mu_i x_i$ such that $\|x^*\| \le 1$, $\|x\| \le 1$ and $\langle \mu_i x_i, x^* \rangle = \alpha_i$ for all i. Thus

$$\begin{aligned}
&\|x \otimes x^* - \mathcal{E}_0\| = \sup_{\|p\|=1} |\langle p(T)x, x^* \rangle - p(0)| \\
= &\sup_{\|p\|=1} \Big|\Big\langle \sum_i \mu_i p(T) x_i, x^* \Big\rangle - p(0)\Big| \approx \sup_{\|p\|=1} \Big|\Big\langle \sum_i \mu_i p(\lambda_i) x_i, x^* \Big\rangle - p(0)\Big| \\
= &\sup_{\|p\|=1} \Big|\sum_i \alpha_i p(\lambda_i) - p(0)\Big| = \Big\|\sum_i \alpha_i \mathcal{E}_{\lambda_i} - \mathcal{E}_0\Big\| \le \Big\|\sum_i \alpha_i P_{\lambda_i} - 1\Big\|_1 < c.
\end{aligned}$$

A technical problem here is that the Zenger theorem gives no estimate on the coefficients μ_i. Such an estimate, which is essential in the above calculations, will be obtained by an application of the classical Carleson interpolation theorem.

Having an approximate solution of (2), it is now necessary to improve it by finding perturbations y and y^* of x and x^*, respectively, such that $y \otimes y^*$

approximates $\mathcal{E}_0$ better than $x \otimes x^*$; moreover, $\|y - x\|$ and $\|y^* - x^*\|$ should be small.

This step is much easier if T is of class C_{00}, i.e., if T satisfies both $T^n x \to 0$ and $T^{*n} x^* \to 0$ for all $x \in X$ and $X^* \in X^*$. In this case it is sufficient to use the classical form of the Zenger theorem 3.1 (and even the Carleson theorem can be avoided). Since in general we can assume only one of these conditions, the technical difficulties are solved by an improved form of the Zenger theorem, see Proposition 3.5, which is of independent interest.

The paper is organized as follows. The following three sections discuss the theorems of Apostol, Zenger and Carleson, respectively. These sections are independent of each other.

Section 5 contains the proof of (3). An interested reader can start reading the paper at this section and return to the previous sections 2–4 only for the necessary auxiliary statements.

Section 6 contains the estimate on the coefficients mentioned above. The main theorems A and B are proved in the last section.

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2 Apostol's theorem

Denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ the open unit disc in the complex plane and by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ the unit circle.

Definition. *A subset $\Lambda \subset \mathbb{D}$ is called an Apostol set if*

$$\sup\{r \in [0,1) : re^{i\theta} \in \Lambda\} = 1$$

for all but countably many numbers $\theta \in (-\pi, \pi]$.

Theorem 2.1. (Apostol) [A1] *Let T be a polynomially bounded operator on a Banach space such that $\sigma(T) \supset \mathbb{T}$ and T has no nontrivial invariant subspace. Let $\varepsilon > 0$ and let $n \geq 1$ be an integer. Then the set*

$$\Lambda := \{\lambda \in \mathbb{D} : \textit{there exists } u \in X \textit{ with } \|u\| = 1 \textit{ and } \|Tu - \lambda u\| < \varepsilon(1 - |\lambda|)^n\}$$

is an Apostol set.

This theorem was proved in [A1] for $n = 1$ and T contractive on a Hilbert space. As it was observed in [B], the same proof also works for general exponent n. For our purpose it is sufficient to use the Apostol theorem with $n = 2$. In fact this exponent was used already in [B].

The idea of the proof is to show that if there exists an uncountable set $S \subset \mathbb{T}$ of points that are not radial limits of sequences from Λ, then the kernel of the operator

$$\frac{1}{2\pi i}\int_\gamma (T-\lambda)^{-1}(\lambda-\lambda_1)^n(\lambda-\lambda_2)^n d\lambda$$

is a nontrivial invariant subspace, where γ is a well-chosen simple rectifiable closed path crossing $\mathbb{T}$ at λ_1 and λ_2.

The existing proofs of Theorem 1 [A1], [B], [Be] were formulated for Hilbert space contractions but the proof remains unchanged also for Banach space operators. Therefore we omit it.

3 Zenger's theorem

The Zenger theorem proved to be a useful tool in constructions of invariant subspaces for operators on Banach spaces. The idea of using the Zenger theorem in the Scott Brown technique comes from Eschmeier [E]; some similar ideas were implicitly present already in the pioneering work of Apostol [A2].

The classical version of the Zenger theorem can be found in [BD], p. 18–20.

Theorem 3.1. (Zenger) *Let X be a complex Banach space, let $u_1, \dots, u_n \in X$ be linearly independent. Let $\alpha_j \quad (j = 1, \dots, n)$ be positive numbers with $\sum_{j=1}^n \alpha_j = 1$. Then there exist complex numbers $w_1, \dots, w_n$ and $\varphi \in X^*$ such that $\left\|\sum_{j=1}^n w_j u_j\right\| \le 1$, $\|\varphi\| \le 1$ and $\varphi(w_j u_j) = \alpha_j$ for all $j = 1, \dots, n$.*

As it was mentioned in the Introduction, Theorem 3.1 can be used to show the existence of nontrivial invariant subspaces for polynomially bounded Banach space operators of class C_{00} whose spectrum contains the unit circle. To get rid of the C_{00} condition, we need a stronger version of the Zenger theorem. Roughly speaking, we need to find the functional φ in a ball centered at some given point, not necessarily at the origin.

The next result is the real version of the required generalization (formulated dually).

Proposition 3.2. *Let $\|\cdot\|$ be a (real) norm on $\mathbb{R}^n$, let $\alpha_1, \dots, \alpha_n$ be positive numbers such that $\sum_{j=1}^n \alpha_j = 1$, let $s = (s_1, \dots, s_n) \in \mathbb{R}^n$. Then there exist $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ and $\psi \in (\mathbb{R}^n, \|\cdot\|)^*$ such that $\|\psi\| \le 1$, $\|w - s\| \le 1$ and $\psi(w_j e_j) = \alpha_j \quad (j = 1, \dots, n)$, where $(e_j)_{j=1}^n$ is the standard basis in $\mathbb{R}^n$, $e_j = (\underbrace{0, \dots, 0}_{j-1}, 1, 0, \dots, 0)$.*

Proof. Let $B = \Big\{x \in \mathbb{R}^n : \|x - s\| \le 1\Big\}$ and

$$B_+ = \Big\{(x_1, \dots, x_n) \in B : x_j s_j \ge 0 \quad (j = 1, \dots, n)\Big\}.$$

Clearly, B_+ is a compact subset of $\mathbb{R}^n$. Let $F : \mathbb{R}^n \to \langle 0, \infty)$ be the function defined by $F(x_1, \dots, x_n) = \prod_{j=1}^n |x_j|^{\alpha_j}$. Let $w = (w_1, \dots, w_n) \in B_+$ satisfy $F(w) = \max\{F(z) : z \in B_+\} := m$. Clearly $w_j \ne 0$ for all j.

Let $\psi : \mathbb{R}^n \to \mathbb{R}$ be the functional defined by

$$\psi(x_1, \dots, x_n) = \sum_{j=1}^n \alpha_j x_j w_j^{-1}.$$

Then $\psi(w) = 1$. In a neighbourhood of w we have

$$\frac{\partial F}{\partial x_j}(x) = \alpha_j |x_j|^{\alpha_j - 1} \operatorname{sign} x_j \cdot \prod_{k \ne j} |x_k|^{\alpha_k} = \frac{\alpha_j F(x)}{x_j}.$$

In particular,

$$\frac{\partial F}{\partial x_j}(w) = \frac{\alpha_j F(w)}{w_j} = \frac{m \alpha_j}{w_j}.$$

Thus $F'(w) = m\psi = F(w)\psi$. For $x \to w$ we have

$$F(x) - F(w) = F'(w)(x - w) + o(\|x - w\|) = m\psi(x - w) + o(\|x - w\|),$$

and so

$$F(x) = m\psi(x) + o(\|x - w\|).$$

We prove that $\psi(x) \le 1$ for all $x \in B$. Suppose on the contrary that there is an $x \in B$ with $\psi(x) > 1$. For $t \in (0, 1)$ let $y_t = (1 - t)w + tx = w + t(x - w)$.

Since B is convex, we have $y_t \in B$ for all t. Furthermore, $y_t \in B_+$ for all t small enough. For $t \to 0$ we have

$$\begin{aligned} F(y_t) &= m\psi(y_t) + o(t\|x-w\|) = m(\psi(w) + t\psi(x-w)) + o(t) \\ &= m + mt\psi(x-w) + o(t). \end{aligned}$$

Since $\psi(x-w) > 0$, we have $F(y_t) > m$ for all $t > 0$ small enough, which is a contradiction.

Thus $\psi(x) \le 1$ for all $x \in B$. Note that $\psi(s) = \sum_{j=1}^n \alpha_j s_j w_j^{-1} \ge 0$.

Hence $\|\psi\| = \max\{\psi(x) - \psi(s) : x \in B\} \le 1$.

It is easy to see that $\psi(w_j e_j) = \alpha_j$ for all j. □

The complex version of Proposition 3.2 is an interesting open problem. We prove it under an additional assumption that the norm is rather regular. This weaker version will be sufficient for our main purpose — the construction of invariant subspaces.

Definition. *Let X be a complex Banach space, let $u_1, \dots, u_n \in X$ be non-zero vectors, let $L > 0$. We say that the vectors $u_1, \dots, u_n$ are L-circled if*

$$\Big\|\sum_{j=1}^n \beta_j u_j\Big\| \le L \cdot \Big\|\sum_{j=1}^n \gamma_j u_j\Big\|$$

whenever $\beta_j, \gamma_j \in \mathbb{C}$, $|\beta_j| \le |\gamma_j| \quad (j = 1, \dots, n)$.

It is easy to see that L-circled vectors are linearly independent.

Lemma 3.3. *Let $L > 0$, let $\|\cdot\|$ be a (complex) norm on $\mathbb{C}^n$ such that the standard basis vectors $e_1, \dots, e_n$ are L-circled. Let $\alpha_1, \dots, \alpha_n$ be positive numbers such that $\sum_{j=1}^n \alpha_j = 1$, let $s = (s_1, \dots, s_n) \in \mathbb{C}^n$. Then there exist $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ and a complex-linear functional $\psi \in (\mathbb{C}^n, \|\cdot\|)^*$ such that $\|\psi\| \le L\sqrt{2}$, $\|w - s\| \le 1$ and $\psi(w_j e_j) = \alpha_j \quad (j = 1, \dots, n)$.*

Proof. For $j = 1, \dots, n$ write $s_j = |s_j| \cdot \nu_j$ where $\nu_j \in \mathbb{C}, |\nu_j| = 1$ (if $s_j = 0$ then choose any $\nu_j \in \mathbb{T}$). Consider the real-linear subspace X of $\mathbb{C}^n$ generated by the vectors $\nu_j e_j \quad (j = 1, \dots, n)$. Clearly $s \in X$ and $\mathbb{C}^n = X + iX$.

By Proposition 3.2, there exist real numbers $t_j \quad (j = 1, \dots, n)$ and a real-linear functional $\psi_0 : X \to \mathbb{R}$ such that $\psi_0(t_j \nu_j e_j) = \alpha_j \quad (j = 1, \dots, n)$, $\Big\|\sum_{j=1}^n t_j \nu_j e_j - s\Big\| \le 1$ and $\|\psi_0\| \le 1$.

Set $w_j = t_j \nu_j$. Extend ψ_0 to $\mathbb{C}^n$ by $\psi(x+iy) = \psi_0(x) + i\psi_0(y)$ for all $x, y \in X$. It is easy to verify that ψ is a complex-linear functional and $\psi(w_j e_j) = \alpha_j \quad (j = 1, \ldots, n)$.

Let $b_j, c_j \in \mathbb{R}$. If $\left\|\sum_{j=1}^n (b_j + ic_j)\nu_j e_j\right\| \leq 1$, then $\left\|\sum_{j=1}^n b_j \nu_j e_j\right\| \leq L$ and $\left\|\sum_{j=1}^n c_j \nu_j e_j\right\| \leq L$. So

$$\left|\psi\Big(\sum_{j=1}^n (b_j + ic_j)\nu_j e_j\Big)\right|^2 = \left|\psi_0\Big(\sum_{j=1}^n b_j \nu_j e_j\Big)\right|^2 + \left|\psi_0\Big(\sum_{j=1}^n c_j \nu_j e_j\Big)\right|^2 \leq 2L^2,$$

and so $\|\psi\| \leq L\sqrt{2}$. □

The next result is a dual version of Lemma 3.3.

Lemma 3.4. *Let $\|\cdot\|$ be a norm on $\mathbb{C}^n$ such that the standard basis vectors $e_1, \ldots, e_n$ are L-circled. Let $\alpha_1, \ldots, \alpha_n$ be positive numbers such that $\sum_{j=1}^n \alpha_j = 1$, let $\varphi \in (\mathbb{C}^n, \|\cdot\|)^*$. Then there exist $w = (w_j, \ldots, w_n) \in \mathbb{C}^n$ and a complex-linear functional $\psi \in (\mathbb{C}^n, \|\cdot\|)^*$ such that $\|\psi - \varphi\| \leq 1$, $\|w\| \leq L\sqrt{2}$ and $\psi(w_j e_j) = \alpha_j \quad (j = 1, \ldots, n)$.*

Proof. Let $f_1, \ldots, f_n \in (\mathbb{C}^n, \|\cdot\|)^*$ be defined by $\langle e_j, f_k \rangle = \delta_{j,k}$ (the Kronecker symbol). We prove first that $f_1, \ldots, f_n$ are L-circled in $(\mathbb{C}^n, \|\cdot\|)^*$.

Let $\beta_j, \gamma_j \in \mathbb{C}$, $|\beta_j| \leq |\gamma_j| \quad (j = 1, \ldots, n)$. Let $F = \{j \in \{1, \ldots, n\} : \beta_j \neq 0\}$. Then

$$\begin{aligned}
\Big\|\sum_{j=1}^n \beta_j f_j\Big\| &= \sup\Big\{\Big|\Big\langle \sum_{j \in F} \beta_j f_j, \sum_{j=1}^n \omega_j e_j \Big\rangle\Big| : \Big\|\sum_{j=1}^n \omega_j e_j\Big\| \leq 1\Big\} \\
&= \sup\Big\{\Big|\sum_{j \in F} \beta_j \omega_j\Big| : \Big\|\sum_{j=1}^n \omega_j e_j\Big\| \leq 1\Big\} \\
&= \sup\Big\{\Big|\Big\langle \sum_{j=1}^n \gamma_j f_j, \sum_{j \in F} \omega_j \beta_j \gamma_j^{-1} e_j \Big\rangle\Big| : \Big\|\sum_{j=1}^n \omega_j e_j\Big\| \leq 1\Big\} \\
&\leq \sup\Big\{\Big|\Big\langle \sum_{j=1}^n \gamma_j f_j, \sum_{j=1}^n \mu_j e_j \Big\rangle\Big| : \Big\|\sum_{j=1}^n \mu_j e_j\Big\| \leq L\Big\} = L \cdot \Big\|\sum_{j=1}^n \gamma_j f_j\Big\|.
\end{aligned}$$

By Lemma 3.3, there is a $\psi \in (\mathbb{C}^n, \|\cdot\|)^*$, $\psi = \sum_{j=1}^n \psi_j f_j$ such that $\|\psi - \varphi\| \leq 1$ and $w = (w_1, \ldots, w_n) \in \mathbb{C}^n$ such that $\|w\| \leq L\sqrt{2}$ and $\alpha_l = \langle \sum_{j=1}^n w_j e_j, \psi_l f_l \rangle = w_l \psi_l = \langle w_l e_l, \psi \rangle$ for all $l = 1, \ldots, n$. □

Proposition 3.5. *Let $u_1, \ldots, u_n$ be L-circled vectors in a complex Banach space X. Let $\alpha_1, \ldots, \alpha_n$ be positive numbers such that $\sum_{j=1}^n \alpha_j = 1$, let*

$\varphi \in X^*$. Then there exist complex numbers $w_j \in \mathbb{C}$ $\quad (j = 1, \dots, n)$ and a complex-linear functional $\psi \in X^*$ such that $\|\psi - \varphi\| \le 1$, $\|\sum_{j=1}^n w_j u_j\| \le L\sqrt{2}$ and $\psi(w_j u_j) = \alpha_j \quad (j = 1, \dots, n)$.

Proof. Let X_0 be the subspace of X generated by the vectors $u_1, \dots, u_n$. We can identify X_0 with $\mathbb{C}^n$ with the norm $\|(w_1, \dots, w_n)\| = \|\sum_{j=1}^n w_j u_j\|_X$. By Lemma 3.4, there are $w_j \in \mathbb{C}$ and $\psi \in X_0^*$ such that $\|\sum_{j=1}^n w_j u_j\| \le L\sqrt{2}$, $\langle w_j u_j, \psi \rangle = \alpha_j$ and $\|\psi - \varphi | X_0\|_{X_0^*} \le 1$. By the Hahn-Banach theorem, we can extend ψ to X such that $\|\psi - \varphi\|_{X^*} \le 1$. □

4 Carleson's interpolation theorem

For $\lambda = re^{i\theta} \in \mathbb{D}$, set $I_\lambda = \{e^{it} : |t - \theta| < 2(1 - r)\}$. These sets will play an important role in the proof.

Lemma 4.1. *There is a constant $a > 0$ with the following property: if z, $\lambda \in \mathbb{D}$ satisfy $I_z \cap I_\lambda = \emptyset$ and $|\lambda|, |z| \ge 3/4$, then $|\frac{z-\lambda}{1-\bar{\lambda}z}| \ge a$.*

Proof. We can assume that $\lambda = r > 0$. Write $z = se^{i\theta}$ with $s \ge 3/4$ and $-\pi < \theta \le \pi$.

Since $s, r \ge 3/4$ and $|\sin\frac{\theta}{2}| \ge \frac{|\theta|}{2} \cdot \frac{2}{\pi}$, we have

$$\begin{aligned} |r - se^{i\theta}|^2 &= (r - s\cos\theta)^2 + s^2\sin^2\theta = (r-s)^2 + 2rs(1 - \cos\theta) \\ &\ge 4rs\sin^2\frac{\theta}{2} \ge \frac{9}{4}\Big(\frac{|\theta|}{\pi}\Big)^2 = \Big(\frac{3|\theta|}{2\pi}\Big)^2. \end{aligned}$$

Since $e^{i\theta} \notin I_r$, we have $|\theta| \ge 2(1 - r)$. Thus $|z - r| \ge \frac{3|\theta|}{2\pi} \ge \frac{3(1-r)}{\pi}$. Furthermore,

$$|1 - rz| \le (1 - r^2) + |r^2 - rz| \le 2(1 - r) + r|r - z|.$$

Hence

$$\frac{|z - r|}{|1 - rz|} \ge \frac{|z - r|}{2(1 - r) + r|z - r|} = \frac{1}{\frac{2(1-r)}{|r-z|} + r} \ge \frac{1}{\frac{2\pi}{3} + 1} \ge \frac{3}{2\pi + 3}.$$

The last constant is independent of the choice of λ and z. □

We remind that a positive measure μ on $\mathbb{D}$ is called *Carleson* if there is a constant c_μ such that

$$\mu(S_{\theta,h}) \le c_\mu\, h$$

for every sector $S_{\theta,h}$ of the form

$$S_{\theta,h} = \{re^{it} : 1 - h \le r < 1, |t - \theta| \le h\}. \tag{4}$$

Lemma 4.2. *Let $F \subset \mathbb{D}$ be a finite set such that the sets $I_\lambda \quad (\lambda \in F)$ are pairwise disjoint. Then the measure $\mu = \sum_{\lambda \in F}(1 - |\lambda|)\delta_\lambda$ is Carleson with the constant ≤ 1, where δ_λ denotes the Dirac measure at λ.*

Proof. Let $S_{\theta,h}$ be a sector of the form (4). Let $\Gamma = \{e^{it} : |t - \theta| \le h\}$. Let $\lambda = re^{is} \in F \cap S_{\theta,h}$. Then there are three possible cases: $e^{i(s+2(1-r))} \in \Gamma$, $e^{i(s-2(1-r))} \in \Gamma$, or $I_\lambda \subset \Gamma$. In all three cases we have

$$m(I_\lambda \cap \Gamma) \ge 2(1 - |\lambda|),$$

where m denotes the Lebesgue measure on $\mathbb{T}$. Thus

$$\mu(S_{\theta,h}) = \sum_{\lambda \in S_{\theta,h} \cap F} (1 - |\lambda|) \le \frac{1}{2}\, m\Big(\Gamma \cap \bigcup_{\lambda \in F \cap S_{\theta,h}} I_\lambda\Big) \le \frac{1}{2} m(\Gamma) = h.$$

□

As usually, denote by H^∞ the algebra of all bounded analytic functions on $\mathbb{D}$ with the norm $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$.

It follows from the Carleson interpolation theory, see [G], Section VII.1, that, given a finite set $F \subset \mathbb{D}$ such that $I_\lambda \quad (\lambda \in F)$ are pairwise disjoint and $|\lambda| \ge 3/4 \quad (\lambda \in F)$, and numbers $c_\lambda \in \mathbb{C}$, then there exists $f \in H^\infty$ such that $f(\lambda) = c_\lambda \quad (\lambda \in F)$ and $\|f\| \le b \cdot \sup_{\lambda \in F} |c_\lambda|$, where b is an absolute constant, independent of F and c_λ. Since the results in [G] are formulated for the upper half-plane, we indicate briefly the argument in the disc case, following the comments preceding Theorem VII.1.1.

Lemma 4.3. *There is a constant $\delta > 0$ with the following property: if $F \subset \mathbb{D}$ is a finite set such that the sets $I_\lambda \quad (\lambda \in F)$ are pairwise disjoint and $|\lambda| \ge 3/4 \quad (\lambda \in F)$, then*

$$\prod_{\lambda \in F \setminus \{\lambda_0\}} \Big| \frac{\lambda_0 - \lambda}{1 - \bar{\lambda}\lambda_0} \Big| \ge \delta$$

for each $\lambda_0 \in F$.

Proof. Let $\mu = \sum_{\lambda \in F}(1-|\lambda|)\delta_\lambda$. Since μ is a Carleson measure with the constant ≤ 1, by [G], lemma VI.3.3 we have

$$\sup_{w\in\mathbb{D}} \int \frac{1-|w|^2}{|1-\overline{w}z|^2} d\mu(z) = \sup_{w\in\mathbb{D}} \sum_{\lambda\in F} \frac{(1-|\lambda|)(1-|w|^2)}{|1-\overline{w}\lambda|^2} \le \sigma < \infty,$$

where σ is a universal constant independent of F. In particular,

$$\sum_{\lambda\in F} \frac{(1-|\lambda|)(1-|\lambda_0|^2)}{|1-\overline{\lambda}_0\lambda|^2} \le \sigma$$

for each $\lambda_0 \in F$.

Let a be the constant from Lemma 4.1. Note that $a<1$, and so $\ln a < 0$. Since $\ln t$ is a concave function, for any $t \in (a^2, 1)$ we have

$$\ln t \ge \frac{2\ln a}{1-a^2}(1-t).$$

Let $\lambda \in F\setminus\{\lambda_0\}$. Using the identity $|1-\overline{\lambda}\lambda_0|^2 - |\lambda_0-\lambda|^2 = (1-|\lambda|^2)(1-|\lambda_0|^2)$ we have

$$\begin{aligned} \ln\Big|\frac{\lambda_0-\lambda}{1-\overline{\lambda}\lambda_0}\Big|^2 &\ge \frac{2\ln a}{1-a^2}\Big(1-\Big|\frac{\lambda_0-\lambda}{1-\overline{\lambda}\lambda_0}\Big|^2\Big) = \frac{2\ln a}{1-a^2}\cdot\frac{(1-|\lambda|^2)(1-|\lambda_0|^2)}{|1-\overline{\lambda}\lambda_0|^2} \\ &\ge \frac{4\ln a}{1-a^2}\cdot\frac{(1-|\lambda|)(1-|\lambda_0|^2)}{|1-\overline{\lambda}\lambda_0|^2}. \end{aligned}$$

Set $B(z) = \prod_{\lambda\in F\setminus\{\lambda_0\}} \frac{z-\lambda}{1-\overline{\lambda}z}$. Then

$$\ln|B(\lambda_0)|^2 \ge \frac{4\ln a}{1-a^2} \sum_{\lambda\in F\setminus\{\lambda_0\}} \frac{1-|\lambda_0|^2}{|1-\overline{\lambda_0}\lambda|^2}(1-|\lambda|) \ge \frac{4\ln a}{1-a^2}\,\sigma.$$

Thus $|B(\lambda_0)| \ge \delta$ for some constant δ independent of F. □

Proposition 4.4. *There is a constant b with the following property: if $F \subset \mathbb{D}$ is a finite set such that the sets I_λ are pairwise disjoint and $|\lambda| \ge 3/4 \quad (\lambda \in F)$, and $c_\lambda \in \mathbb{C} \quad (\lambda \in F)$ are given, then there exists $f \in H^\infty$ such that $f(\lambda) = c_\lambda \quad (\lambda\in F)$ and $\|f\| \le b\cdot\sup_{\lambda\in F}|c_\lambda|$.*

Proof. The proof follows from [SS], Theorem 1. More precisely, it is possible to take $b = \frac{2}{\delta^5}(1-2\ln\delta)$, where δ is the constant from Lemma 4.3. □

5 Preliminary steps

For every $\lambda \in \mathbb{D}$, let $P_\lambda(t) = \frac{1-|\lambda|^2}{|\lambda - e^{it}|^2}$ $(t \in \mathbb{R})$ denote the Poisson kernel. It is well known that $\int_{-\pi}^{\pi} P_\lambda dt = 2\pi$ and $\max_t P_\lambda(t) = \frac{1+|\lambda|}{1-|\lambda|}$.

Recall that for $\lambda = re^{i\theta} \in \mathbb{D}$ we write $I_\lambda = \{e^{it} : |t-\theta| < 2(1-r)\}$.

Notation. *For $\lambda \in \mathbb{D}$ define the 2π–periodic function Q_λ on $\mathbb{R}$ by: $Q_\lambda(t) = P_\lambda(t)$ if $e^{it} \in I_\lambda$, and $Q_\lambda(t) = 0$ otherwise. Denote by m the Lebesgue measure both on the real line $\mathbb{R}$ and on the unit circle $\mathbb{T}$.*

Lemma 5.1. *For any $\lambda \in \mathbb{D}$ with $|\lambda| \geq 3/4$ we have $\int_{-\pi}^{\pi} Q_\lambda(t)dt \geq \frac{7\pi}{6}$.*

Proof. Without loss of generality we can suppose that $\lambda = r \geq 3/4$. We have $\sin^2(1-r) \leq \sin(1-r) \leq 1-r$. If $e^{it} \in I_\lambda$ then

$$\cos t \geq \cos 2(1-r) = 1 - 2\sin^2(1-r) \geq 2r-1,$$

and so $1 - r \geq \cos t - r \geq r - 1$. Thus

$$|r - e^{it}|^2 = (r-\cos t)^2 + \sin^2 t \leq (1-r)^2 + t^2.$$

Hence

$$\begin{aligned}
\int_{-\pi}^{\pi} Q_\lambda(t)dt &= \int_{-2(1-r)}^{2(1-r)} \frac{1-r^2}{|r-e^{it}|^2}dt = 2(1-r^2)\int_0^{2(1-r)} \frac{dt}{|r-e^{it}|^2} \\
&\geq 2(1-r^2)\int_0^{2(1-r)} \frac{dt}{(1-r)^2+t^2} = 2(1-r^2)\Big[\frac{1}{1-r}\tan^{-1}\frac{t}{1-r}\Big]_0^{2(1-r)} \\
&= 2(1+r)\tan^{-1} 2 \geq \frac{7}{2}\cdot\tan^{-1}\sqrt{3} = \frac{7\pi}{6}.
\end{aligned}$$

$\square$

Corollary 5.2. *For each $\lambda \in \mathbb{D}$ with $|\lambda| \geq 3/4$ we have*

$$\int_{-\pi}^{\pi} (P_\lambda(t) - Q_\lambda(t))dt \ \leq \ \frac{5}{7}\int_{-\pi}^{\pi} Q_\lambda(t)dt.$$

Proof. By Lemma 5.1, we have the estimates

$$\frac{\int_{-\pi}^{\pi}(P_\lambda(t)-Q_\lambda(t))dt}{\int_{-\pi}^{\pi} Q_\lambda(t)dt} = \frac{\int_{-\pi}^{\pi} P_\lambda(t)dt}{\int_{-\pi}^{\pi} Q_\lambda(t)dt} - 1 \leq 2\pi\cdot\left(\frac{7\pi}{6}\right)^{-1} - 1 = \frac{5}{7}.$$

$\square$

Theorem 5.3. *Let $\Lambda \subset \mathbb{D}$ be an Apostol set. Let $t_1, t_2 \in \mathbb{R}$ satisfy $-\pi \le t_1 < t_2 \le \pi$. Let $f(t) := 1$ if $t_1 \le t \le t_2$, and $f(t) := 0$ otherwise. Then there is an $n_0 \ge 1$ such that for every $n \ge n_0$ there exist a finite set $F \subset \Lambda$ and positive real numbers $\alpha_\lambda \quad (\lambda \in F)$ with the following properties:*

(i) $I_\lambda \subset \{e^{it} : t_1 < t < t_2\}$ *for all* $\lambda \in F$;

(ii) *the sets* $I_\lambda \quad (\lambda \in F)$ *are pairwise disjoint;*

(iii) $m(\cup_{\lambda \in F} I_\lambda) \ge \frac{1}{40\pi}(t_2 - t_1)$;

(iv) $|\lambda| \ge 3/4$ *and* $|\lambda^n - 1| < \frac{1}{9}$ *for all* $\lambda \in F$;

(v) $\sum_{\lambda \in F} \alpha_\lambda \le \frac{t_2 - t_1}{7}$;

(vi) $\int_{-\pi}^{\pi} |\sum_{\lambda \in F} \alpha_\lambda \lambda^n P_\lambda(t) - f(t)| dt \le c_1 (t_2 - t_1)$, *where* $c_1 = 1 - \frac{1}{1920}$.

Proof. For every $n \ge 1$, set $M_n = \{t \in (t_1, t_2) : |e^{int} - 1| \le 1/10\}$. Clearly for all n sufficiently large we have

$$m(M_n) > \frac{t_2 - t_1}{10 \cdot 2\pi}. \tag{5}$$

Fix n satisfying (5). Let $\varepsilon > 0$ satisfy $m(M_n) - \varepsilon > (t_2 - t_1)/20\pi$. Let $S \subset (t_1, t_2)$ be the exceptional set of the Apostol set Λ, i.e., $\sup\{0 \le r < 1 : re^{i\theta} \in \Lambda\} = 1$ for all $\theta \in (t_1, t_2) \setminus S$. Since S is at most countable, it can be covered by a countable union U of open intervals with $m(U) < \varepsilon/2$. Then the set M' defined by

$$M' = \Big(M_n \cap [t_1 + \varepsilon/4, t_2 - \varepsilon/4] \Big) \setminus U$$

is compact with $m(M') > (t_2 - t_1)/20\pi$. For each $t \in M'$ we can find $r_t \ge 3/4$ such that $\lambda_t := r_t e^{it} \in \Lambda$, $|\lambda_t^n - 1| < 1/9$ and $I_{\lambda_t} \subset \{e^{is} : t_1 < s < t_2\}$. Then $\{e^{is} : s \in M'\} \subset \cup_{t \in M'} I_{\lambda_t}$. Since $\{e^{is} : s \in M'\}$ is a compact subset of the 1–dimensional set $\mathbb{T}$, there exists a finite subcover of $(I_{\lambda_t})_{t \in M'}$ such that any three of these subsets have empty intersection. Considering a cover of the minimal cardinality with this property it is easy to see that there are numbers $\lambda_1, \dots, \lambda_k \in \Lambda$ with $\lambda_j = |\lambda_j| e^{is_j}$ such that $t_1 < s_1 < \cdots < s_k < t_2$, $\bigcup_{j=1}^k I_{\lambda_j} \supset \{e^{is} : s \in M'\}$ and $I_{\lambda_j} \cap I_{\lambda_{j'}} = \emptyset$ if $|j' - j| \ge 2$. Let $F_1 = \{\lambda_1, \lambda_3, \dots\}$ and $F_2 = \{\lambda_2, \lambda_4, \dots\}$. Let F be one of the sets F_1, F_2 such that

$$m\Big(\bigcup_{\lambda \in F} I_\lambda\Big) = \max\Big\{ m\Big(\bigcup_{\lambda \in F_1} I_\lambda\Big), m\Big(\bigcup_{\lambda \in F_2} I_\lambda\Big)\Big\}.$$

Then $I_\lambda \cap I_{\lambda'} = \emptyset$ for all distinct λ, λ' in F, and $m\Big(\bigcup_{\lambda\in F} I_\lambda\Big) \geq m(M')/2 > (t_2 - t_1)/\,40\pi$. For any $\lambda \in F$, set $\alpha_\lambda = (1 - |\lambda|)(1 + |\lambda|)^{-1}$. Then $\alpha_\lambda > 0$ and

$$\sum_{\lambda\in F} \alpha_\lambda \leq \frac{4}{7} \sum_{\lambda\in F} (1 - |\lambda|) = \frac{1}{7} \sum_{\lambda\in F} m(I_\lambda) \leq \frac{t_2 - t_1}{7}.$$

Finally,

$$\begin{aligned}
& \int_{-\pi}^{\pi} \Big|\sum_{\lambda\in F} \alpha_\lambda \lambda^n P_\lambda(t) - f(t)\Big| dt \\
\leq\ & \int_{-\pi}^{\pi} \Big|\sum_{\lambda\in F} \alpha_\lambda \lambda^n (P_\lambda(t) - Q_\lambda(t))\Big| dt + \int_{-\pi}^{\pi} \sum_{\lambda\in F} \alpha_\lambda |\lambda^n - 1| Q_\lambda(t) dt \\
& \qquad + \int_{t_1}^{t_2} \Big(1 - \sum_{\lambda\in F} \alpha_\lambda Q_\lambda(t)\Big) dt \leq \sum_{\lambda\in F} \alpha_\lambda \int_{-\pi}^{\pi} (P_\lambda(t) - Q_\lambda(t)) dt \\
& \qquad + \frac{1}{9} \int_{t_1}^{t_2} \sum_{\lambda\in F} \alpha_\lambda Q_\lambda(t) dt + (t_2 - t_1) - \int_{t_1}^{t_2} \sum_{\lambda\in F} \alpha_\lambda Q_\lambda(t) dt \\
\leq\ & t_2 - t_1 + \Big(\frac{5}{7} + \frac{1}{9} - 1\Big) \int_{t_1}^{t_2} \sum_{\lambda\in F} \alpha_\lambda Q_\lambda(t) dt \leq t_2 - t_1 - \frac{1}{7} \int_{t_1}^{t_2} \sum_{\lambda\in F} \alpha_\lambda Q_\lambda(t) dt \\
\leq\ & t_2 - t_1 - \frac{1}{7} \sum_{\lambda\in F} \frac{1 - |\lambda|}{1 + |\lambda|} \cdot \frac{7\pi}{6} \leq t_2 - t_1 - \frac{\pi}{12} \sum_{\lambda\in F} (1 - |\lambda|) \\
=\ & t_2 - t_1 - \frac{\pi}{48} \cdot m\Big(\bigcup_{\lambda\in F} I_\lambda\Big) \leq c_1 (t_2 - t_1),
\end{aligned}$$

where $c_1 = 1 - \frac{1}{1920}$. □

Corollary 5.4. *Let c_1 be the constant from the previous lemma and let $c_2 \in (c_1, 1)$. Let $f : (-\pi, \pi] \to [0, \infty)$ be an integrable function and let Λ be an Apostol set. Then for any n sufficiently large there are a finite set $F \subset \Lambda$ and positive numbers $\alpha_\lambda \quad (\lambda \in F)$ such that:*

(i) *the sets $(I_\lambda)_{\lambda\in F}$ are pairwise disjoint;*

(ii) *$|\lambda| \geq 3/4$ and $|\lambda^n - 1| \leq \frac{1}{9}$ for all $\lambda \in F$;*

(iii) *$\sum_{\lambda\in F} \alpha_\lambda \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$;*

(iv) *$\int_{-\pi}^{\pi} \Big|\sum_{\lambda\in F} \alpha_\lambda \lambda^n P_\lambda(t) - f(t)\Big| dt \leq c_2 \int_{-\pi}^{\pi} f(t) dt$.*

Proof. Let $\varepsilon > 0$ be sufficiently small ($\varepsilon < \min\{\frac{c_2 - c_1}{2}, \frac{7}{2\pi} - 1\}$). Let g be a step function $g : (-\pi, \pi] \to [0, \infty)$ such that $\int_{-\pi}^{\pi} |f - g| dt \leq \varepsilon \int_{-\pi}^{\pi} f(t) dt$.

By Theorem 5.3 applied to each interval where g is constant, we can find a finite set $F \subset \Lambda$ and positive numbers $\alpha_\lambda \quad (\lambda \in F)$ satisfying (i), (ii) and

$$\begin{aligned}
&\sum_{\lambda \in F} \alpha_\lambda \le \frac{1}{7} \int_{-\pi}^{\pi} g(t)dt \le \frac{1}{7}\Big(\int_{-\pi}^{\pi} f(t)dt + \int_{-\pi}^{\pi} |f-g|dt\Big) \\
\le\ & \frac{1}{7}(1+\varepsilon) \int f(t)dt \le \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt.
\end{aligned}$$

Further,

$$\int_{-\pi}^{\pi} \Big| \sum_{\lambda \in F} \alpha_\lambda \lambda^n P_\lambda(t) - g(t) \Big| dt \le c_1 \int_{-\pi}^{\pi} g(t)dt.$$

Then we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} \Big| \sum_{\lambda \in F} \alpha_\lambda \lambda^n P_\lambda(t) - f(t) \Big| dt \\
\le\ & \int_{-\pi}^{\pi} \Big| \sum_{\lambda \in F} \alpha_\lambda \lambda^n P_\lambda(t) - g(t) \Big| dt + \int_{-\pi}^{\pi} |f(t) - g(t)| dt \\
\le\ & c_1 \int_{-\pi}^{\pi} g(t)dt + \varepsilon \int_{-\pi}^{\pi} f(t)dt \le (c_1 + 2\varepsilon) \int_{-\pi}^{\pi} f(t)dt \le c_2 \int_{-\pi}^{\pi} f(t)dt. \qquad \square
\end{aligned}$$

6 Polynomially bounded operators

Let $T \in B(X)$ be a polynomially bounded operator with polynomial bound k.

Denote by $A(\mathbb{D})$ the disc algebra consisting of all functions continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$, with the norm $\|f\| = \sup\{|f(z)| : z \in \mathbb{D}\}$. It is well known that functions from $A(\mathbb{D})$ are uniform limits of polynomials. Therefore we can extend the polynomial calculus for T to functions from $A(\mathbb{D})$ with the same constant k, i.e.,

$$\|f(T)\| \le k \cdot \|f\| \qquad (f \in A(\mathbb{D})).$$

Lemma 6.1. *Let $T \in B(X)$ be a polynomially bounded operator with polynomial bound k. Let b be the constant from Proposition 4.4. Let $F \subset \mathbb{D}$ be a finite set with $(I_\lambda)_{\lambda \in F}$ pairwise disjoint and $|\lambda| \ge 3/4 \quad (\lambda \in F)$.*

Suppose that there are vectors $u_\lambda \in X$ and complex numbers $\mu_\lambda \quad (\lambda \in F)$ such that $\|u_\lambda\| = 1$, $\|(T-\lambda)u_\lambda\| < \frac{1}{2kb\pi}(1-|\lambda|)^2$ and $\|\sum_{\lambda\in F}\mu_\lambda u_\lambda\| = 1$. Then $|\mu_\lambda| \le 2kb$ for all $\lambda \in F$.

Proof. Let $\lambda_0 \in F$ satisfy $|\mu_{\lambda_0}| = \max_{\lambda\in F}|\mu_\lambda|$. By Proposition 4.4, there is a function $f \in H^\infty$ such that $\|f\| \le b$, $f(\lambda_0) = 1$ and $f(\lambda) = 0$ for $\lambda \in F \setminus \{\lambda_0\}$.

For $r \in (0,1)$ and $z \in \mathbb{D}$ define $f_r(z) = f(rz)$. Clearly $\|f_r\| \le \|f\| \le b$ and f_r is a function analytic on a neighbourhood of $\overline{\mathbb{D}}$, and so $f_r \in A(\mathbb{D})$. Thus we can define $f_r(T)$ and $\|f_r(T)\| \le kb$ for all r.

Let $u = \sum_{\lambda\in F}\mu_\lambda u_\lambda$. Then $\|f_r(T)u\| \le kb\|u\| = kb$ for all r.

For $\lambda \in F$ define $g_{r,\lambda}(z) = \frac{f_r(z)-f_r(\lambda)}{z-\lambda}$. Clearly $g_{r,\lambda}$ is analytic on a neighbourhood of $\overline{\mathbb{D}}$ and $\|g_{r,\lambda}\| \le 2\|f_r\|(1-|\lambda|)^{-1} \le 2b(1-|\lambda|)^{-1}$. Hence

$$\begin{aligned}
kb &\ge \limsup_{r\to 1_-}\|f_r(T)u\| \\
&\ge \limsup_{r\to 1_-}\Big(\Big\|\sum_{\lambda\in F} f_r(\lambda)\mu_\lambda u_\lambda\Big\| - \Big\|\sum_{\lambda\in F}\mu_\lambda\big(f_r(\lambda)-f_r(T)\big)u_\lambda\Big\|\Big) \\
&\ge \|\mu_{\lambda_0}u_{\lambda_0}\| - \liminf_{r\to 1_-}\sum_{\lambda\in F}|\mu_\lambda|\cdot\|g_{r,\lambda}(T)(T-\lambda)u_\lambda\| \\
&\ge |\mu_{\lambda_0}| - |\mu_{\lambda_0}|\sum_{\lambda\in F}2kb(1-|\lambda|)^{-1}\frac{1}{2kb\pi}(1-|\lambda|)^2 \\
&\ge |\mu_{\lambda_0}|\Big(1-\sum_{\lambda\in F}\pi^{-1}(1-|\lambda|)\Big) \ge \frac{|\mu_{\lambda_0}|}{2},
\end{aligned}$$

since $\sum_{\lambda\in F}(1-|\lambda|) \le \frac{1}{4}m\Big(\bigcup_{\lambda\in F}I_\lambda\Big) \le \frac{\pi}{2}$. Hence $|\mu_\lambda| \le |\mu_{\lambda_0}| \le 2kb$ for each $\lambda \in F$. □

Proposition 6.2. *Let $T \in B(X)$ be a polynomially bounded operator with polynomial bound k. Suppose that $\sigma(T) \supset \mathbb{T}$ and that T has no nontrivial invariant subspace. Let b be the constant constructed in Proposition 4.4. Then there is a positive constant c_2, $c_2 < 1$ with the following property: if $f : (-\pi,\pi] \to [0,\infty)$ is an integrable function and $0 < \varepsilon < \frac{1}{2kb\pi}$, then for any n sufficiently large there are a finite set $F \subset \mathbb{D}$, vectors $u_\lambda \in X$ and positive numbers $\alpha_\lambda \quad (\lambda \in F)$ such that*

(i) *the sets $(I_\lambda)_{\lambda\in F}$ are pairwise disjoint;*

(ii) *$|\lambda| \ge 3/4$ and $|\lambda^n - 1| \le \frac{1}{9}$ for all $\lambda \in F$;*

(iii) *$\sum_{\lambda\in F}\alpha_\lambda \le \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt$;*

(iv) $\int_{-\pi}^{\pi}\Big|\sum_{\lambda\in F}\alpha_\lambda\lambda^n P_\lambda(t)-f(t)\Big|dt\le c_2\int_{-\pi}^{\pi}f(t)dt$;
(v) $\|u_\lambda\|=1$ *and* $\|(T-\lambda)u_\lambda\|<\varepsilon(1-|\lambda|)^2$ *for all* $\lambda\in F$;
(vi) *the vectors* $u_\lambda \quad (\lambda\in F)$ *are* $2kb$*-circled.*

Proof. Properties (i) – (iv) were proved in Corollary 5.4. Property (v) follows from the Apostol theorem, see Theorem 2.1.

To show property (vi), let $\beta_\lambda,\gamma_\lambda\in\mathbb{C}$, $|\beta_\lambda|\le|\gamma_\lambda| \quad (\lambda\in F)$. Suppose that $\|\sum_{\lambda\in F}\gamma_\lambda u_\lambda\|\le 1$. By Proposition 4.4, there is a function $q\in H^\infty$ such that $\|q\|\le b$, $q(\lambda)=\beta_\lambda\gamma_\lambda^{-1}$ for all $\lambda\in F$ with $\gamma_\lambda\ne 0$, and $q(\lambda)=0$ if $\gamma_\lambda=0$.

For $r\in(0,1)$ and $z\in\mathbb{D}$ define q_r by $q_r(z)=q(rz)$. Then $q_r\in A(\mathbb{D})$, $\|q_r\|\le\|q\|\le b$ for all r and $\lim_{r\to 1_-}q_r(\lambda)=q(\lambda) \quad (\lambda\in F)$. Write $g_{r,\lambda}(z)=\frac{q_r(z)-q_r(\lambda)}{z-\lambda}$. Then $\|g_{r,\lambda}\|\le 2\|q_r\|(1-|\lambda|)^{-1}\le 2b(1-|\lambda|)^{-1}$.

Using Lemma 6.1, we have

$$\begin{aligned}
\Big\|\sum_{\lambda\in F}\beta_\lambda u_\lambda\Big\| &= \Big\|\sum_{\lambda\in F}q(\lambda)\gamma_\lambda u_\lambda\Big\|=\lim_{r\to 1_-}\Big\|\sum_{\lambda\in F}q_r(\lambda)\gamma_\lambda u_\lambda\Big\|\\
&\le \limsup_{r\to 1_-}\Big(\Big\|\sum_{\lambda\in F}q_r(T)\gamma_\lambda u_\lambda\Big\|+\Big\|\sum_{\lambda\in F}(q_r(T)-q_r(\lambda))\gamma_\lambda u_\lambda\Big\|\Big)\\
&\le kb\Big\|\sum_{\lambda\in F}\gamma_\lambda u_\lambda\Big\|+\limsup_{r\to 1_-}\sum_{\lambda\in F}\|g_{r,\lambda}(T)\|\cdot\|(T-\lambda)u_\lambda\|\cdot|\gamma_\lambda|\\
&\le kb+\sum_{\lambda\in F}2kb(1-|\lambda|)^{-1}\varepsilon(1-|\lambda|)^2|\gamma_\lambda|\\
&\le kb+\frac{(2kb)^2}{2kb\pi}\sum_{\lambda\in F}(1-|\lambda|)\le kb+\frac{2kb}{\pi}\cdot\frac{\pi}{2}=2kb.
\end{aligned}$$

Hence the vectors $u_\lambda \quad (\lambda\in F)$ are $2kb$-circled. □

7 Invariant subspaces

Denote by $\mathcal{P}$ the normed space of all polynomials with the norm $\|p\|=\sup\{|p(z)|: z\in\mathbb{D}\}$. Let $\mathcal{P}^*$ be its dual with the usual dual norm.

Let $\varphi\in\mathcal{P}^*$. By the Hahn-Banach theorem, φ can be extended without changing the norm to a functional on the space of all continuous function on $\mathbb{T}$ with the sup-norm. By the Riesz theorem, there exists a Borel measure μ on $\mathbb{T}$ such that $\|\mu\|=\|\varphi\|$ and $\varphi(p)=\int p\,d\mu$ for all polynomials p.

Let L^1 be the Banach space of all complex integrable functions on $\mathbb{T}$ with the norm $\|f\|_1=(2\pi)^{-1}\int_{-\pi}^{\pi}|f(e^{it})|dt$.

Of particular interest are the following functionals on $\mathcal{P}$:

(i) Let $\lambda \in \mathbb{D}$. Denote by $\mathcal{E}_\lambda$ the evaluation functional defined by $\mathcal{E}_\lambda(p) = p(\lambda) \quad (p \in \mathcal{P})$. Clearly $\|\mathcal{E}_\lambda\| = 1$.

(ii) Let $f \in L^1$. Denote by $M_f \in \mathcal{P}^*$ the functional defined by

$$M_f(p) = (2\pi)^{-1} \int_{-\pi}^{\pi} p(e^{it}) f(e^{it}) dt \qquad (p \in \mathcal{P}).$$

Then $\|M_f\| \le \|f\|_1$.

In particular, if $g = 1$ then $M_g(p) = p(0)$ for all p and M_g is the evaluation at the origin. More generally, if $\lambda \in \mathbb{D}$ and $g(e^{it}) = P_\lambda(t)$ then M_g is the evaluation at the point λ.

(iii) Let $T : X \to X$ be a polynomially bounded operator with polynomial bound k, let $x \in X$ and $x^* \in X^*$. Let $x \otimes x^* \in \mathcal{P}^*$ be the functional defined by

$$(x \otimes x^*)(p) = \langle p(T)x, x^* \rangle \qquad (p \in \mathcal{P}).$$

Since T is polynomially bounded, $x \otimes x^*$ is a bounded functional and we have $\|x \otimes x^*\| \le k\|x\| \cdot \|x^*\|$.

Of course the definition of $x \otimes x^*$ depends on the operator T but since we are going to consider only one operator T, this cannot lead to a confusion.

Suppose that T also satisfies the condition that $\|T^n u\| \to 0$ for all $u \in X$. It is a folklore result that then all the functionals $x \otimes x^*$ where $x \in X$ and $x^* \in X^*$ can be represented by absolutely continuous measures, and so these functionals are of the form (ii). Various versions of this result can be found in [Sz], [KO], [A2], [E]. Usually such results are proved by defining the H^∞ calculus for T (by means of radial limits) and by showing that this functional calculus is (w^*, SOT) continuous. Since we have not found the precise form of the necessary statement, we include the proof below; we present a more direct argument using some classical results from measure theory.

Lemma 7.1. *Let T be a polynomially bounded operator on a Banach space X. Suppose that $\|T^n u\| \to 0$ for all $u \in X$. Let $x \in X$, $x^* \in X^*$. Then there exists $f \in L^1$ such that $\langle p(T)x, x^* \rangle = \int_{-\pi}^{\pi} p(e^{it}) f(e^{it}) dt$ for all polynomials p.* Moreover, it is possible to choose $f \in L^1$ such that $\|f\|_1 = \|x \otimes x^*\|$.

Proof. Recall that a sequence $(f_n)_n \subset A(\mathbb{D})$ is called Montel if $\sup \|f_n\| < \infty$ and $\lim_{n\to\infty} f_n(z) = 0$ for all $z \in D$.

We show that $\langle f_n(T)x, x^*\rangle \to 0$ for any Montel sequence (f_n).

Without loss of generality we can assume that $\sup \|f_n\| \le 1$, $\|x\| \le 1$ and $\|x^*\| \le 1$. Let $f_n(z) = \sum_{j=0}^{\infty} c_{n,j} z^j$ be the Taylor expansion of f_n. By the Cauchy formula and the Lebesgue domination theorem, we have $\lim_{n\to\infty} c_{n,j} = 0$ for each $j \ge 0$.

Let ε be a positive number such that $\varepsilon < 2k$, where k is the polynomial bound of T. Choose l such that $\|T^l x\| \le \varepsilon / 4k$. There exists n_0 sufficiently large such that for every $n \ge n_0$ we have $|c_{n,j}| < \varepsilon / 2lk \quad (j = 0, \dots, l)$. Fix such an n and write $g(z) = \sum_{j=0}^{l-1} c_{n,j} z^j$. Then $f_n(z) = g(z) + z^l h(z)$ for some function $h \in A(\mathbb{D})$. Clearly $\|g\| \le \sum_{j=0}^{l-1} |c_{n,j}| \le \varepsilon / 2k$ and $\|h\| = \|f_n - g\|$. Thus

$$
\begin{aligned}
& |\langle f_n(T)x, x^*\rangle| \le \|f_n(T)x\| \le \|g(T)x\| + \|(f_n - g)(T)x\| \\
\le \; & k\|g\| + \|h(T)\| \cdot \|T^l x\| \le \frac{\varepsilon}{2} + k\|f_n - g\| \cdot \frac{\varepsilon}{4k} \\
\le \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \cdot (\|f_n\| + \|g\|) < \varepsilon.
\end{aligned}
$$

Thus $\langle f_n(T)x, x^*\rangle \to 0$.

Now let μ be a measure representing the functional $x \otimes x^*$ such that $\|\mu\| = \|x \otimes x^*\|$. Since we have $(x \otimes x^*)(f_n) \to 0$ for each Montel sequence (f_n), μ is a Henkin measure. By the Val'skii theorem and the M. and F. Riesz theorem, μ is absolutely continuous with respect to the Lebesgue measure. For details we refer to [R], Theorem 9.2.1 and Remark 9.2.2 (c).

The Radon-Nikodym theorem now implies the statement of the Lemma.

□

Let c_3 be a constant satisfying $c_2 < c_3 < 1$, where c_2 is the constant from Proposition 6.2. Let b be the universal constant form Proposition 4.4.

Theorem 7.2. *Let $T : X \to X$ be a polynomially bounded operator with constant k, such that $\sigma(T) \supset \mathbb{T}$ and T has no nontrivial invariant subspace. Let $f \in L^1$ be non–negative with $\|f\|_1 = 1$, and let $y^* \in X^*$ be arbitrary. Then for every positive integer n sufficiently large there exist $x \in X$ and $x^* \in X^*$ such that $\|x\| \le 2kb\sqrt{2}$, $\|x^*\| \le 1$ and $\|x \otimes (T^{*n}x^* + y^*) - M_f\| < c_3$.*

Proof. Let ε be a positive number satisfying $\varepsilon < \frac{1}{2kb\pi}$, $\varepsilon\|y^*\|^2 < 1$ and $12k^3b^2\pi\sqrt{\varepsilon} < c_3 - c_2$.

For any n sufficiently large there exist, by Proposition 6.2, a finite set $F \subset \mathbb{D}$ and positive numbers $\alpha_\lambda \quad (\lambda \in F)$ such that the intervals $(I_\lambda)_{\lambda \in F}$ are pairwise disjoint and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\Big|\sum_{\lambda\in F}\alpha_\lambda \lambda^n P_\lambda(t) - f(t)\Big|\, dt \le c_2.$$

Also, there exist $2kb$-circled vectors $u_\lambda \in X \quad (\lambda \in F)$ such that $\|u_\lambda\| = 1$ and $\|(T-\lambda)u_\lambda\| < \varepsilon(1-|\lambda|)^2$ for all $\lambda \in F$.

We define, on the linear span of $(u_\lambda)_\lambda$, the linear functional φ by $\varphi(u_\lambda) = \lambda^{-n}y^*(u_\lambda) \quad (\lambda \in F)$. By the Hahn-Banach theorem, we can extend it to a bounded functional on X denoted by the same symbol φ. By Proposition 3.5, there exist complex numbers μ_λ and a functional $\psi \in X^*$ such that $\|\sum_{\lambda\in F}\mu_\lambda u_\lambda\| \le 2kb\sqrt{2}$, $\|\psi - \varphi\| \le 1$ and $\psi(\mu_\lambda u_\lambda) = \alpha_\lambda$ for every $\lambda \in F$. Note that we have the estimates $|\mu_\lambda| \le 4k^2b^2\sqrt{2} < 6k^2b^2$ by Lemma 6.1. We take $x = \sum_{\lambda\in F}\mu_\lambda u_\lambda$ and $x^* = \psi - \varphi$.

Let $g \in L^1$ be defined by $g(e^{it}) = \sum_{\lambda\in F}\alpha_\lambda\lambda^n P_\lambda(t)$. Thus

$$\|M_g - M_f\| \le \|g - f\|_1 \le c_2$$

and for any polynomial $p \in \mathcal{P}$ we have

$$M_g p = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{it})p(e^{it})\,dt = \frac{1}{2\pi}\sum_{\lambda\in F}\alpha_\lambda\lambda^n\int_{-\pi}^{\pi}P_\lambda(t)p(e^{it})\,dt = \sum_{\lambda\in F}\alpha_\lambda\lambda^n p(\lambda).$$

Therefore

$$\begin{aligned} & \|x\otimes(T^{*n}x^* + y^*) - M_f\| \le \|T^n x \otimes x^* + x\otimes y^* - M_g\| + \|M_g - M_f\| \\ \le\ & \sup_{\|p\|\le 1}\Big|\langle p(T)T^n x, x^*\rangle + \langle p(T)x, y^*\rangle - M_g p\Big| + c_2. \end{aligned}$$

Now

$$\begin{aligned} \langle p(T)T^n x, x^*\rangle &= \sum_{\lambda\in F}\mu_\lambda\langle T^n p(T)u_\lambda, x^*\rangle \\ = & \sum_{\lambda\in F}\mu_\lambda\Big\langle (T^n p(T) - \lambda^n p(\lambda))u_\lambda, x^*\Big\rangle + \sum_{\lambda\in F}\mu_\lambda\Big\langle \lambda^n p(\lambda)u_\lambda, x^*\Big\rangle \end{aligned}$$

and

$$\langle p(T)x, y^*\rangle = \sum_{\lambda\in F}\mu_\lambda\langle p(T)u_\lambda, y^*\rangle$$

$$= \sum_{\lambda\in F} \mu_\lambda \Big\langle (p(T)-p(\lambda))u_\lambda, y^* \Big\rangle + \sum_{\lambda\in F} \mu_\lambda \langle p(\lambda)u_\lambda, y^*\rangle.$$

Using the equalities $\langle u_\lambda, y^*\rangle = \lambda^n\langle u_\lambda, \varphi\rangle$, $x^*+\varphi=\psi$ and $\langle \mu_\lambda u_\lambda, \psi\rangle = \alpha_\lambda$, we obtain that

$$\sum_{\lambda\in F} \mu_\lambda\langle \lambda^n p(\lambda)u_\lambda, x^*\rangle + \sum_{\lambda\in F}\mu_\lambda\langle p(\lambda)u_\lambda, y^*\rangle = \sum_{\lambda\in F}\mu_\lambda\langle\lambda^n p(\lambda)u_\lambda, x^*+\varphi\rangle$$

$$= \sum_{\lambda\in F}\mu_\lambda\langle \lambda^n p(\lambda)u_\lambda, \psi\rangle = \sum_{\lambda\in F}\alpha_\lambda\lambda^n p(\lambda) = M_g p.$$

Therefore we have

$$\begin{aligned} & \Big|\langle p(T)T^n x, x^*\rangle + \langle p(T)x, y^*\rangle - M_g p\Big| \\ = & \Big|\sum_{\lambda\in F}\mu_\lambda\Big(\Big\langle (T^n p(T)-\lambda^n p(\lambda))u_\lambda, x^*\Big\rangle + \sum_{\lambda\in F}\Big\langle (p(T)-p(\lambda))u_\lambda, y^*\Big\rangle\Big)\Big|. \end{aligned} \tag{6}$$

We estimate the right-hand side of (6) in a standard way. Write $q(z) = \frac{z^n p(z)-\lambda^n p(\lambda)}{z-\lambda}$. Clearly $\|q\| \le 2\|p\|(1-|\lambda|)^{-1} \le 2(1-|\lambda|)^{-1}$. Then $\|q(T)\| \le 2k(1-|\lambda|)^{-1}$. Hence

$$\begin{aligned} & \|(T^n p(T)-\lambda^n p(\lambda))u_\lambda\| = \|q(T)(T-\lambda)u_\lambda\| \\ \le\ & 2k(1-|\lambda|)^{-1}\varepsilon(1-|\lambda|)^2 \le 2k\varepsilon(1-|\lambda|). \end{aligned}$$

Similarly, one obtains the estimate $\|(p(T)-p(\lambda))u_\lambda\| \le 2k\varepsilon(1-|\lambda|)$. Since $\|x^*\|\le 1$ and $\varepsilon\|y^*\|^2 < 1$, from (6) we obtain

$$\begin{aligned} & \Big|\langle p(T)T^n x, x^*\rangle + \langle p(T)x, y^*\rangle - M_g p\Big| \\ \le\ & \sum_{\lambda\in F}|\mu_\lambda|\Big(2k\varepsilon(1-|\lambda|) + 2k\sqrt{\varepsilon}(1-|\lambda|)\Big) \le 6k^2b^2\cdot 4k\sqrt{\varepsilon}\cdot\frac{\pi}{2} \end{aligned}$$

because $|\mu_\lambda| \le 6k^2b^2$ and

$$4\sum_{\lambda\in F}(1-|\lambda|) = m(\bigcup_{\lambda\in F} I_\lambda) \le 2\pi.$$

Thus

$$\|x\otimes(T^{*n}x^*+y^*) - M_f\| \le 12k^3b^2\pi\sqrt{\varepsilon} + c_2 < c_3.$$

This completes the proof. □

Theorem 7.3. *Let $T: X\to X$ be a polynomially bounded operator with constant k. Suppose that $\sigma(T)\supset\mathbb{T}$ and that T has no nontrivial invariant* *subspace. Assume that $\|T^n x\| \to 0$ for all $x \in X$. Let $w \in X$, $z^* \in X^*$, $\delta > 0$ and $f \in L^1$ with $f \geq 0$. Then there exist $u \in X$ and $u^* \in X^*$ such that*

(i) $\|u \otimes (u^* + z^*) - M_f\| \leq c_3 \cdot \|f\|_1$;

(ii) $\|w \otimes u^*\| < \delta$;

(iii) $\|u\| \leq 2kb\sqrt{2}\|f\|_1^{1/2}$ *and* $\|u^*\| \leq k\|f\|_1^{1/2}$.

Proof. The statement is trivial if $\|f\|_1 = 0$. Assume that $\|f\|_1 \neq 0$. Choose n large enough such that $\|T^n w\| < \delta \|f\|_1^{-1/2} k^{-1}$ and such that, by Theorem 7.2 applied to the function $f \cdot \|f\|_1^{-1}$ and the functional $z^* \|f\|_1^{-1/2}$, there exist $v \in X$ and $v^* \in X^*$ with $\|v\| \leq 2kb\sqrt{2}$, $\|v^*\| \leq 1$ and

$$\left\| v \otimes (T^{*n} v^* + z^* \|f\|_1^{-1/2}) - M_{f\|f\|^{-1}} \right\| \leq c_3.$$

Set $u = \|f\|_1^{1/2} v$ and $u^* = \|f\|_1^{1/2} T^{*n} v^*$. Then $\|u\| \leq 2kb\sqrt{2}\|f\|_1^{1/2}$ and $\|u^*\| \leq k\|f\|_1^{1/2}$.

Furthermore,

$$\|w \otimes u^*\| = \|f\|_1^{1/2} \cdot \|w \otimes T^{*n} v^*\| = \|f\|_1^{1/2} \cdot \|T^n w \otimes v^*\| \leq \|f\|_1^{1/2} k \cdot \|T^n w\| < \delta.$$

Finally,

$$\|u \otimes (u^* + z^*) - M_f\| = \|f\|_1 \cdot \left\| v \otimes (T^{*n} v^* + z^* \|f\|_1^{-1/2}) - M_{f\|f\|_1^{-1}} \right\| \leq c_3 \|f\|_1.$$

□

We fix an integer N such that $c_3 + \pi N^{-1} < 1$ and a positive constant c such that $1 - N^{-1}(1 - c_3 - \pi N^{-1}) < c < 1$.

Theorem 7.4. *Let $T : X \to X$ be a polynomially bounded operator with constant k. Assume $\|T^n u\| \to 0$ for all $u \in X$. Suppose that $\sigma(T) \supset \mathbb{T}$ and T has no nontrivial invariant subspace. Let $x \in X$, $x^* \in X^*$ and $h \in L^1$. Then there exist $y \in X$ and $y^* \in X^*$ such that*

(i) $\|y - x\| \leq 2kb\sqrt{2}\|h\|_1^{1/2}$;

(ii) $\|y^* - x^*\| \leq k\|h\|_1^{1/2}$;

(iii) $\|y \otimes y^* - x \otimes x^* - M_h\| \leq c\|h\|_1$.

Proof. For $j = 0, \ldots, N-1$ let B_j be the set of all complex numbers that are of the form re^{it} with $r > 0$ and $-\frac{\pi}{N} \le t - \frac{2\pi j}{N} < \frac{\pi}{N}$. Fix a representative of h and define $A_j = h^{-1}(B_j) \quad (j = 0, \ldots, N-1)$. Then $\|h\|_1 = \sum_{j=0}^{N-1} \|h\chi_j\|_1$ where χ_j is the characteristic function of $A_j \quad (j = 0, \ldots, N-1)$.

Fix j_0, $0 \le j_0 \le N-1$ such that $\|h\chi_{j_0}\|_1 \ge N^{-1}\|h\|_1$.

Set $\nu = e^{2\pi j_0 i/N}$. For each $z \in A_{j_0}$ we have

$$\Big||h(z)|\nu - h(z)\Big| = |h(z)| \cdot \Big|\nu - \frac{h(z)}{|h(z)|}\Big| \le |h(z)|\pi N^{-1},$$

and so $\Big\||h|\nu\chi_{j_0} - h\chi_{j_0}\Big\|_1 \le \pi N^{-1}\|h\chi_{j_0}\|_1$.

Without loss of generality we can assume that $\|h\|_1 \ne 0$. Let δ be a positive number such that $\delta\|h\|_1^{-1} + 1 - N^{-1}(1 - c_3 - \pi N^{-1}) < c$.

By Theorem 7.3, there are vectors $u \in X$ and $u^* \in X^*$ such that $\|u\| \le 2kb\sqrt{2}\|h\chi_{j_0}\|_1^{1/2}$, $\|u^*\| \le k\|h\chi_{j_0}\|_1^{1/2}$, $\|x \otimes u^*\| < \delta$ and

$$\|u \otimes (u^* + x^*) - M_{|h|\chi_{j_0}}\| \le c_3\|h\chi_{j_0}\|_1.$$

Set $y = x + \nu u$ and $y^* = x^* + u^*$. Then $\|y - x\| = \|\nu u\| \le 2kb\sqrt{2}\|h\chi_{j_0}\|_1^{1/2} \le 2kb\sqrt{2}\|h\|_1^{1/2}$ and $\|y^* - x^*\| = \|u^*\| \le k\|h\|_1^{1/2}$.

Furthermore,

$$\begin{aligned}
& \|y \otimes y^* - x \otimes x^* - M_h\| \le \|x \otimes y^* - x \otimes x^*\| + \|\nu u \otimes y^* - M_h\| \\
\le\ & \|x \otimes u^*\| + \Big\|\nu\Big(u \otimes (x^* + u^*) - M_{|h|\chi_{j_0}}\Big)\Big\| + \|\nu M_{|h|\chi_{j_0}} - M_h\| \\
\le\ & \delta + c_3\|h\chi_{j_0}\|_1 + \Big\|\nu|h|\chi_{j_0} - h\chi_{j_0}\Big\|_1 + \sum_{j \ne j_0} \|h\chi_j\|_1 \\
\le\ & \delta + (c_3 + \pi N^{-1})\|h\chi_{j_0}\|_1 + \|h\|_1 - \|h\chi_{j_0}\|_1 \\
\le\ & \|h\|_1 - \|h\chi_{j_0}\|_1\Big(1 - c_3 - \pi N^{-1}\Big) + \delta \\
\le\ & \|h\|_1 \cdot \Big(1 - N^{-1}(1 - c_3 - \pi N^{-1})\Big) + \delta \le c\|h\|_1.
\end{aligned}$$

□

Now we are ready to prove the main theorem B.

Theorem B. *Let T be a polynomially bounded operator on a complex Banach space X. Assume that $\|T^n x\| \to 0$ for all $x \in X$ and that the spectrum of T contains the unit circle. Then T has a nontrivial invariant subspace.*

Proof. Suppose on the contrary that T has no nontrivial invariant subspace. We construct inductively convergent sequences $(x_j) \subset X$ and $(x_j^*) \subset X^*$ such that $\|x_j \otimes x_j^* - M_1\| \to 0$, where 1 denotes the constant function equal to 1 on $\mathbb{T}$.

Set $x_0 = 0$ and $x_0^* = 0$. Let $\varphi_0 = x_0 \otimes x_0^* - M_1$. Then $\|\varphi_0\| = 1$.

Suppose that we have already constructed vectors $x_j \in X$ and $x_j^* \in X^*$ such that $\|\varphi_j\| \le c^j$ where $\varphi_j = x_j \otimes x_j^* - M_1$. Let $h_j \in L^1$ be a function representing the functional φ_j such that $\|h_j\|_1 = \|\varphi_j\| \le c^j$. Let k be the polynomial bound of T. By Theorem 7.4, there are $x_{j+1} \in X$ and $x_{j+1}^* \in X^*$ such that

$$\begin{aligned} &\|x_{j+1} - x_j\| \le 2kb\sqrt{2}\|h_j\|_1^{1/2} \le 2\sqrt{2}kbc^{j/2}, \\ &\|x_{j+1}^* - x_j^*\| \le k\|h_j\|_1^{1/2} \le kc^{j/2} \end{aligned}$$

and for $\varphi_{j+1} := x_{j+1} \otimes x_{j+1}^* - M_1$ we have

$$\begin{aligned} \|\varphi_{j+1}\| &= \|x_{j+1} \otimes x_{j+1}^* - x_j \otimes x_j^* + \varphi_j\| \\ &= \|x_{j+1} \otimes x_{j+1}^* - x_j \otimes x_j^* + M_{h_j}\| \le c\|h_j\|_1 \le c^{j+1}. \end{aligned}$$

Clearly (x_j) and (x_j^*) are Cauchy sequences. Let $x = \lim_{j\to\infty} x_j$ and $x^* = \lim_{j\to\infty} x_j^*$. For each polynomial p with $\|p\| \le 1$ we have

$$\begin{aligned} &\Big|\langle p(T)x_j, x_j^*\rangle - \langle p(T)x, x^*\rangle\Big| \\ \le\ &\Big|\langle p(T)x_j, x_j^*\rangle - \langle p(T)x_j, x^*\rangle\Big| + \Big|\langle p(T)x_j, x^*\rangle - \langle p(T)x, x^*\rangle\Big| \\ \le\ &k\|x_j\| \cdot \|x^* - x_j^*\| + k\|x_j - x\| \cdot \|x^*\| \to 0 \end{aligned}$$

uniformly on the unit ball in $\mathcal{P}$. Thus $x \otimes x^* = \lim_{j\to\infty} x_j \otimes x_j^* = M_1$ and $\langle p(T)x, x^*\rangle = p(0)$ for each polynomial p. It is well known that this implies that T has a nontrivial invariant subspace. Indeed, either $Tx = 0$ (in this case x generates a 1-dimensional invariant subspace) or the vectors $T^k x \quad (k \ge 1)$ generate a nontrivial closed invariant subspace. $\square$

The condition $T^n x \to 0 \quad (x \in X)$ in the previous theorem can be omitted. However, in this case we obtain an invariant subspace for T^* instead of T.

Theorem A. *Let T be a polynomially bounded operator on a Banach space X such that $\sigma(T) \supset \mathbb{T}$. Then T^* has a nontrivial closed invariant subspace.*

In particular, if X is reflexive, then T itself has a nontrivial closed invariant subspace.

Proof. We reduce the problem in a standard way. Let $X_1 = \{x \in X : \|T^n x\| \to 0\}$ and $Y_1 = \{x^* \in X^* : \|T^{*n}x^*\| \to 0\}$. Then X_1 and Y_1 are closed subspaces invariant with respect to T and T^*, respectively. So $X_1^{\perp}$ is invariant with respect to T^*. Thus it is sufficient to consider only the cases that X_1 and Y_1 are trivial.

If $X_1 = X$ then T has a nontrivial invariant subspace by Theorem B, and so has T^*. If $Y_1 = X^*$ then T^* has a nontrivial invariant subspace by Theorem B.

The remaining case of $X_1 = \{0\}$ and $Y_1 = \{0\}$ (i.e., the class C_{11} in the terminology of Sz. Nagy and Foiaş [NF]) was discussed in [CF], cf. p. 136. Since in [CF] it was considered only the case of reflexive Banach spaces, we indicate briefly the argument in the general situation as a separate theorem, which will finish the proof of our main result.

Theorem 7.5. *If T is a power bounded operator of class C_{11} on a complex Banach space X, then either T^* has a nontrivial hyperinvariant subspace, or T is a scalar multiple of the identity.*

Proof. We follow closely the lines of the original proof, avoiding the reflexivity assumption on X required in [CF]. For $x \in X$ define

$$\|x\|_1 = \limsup_{n\to\infty} \|T^n x\|.$$

Note that $\|x\|_1 \le k\|x\|$. Let X_1 be the completion of X with respect to the norm $\|\cdot\|_1$. Let $A : X \to X_1$ be the natural embedding of X into X_1. Then A is a quasiaffinity, i.e., it is a bounded injective linear operator with dense range.

Since $\|Tx\|_1 = \|x\|_1$ for all $x \in X$, the operator T extends continuously to an isometry T_1 on X_1. We can assume that TX is dense in X, since otherwise $\ker T^* = (TX)^{\perp}$ is a nontrivial subspace hyperinvariant with respect to T^*. Hence T_1X_1 is dense in X_1. Therefore T_1 is an invertible isometrical operator. By [CF], Proposition 5.1.4, T_1 is $C^2(\mathbb{T})$–unitary, where $C^2(\mathbb{T})$ denotes the algebra of all complex functions of class C^2 on $\mathbb{T}$ (we refer to the Definitions 3.1.3, 3.1.18 and 5.1.1 of [CF]). Hence T_1 is decomposable by Theorem 3.1.19.

It is easy to see that $AT = T_1A$. Hence T is a quasiaffine transformation of T_1. In the standard notation this is denoted by $T \prec T_1$. Consequently, $T_1^* \prec T^*$.

Applying the same argument to T^* instead of T, we get a decomposable operator T_2 such that $T^* \prec T_2$. Thus $T_1^* \prec T^* \prec T_2$ where both T_1^* and T_2 are decomposable, see [LN], Theorem 2.5.3. Now [CF], Theorem 2.4.5 leads to the desired conclusion, except for the case when the spectrum of T_1^* is a single point $\{\lambda\}$. In this case the arguments in the proof of Theorem 5.1.9 and Lemma 4.3.5, show that $T = \lambda I$. □

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