On spectral properties of linear combinations of idempotents

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Abstract. Let P, Q be two linear idempotents on a Banach space. We show that the closeness of the range and complementarity of the kernel (range) of linear combinations of P and Q are independent of the choice of coefficients. This generalizes known results and shows that many spectral properties do not depend on the coefficients.

The non-singularity of the difference and sum of two idempotent matrices P and Q was first studied in [KRS]. In [BB] it was proved that the non-singularity of $P+Q$ is equivalent to the non-singularity of any linear combination c_1P+c_2Q where $c_1, c_2 \neq 0, c_1+c_2 \neq 0$. The result was further generalized [DYD] to Hilbert space operators and in [KR1] the stability of the nullity and the rank of linear combinations of idempotents was proved.

Finally, in [KR2] it was proved (for Banach space operators) that the Fredholmness and semi-Fredholmness of linear combinations of two idempotents is independent of the choice of coefficients.

We improve these results and show that for two idempotents P, Q on a Banach space the closeness of the range of c_1P+c_2Q and the complementarity of its kernel and range are independent of the choice of the coefficients c_1, c_2. Moreover, the kernel and range behave continuously in the gap topology. This implies the independence of many spectral properties of linear combinations c_1P+c_2Q from the coefficients c_1, c_2.

Let $T \in B(X)$ where $B(X)$ denotes the set of all bounded linear operators on a Banach space X. Denote by $N(T)$ and $R(T)$ the kernel and range of T, respectively.

An operator $P \in B(X)$ is called an idempotent if $P^2 = P$. Note that the range of an idempotent is always closed since $R(P) = N(I-P)$, where I is the identity operator.

The main result of this paper is the following theorem:

Main Theorem. *Let $P, Q \in B(X)$ be idempotents. Let $c_1, c_2 \in \mathbb{C} \setminus \{0\}$, $c_1 + c_2 \neq 0$. If $c_1P + c_2Q$ is invertible (left invertible, right invertible, injective, bounded below, surjective, Fredholm, upper semi-Fredholm, lower semi-Fredholm, left essentially invertible, right essentially invertible or has a generalized inverse, respectively), then $z_1P + z_2Q$ has the same property for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$, $z_1 + z_2 \neq 0$.*

Let M, L be closed subspaces of a Banach space X. Let

$$\delta(M, L) = \sup\big\{\text{dist}\,\{x, L\} : x \in M, \|x\| \le 1\big\}.$$

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The gap $\hat{\delta}(M,L)$ between M and L is defined by

$$\hat{\delta}(M,L)=\max\big\{\delta(M,L),\delta(L,M)\big\}.$$

The reduced minimum modulus of an operator $T\in B(X)$ is defined by

$$\gamma(T)=\inf\big\{\|Tx\|:\operatorname{dist}\{x,N(T)\}\le 1\big\}.$$

The most important property of the reduced minimum modulus is that $\gamma(T)>0$ if and only if T has closed range. For basic properties of the gap and reduced minimum modulus see [K], p. 197–201, or [M], Sec. 10.

Let P,Q be idempotents on a Banach space X. It is easy to see that instead of the function $(c_1,c_2)\mapsto c_1P+c_2Q$ of two variables (c_1,c_2), $c_1,c_2\ne 0, c_1+c_2\ne 0$, it is sufficient to study the function $z\mapsto P-zQ$ where $z\ne 0,1$.

For $z,z'\in\mathbb{C}\setminus\{0,1\}$ write $V_{z,z'}=I+\frac{z-z'}{z(z'-1)}P$.

Lemma 1. *Let $z,z'\in\mathbb{C}\setminus\{0,1\}$. Then:*
(i) $V_{z,z'}V_{z',z}=V_{z',z}V_{z,z'}=I$;
(ii) $V_{z,z'}N(P-zQ)=N(P-z'Q)$;
(iii) $\delta\big(N(P-zQ),N(P-z'Q)\big)\le\|P\|\cdot\big|\frac{z-z'}{z(z'-1)}\big|$.

Proof. (i) Clearly $V_{z,z'}V_{z',z}=V_{z',z}V_{z,z'}$ and we have

$$\begin{aligned}V_{z,z'}V_{z',z}&=\Big(I+\frac{z-z'}{z(z'-1)}P\Big)\Big(I+\frac{z'-z}{z'(z-1)}P\Big)\\&=I+\Big(\frac{z-z'}{z(z'-1)}+\frac{z'-z}{z'(z-1)}+\frac{(z-z')(z'-z)}{zz'(z-1)(z'-1)}\Big)P\\&=I+\frac{(z-z')}{zz'(z-1)(z'-1)}\big(z'(z-1)-z(z'-1)+z'-z\big)P=I.\end{aligned}$$

(ii) Let $x\in N(P-zQ)$, $\|x\|=1$. Then $Qx=\frac{1}{z}PX$ and $QPx=Px$. We have

$$\begin{aligned}(P-z'Q)V_{z,z'}x&=Px+\frac{z-z'}{z(z'-1)}Px-\frac{z'}{z}Px-\frac{z'(z-z')}{z(z'-1)}Px\\&=\Big(\frac{zz'-z+z-z'-z'^2+z'-z'z+z'^2}{z(z'-1)}\Big)Px=0.\end{aligned}$$

Hence $V_{z,z'}N(P-zQ)\subset N(P-z'Q)$.

Similarly, $V_{z',z}N(P-z'Q)\subset N(P-zQ)$ and $N(P-z'Q)=V_{z,z'}V_{z',z}N(P-z'Q)\subset V_{z,z'}N(P-zQ)$. Hence $V_{z,z'}N(P-zQ)=N(P-z'Q)$.

(iii) Let $x\in N(P-zQ)$, $\|x\|=1$. By (ii), $V_{z,z'}x\in N(P-z'Q)$, and so $\operatorname{dist}\{x,N(P-z'Q)\}\le\|x-V_{z,z'}x\|\le\Big\|\frac{z-z'}{z(z'-1)}Px\Big\|\le\|P\|\cdot\Big|\frac{z-z'}{z(z'-1)}\Big|$. So

$$\delta\big(N(P-zQ),N(P-z'Q)\big)\le\|P\|\cdot\Big|\frac{z-z'}{z(z'-1)}\Big|.$$

$\square$

Corollary 2. *The function $z \mapsto N(P - zQ)$ is continuous in the gap topology for $z \in \mathbb{C} \setminus \{0,1\}$. Consequently, the function $z \mapsto \dim N(P - zQ)$ is constant for $z \in \mathbb{C} \setminus \{0,1\}$.*

Proposition 3. *Let $P, Q \in B(X)$, $P^2 = P$, $Q^2 = Q$. Let $z \in \mathbb{C} \setminus \{0,1\}$ and $0 < \varepsilon < 1/3$. Then there exists a neighbourhood U of z such that*

$$\frac{1}{1+\varepsilon}\gamma(P - zQ) \le \gamma(P - z'Q) \le (1+\varepsilon)\gamma(P - zQ)$$

for all $z' \in U$.

Proof. Let U be the set of all $z' \in \mathbb{C} \setminus \{0,1\}$ such that

$$\hat{\delta}\big(N(P - zQ), N(P - z'Q)\big) < \varepsilon/6$$

and

$$|z - z'| < \frac{\varepsilon}{6\max\{1, \|P\|, \|Q\|\}} \cdot \min\Big\{|z(z'-1)|, |z'(z-1)|, \Big|\frac{z(z'-1)}{z'}\Big|, \Big|\frac{z'(z-1)}{z}\Big|\Big\}.$$

It is sufficient to show that $\gamma(P - z'Q) \le (1+\varepsilon)\gamma(P - zQ)$ for all $z' \in U$ since the conditions are symmetrical for z and z'.

Let $z' \in U$. Let (x_n) be a sequence of vectors in X satisfying

$$\operatorname{dist}\{x_n, N(P - zQ)\} = 1$$

for all n and $\|(P - zQ)x_n\| \to \gamma(P - zQ)$. Without loss of generality we may assume that $\|x_n\| \to 1$.

For each n set $x'_n = V_{z,z'}x_n$. We have

$$\begin{aligned}
\limsup_{n\to\infty} \|(P - z'Q)x'_n\| &= \limsup_{n\to\infty}\Big\|Px_n - z'Qx_n + \frac{z - z'}{z(z'-1)}Px_n - \frac{z'(z-z')}{z(z'-1)}QPx_n\Big\| \\
= \limsup_{n\to\infty}&\Big\|(Px_n - zQx_n) + (z - z')Qx_n + \frac{z - z'}{z(z'-1)}Px_n \\
&\quad - \frac{z'(z-z')}{z'-1}Qx_n + \frac{z'(z-z')}{z(z'-1)}(zQx_n - QPx_n)\Big\| \\
\le \gamma(P - zQ) &+ \|Q\|\Big|\frac{z'(z-z')}{z(z'-1)}\Big|\gamma(P - zQ) \\
&\quad + \Big|\frac{z - z'}{z'-1}\Big| \limsup_{n\to\infty}\Big\|(z'-1)Qx_n + \frac{Px_n}{z} - z'Qx_n\Big\| \\
\le (1+\varepsilon/6)&\gamma(P - zQ) + \Big|\frac{z - z'}{z(z'-1)}\Big| \limsup_{n\to\infty}\|Px_n - zQx_n\| \le (1+\varepsilon/3)\gamma(P - zQ).
\end{aligned}$$

We estimate $\operatorname{dist}\{x'_n, N(P - z'Q)\}$. For all n large enough we have

$$\operatorname{dist}\{x'_n, N(P - z'Q)\} \ge \operatorname{dist}\{x_n, N(P - z'Q)\} - \|x_n - x'_n\| \ge \operatorname{dist}\{x_n, N(P - z'Q)\} - \varepsilon/6.$$

For each n there is a $y_n \in N(P-z'Q)$ with $\|x_n-y_n\| < \operatorname{dist}\{x_n, N(P-z'Q)\}+\frac{1}{n} \le \|x_n\|+\frac{1}{n}$. Hence

$$\begin{aligned}
1 = \operatorname{dist}\{x_n, N(P-zQ)\} &\le \|x_n - y_n\| + \operatorname{dist}\{y_n, N(P-zQ)\} \\
&\le \|x_n - y_n\| + \|y_n\|\delta\Big(N(P-z'Q), N(P-zQ)\Big) \\
&\le \operatorname{dist}\{x_n, N(P-z'Q)\} + \frac{1}{n} + \Big(2\|x_n\| + \frac{1}{n}\Big)\delta\Big(N(P-z'Q), N(P-zQ)\Big)
\end{aligned}$$

and

$$\liminf_{n\to\infty} \operatorname{dist}\{x_n, N(P-z'Q)\} \ge 1 - 2\delta\Big(N(P-z'Q), N(P-zQ)\Big) \ge 1-\varepsilon/3.$$

Hence

$$\liminf_{n\to\infty} \operatorname{dist}\{x'_n, N(P-z'Q)\} \ge 1-\varepsilon/2$$

and

$$\gamma(P-z'Q) \le \frac{1+\varepsilon/3}{1-\varepsilon/2}\gamma(P-zQ) \le (1+\varepsilon)\gamma(P-zQ).$$

□

Corollary 4. *The function $z \mapsto \gamma(P-zQ)$ is continuous in $\mathbb{C}\setminus\{0,1\}$. The set $\{z \in \mathbb{C}\setminus\{0,1\} : \gamma(P-zQ)=0\}$ is both open and closed, so it is either empty or equal to $\mathbb{C}\setminus\{0,1\}$.*

Proof. Follows from the previous proposition and the connectivity of $\mathbb{C}\setminus\{0,1\}$. □

Recall that a closed subspace M of a Banach space X is called complemented if there exists a closed subspace $L \subset X$ such that $X = M \oplus L$. Equivalently, M is complemented if and only if there exists a bounded linear idempotent $P \in B(X)$ with $R(P) = M$.

Corollary 5. *Let $P, Q \in B(X)$ be idempotents. Let $z_0 \in \mathbb{C}\setminus\{0,1\}$. Then:*
(i) $\dim N(P-zQ) = \dim N(P-z_0Q)$ *for all* $z \in \mathbb{C}\setminus\{0,1\}$*;*
(ii) if $N(P-z_0Q)$ is complemented, then $N(P-zQ)$ is complemented for all $z \in \mathbb{C}\setminus\{0,1\}$;
(iii) if $R(P-z_0Q)$ is closed then $R(P-zQ)$ is closed for all $z \in \mathbb{C}\setminus\{0,1\}$. Moreover, the function $z \mapsto R(P-zQ)$ is continuous in the gap topology. In particular, $\operatorname{codim} R(P-zQ) = \operatorname{codim} R(P-z_0Q)$*;*
(iv) if $R(P-z_0Q)$ is complemented then $R(P-zQ)$ is complemented for all $z \in \mathbb{C}\setminus\{0,1\}$.

Proof. (i) was proved in Corollary 2.

(ii) Let $N(P-z_0Q)$ be complemented and $z \in \mathbb{C}\setminus\{0,1\}$. By Lemma 1 (ii), we have $N(P-zQ) = V_{z_0,z}N(P-z_0Q)$ where $V_{z_0,z}$ is an invertible operator. So $N(P-zQ)$ is complemented.

(iii) Suppose that $R(P-z_0Q)$ is closed. Then $\gamma(P-z_0Q) > 0$ and, by Corollary 4, $\gamma(P-zQ) > 0$ for all $z \in \mathbb{C}\setminus\{0,1\}$. Hence $R(P-zQ)$ is closed. By Corollary 2 for $P^*, Q^* \in B(X^*)$, we have

$$\operatorname{codim} R(P-zQ) = \dim N(P^*-zQ^*) = \dim N(P^*-z_0Q^*) = \operatorname{codim} R(P-z_0Q).$$

Similarly, the function $z \mapsto R(P - zQ)$ is continuous in the gap topology by duality.

(iv) Suppose that $R(P - z_0Q)$ is complemented. Let $X = R(P - z_0Q) \oplus L_0$ and $z \in \mathbb{C} \setminus \{0, 1\}$. Then $N(P^* - z_0Q^*) = R(P - z_0Q)^\perp$ and $X^* = N(P^* - z_0Q^*) \oplus L_0^\perp$. Note that $L_0^\perp$ is w^*-closed. By (ii), $N(P^* - zQ^*)$ is complemented in X^*. Moreover, by the proof of (ii), $N(P^* - zQ^*) = V'N(P^* - z_0Q^*)$ where $V' = I + \frac{z_0 - z}{z_0(z-1)}P^*$ is invertible. Hence $X^* = N(P^* - zQ^*) \oplus L'$ where $L' = V'L_0^\perp$ and L' is w^*-closed.

Let $L = {}^\perp L'$. Since $R(P - zQ)^\perp + L^\perp = N(P^* - zQ^*) + L' = X^*$, which is closed, $R(P - zQ) + L$ is a closed subspace of X, see [LN], Theorem A.1.9. We have $(L \cap R(P - zQ))^\perp = L^\perp + R(P - zQ)^\perp = L' + N(P^* - zQ^*) = X^*$, and so $L \cap R(P - zQ) = \{0\}$. Furthermore,

$$(L + R(P - zQ))^\perp = L^\perp \cap R(P - zQ)^\perp = L' \cap N(P^* - zQ^*) = \{0\},$$

and so $L + R(P - zQ) = X$.

Hence $R(P - zQ)$ is complemented. $\square$

Recall that $T \in B(X)$ is left (right) invertible if there exists $S \in B(X)$ such that $ST = I$ $(TS = I)$. It is well known that T is left (right) invertible if and only if T is injective and $R(T)$ is complemented (T is surjective and $N(T)$ is complemented, respectively). T has a generalized inverse if there exists $S \in B(X)$ such that $TST = T$. Equivalently, T has a generalized inverse if and only if T has closed range and both $N(T)$ and $R(T)$ are complemented.

$T \in B(X)$ is called upper (lower) semi-Fredholm if $R(T)$ is closed and $\dim N(T) < \infty$ $(\operatorname{codim} R(T) < \infty$, respectively). T is left (right) essentially invertible if there are $S, K \in B(X)$, K compact and $ST = I + K$ $(TS = I + K$, respectively). It is well known that T is left (right) essentially invertible if and only if T is upper (lower) semi-Fredholm and $R(T)$ is complemented ($N(T)$ is complemented, respectively).

The Main Theorem is now an easy consequence of Corollary 5.

References

[BB] J.K. Baksalary, O.M. Baksalary, Nonsingularity of linear combinations of idempotent matrices, Linear Algebra Appl. 388 (2004), 25–29.

[DYD] H. Du, X. Yao, C. Deng, Invertibility of linear combinations of two idempotents, Proc. Amer. Math. Soc. 134 (2006), 1451–1457.

[K] T. Kato, Perturbation Theory for Linear Operators (Second edition), Springer-Verlag, Berlin-Heidelberg-New York, 1976.

[KR1] J.J. Koliha, V. Rakočevič, The nullity and rank of linear combinations of idempotent matrices, Linear Algebra Appl. 418 (2006), 11–14.

[KR2] J.J. Koliha, V. Rakočevič, Stability theorems for linear combinations of idempotents, Integral Equations Operator Theory, to appear.

[KRS] J.J. Koliha, V. Rakočevič, I. Straškraba, The difference and sum of projectors, Linear Algebra Appl. 388 (2004), 279–288.

[LN] K.B. Laursen, M.M. Neumann: An Introduction to Local Spectral theory, London Mathematical Society Monographs, New Series 20, Oxford Science Publications, Clarendon Press, Oxford, 2000.

[M] V. Müller, Spectral Theory of Linear Operators, Operator Theory Advances and Applications vol. 139, Birkhäuser Verlag, Basel-Boston-New York, 2003.

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