SPHERICAL ISOMETRIES ARE HYPOREFLEXIVE

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ABSTRACT. The result from the title is shown.

Let $L(H)$ denote the algebra of all bounded linear operators on a complex Hilbert space H. If $\mathcal{M} \subset L(H)$, then we denote by $\mathcal{M}'$ the commutant of $\mathcal{M}$, $\mathcal{M}' = \{S \in L(H) : TS = ST \text{ for every } T \in \mathcal{M}\}$. The second commutant is denoted by $\mathcal{M}'' = (\mathcal{M}')'$. Denote further by $\mathcal{W}(\mathcal{M})$ the smallest weakly closed subalgebra of $L(H)$ containing $\mathcal{M}$ and by $AlgLat\mathcal{M}$ the algebra of all operators leaving invariant all subspaces which are invariant for all operators from $\mathcal{M}$. Recall that $\mathcal{M}$ is said to be *reflexive* if $\mathcal{W}(\mathcal{M}) = AlgLat\mathcal{M}$. For a commutative set $\mathcal{M}$, there is also a weaker version of the reflexivity: $\mathcal{M}$ is called *hyporeflexive* if $\mathcal{W}(\mathcal{M}) = AlgLat\mathcal{M} \cap \mathcal{M}'$.

Reflexivity and hyporeflexivity have been studied intensely by many authors. Deddens in [D] proved the reflexivity of a single isometry. The result was extended to sets of commuting isometries in [B] (see also [LM]).

An analogy and, in some sense, a generalization of commuting N-tuple of isometries are spherical isometries. A *spherical isometry* is an N-tuple $T = (T_1, \dots, T_N)$ of mutually commuting operators on H satisfying $T_1^*T_1 + \cdots + T_N^*T_N = I_H$.

The reflexivity of doubly commuting spherical isometries was mentioned in [P]. The aim of the paper is to show the hyporeflexivity of spherical isometries.

If μ is a positive Borel measure on the unit sphere $\partial\mathbb{B}_N = \{(z_1, \dots, z_N) \in \mathbb{C}^N : |z_1|^2 + \cdots + |z_N|^2 = 1\}$, then denote by $H^2(\mu)$ the closure of polynomials in $L^2(\mu)$. We start with the following

Lemma 1. *Let μ be finite positive Borel measure supported on $\partial\mathbb{B}_N$. Then, for all non-negative function $h \in L^1(\mu)$ and $\varepsilon > 0$, there exists a polynomial p such that*

$$\|h - |p|^2\|_1 < \varepsilon.$$

Proof. First note that there is a non–negative continuous function g such that $\|h - g\|_1 < \frac{\varepsilon}{2}$. Moreover, since μ is finite, we can also assume that $g > 0$. By Theorem 3.5 of [R], there exists a sequence p_n of of polynomials such that $|p_n| < \sqrt{g}$ and $|p_n(z)| \to \sqrt{g(z)}$ a.e. μ on $\partial\mathbb{B}_N$. Hence $g - |p_n|^2 \le g \in L^1(\mu)$ and

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$g(z) - |p_n(z)|^2 \to 0$ a.e. μ on $\partial\mathbb{B}_N$. By the Lebesgue's Dominated Convergence Theorem there exists n such that $\|g - |p_n|^2\|_1 < \frac{\varepsilon}{2}$. Hence $\|h - |p|^2\|_1 < \varepsilon$ for $p = p_n$.

Using the terminology of Bercovici [B], the result of Lemma 1 exactly means that the subspace $H^2(\mu) \subset L^2(\mu)$ has the *approximate factorization property.* Thus, by Corollary 1.2 of [B], we have

Corollary 2. *Let μ be a finite positive Borel measure on $\partial\mathbb{B}_N$ and $h \in L^1(\mu)$. Then there exist $f \in H^2(\mu)$ and $g \in L^2(\mu)$ such that $h(z) = f(z)\overline{g(z)}$ a.e. μ.*

Let us fix from now on a spherical isometry $T = (T_1, \dots, T_N) \subset L(H)$. Then T is jointly subnormal by Proposition 2 of [A]. It means that there exist a Hilbert space $K \supset H$ and an N-tuple $\tilde{T} = (\tilde{T}_1, \dots, \tilde{T}_N)$ of mutually commuting normal operators on K such that $T_i = \tilde{T}_i|_H$ for $i = 1, \dots, N$. Further, the joint spectral measure of $\tilde{T}$ is supported on $\partial\mathbb{B}_N$, equivalently $\tilde{T}_1^*\tilde{T}_1 + \cdots + \tilde{T}_N^*\tilde{T}_N = I$. We can assume that $\tilde{T}$ is the minimal normal extension of T in the sense that $K = \bigvee_{\alpha\in\mathbb{Z}_+^N} \tilde{T}^{*\alpha}H$.

If $S \in T'$, then, by Proposition 8 of [A] there exists $\tilde{S} \in L(K)$ commuting with $\tilde{T}_i$ $(i = 1, \dots, N)$ such that $\|\tilde{S}\| = \|S\|$ and $S = \tilde{S}|_H$. In fact $\tilde{S}$ is uniquely determined for the minimal extension $\tilde{T}$. Indeed, $\tilde{S}$ commutes also with $\tilde{T}_i^*$ for $i = 1, \dots, N$ by Fuglede's Theorem and $\tilde{S}\tilde{T}^{*\alpha}x = \tilde{T}^{*\alpha}\tilde{S}x = \tilde{T}^{*\alpha}Sx$ for $\alpha \in \mathbb{Z}_+^N$ and $x \in H$. Hence the uniqueness of $\tilde{S}$ follows from the minimality of the normal extension.

Proposition 3. *Let T be a spherical isometry. If $S \in AlgLatT \cap T'$, then $\tilde{S} \in \tilde{T}''$.*

Proof. Proposition 3, formulated for commuting N-tuples of isometries, was the main result of [HM], but it remains true with the same proof in our situation. For convenience of the reader we indicate briefly the main steps of the proof.

Let $E(\cdot)$ be the spectral measure of the normal N-tuple $\tilde{T}$. Denote by $\mu_x = \|\mu_x(\cdot)x\|^2$ the positive scalar measure corresponding to $x \in K$.

A. (Lemma 4 of [HM]) For $x, y \in H$ there exists a complex number λ such that the measures $\mu_x \vee \mu_y$ and $\mu_{x+\lambda y}$ are equivalent (i.e., absolutely continuous with respect to each other).

In fact, all but countably many complex numbers satisfy the property of A.

For $x \in H$ denote by $Z_+(x)$ $\big(Z(x)\big)$ the smallest subspace of K containing x which is invariant (reducing, respectively) with respect to all $\tilde{T}_1, \dots, \tilde{T}_N$.

By the assumption, $\tilde{S}$ commutes with $\tilde{T}_i$ and $\tilde{T}_i^*$ $(i = 1, \dots, N)$. Further $SZ_+(x) \subset Z_+(x)$. Since $Z(x) = \bigvee_{\alpha\in\mathbb{Z}_+^n} T^{*\alpha}Z_+(x)$, we have $\tilde{S}Z(x) \subset Z(x)$. Consequently $\tilde{S}|Z(x) = t_x(\tilde{T})|Z(x)$ for some function $t_x \in L^\infty(\mu_x)$.

B. (Lemma 5 of [HM]) If $x, y \in H$ and $\mu_x \prec \mu_y$, then $t_x = t_y$ a.e. μ_x.

By induction it is easy to generalize A. and B. to finite families of vectors.

Proof of Proposition 3. Let $\tilde{V} \in L(K)$ commute with $\tilde{T}_1, \dots, \tilde{T}_N$, let $u \in K$ and $\varepsilon > 0$.

Since $K = \bigvee_{x\in H} Z(x)$, we can find vectors $x_1,\dots,x_n, x'_1,\dots,x'_m \in H$ and $u_i \in Z(x_i) \quad (i=1,\dots,n)$, $u'_j \in Z(x'_j) \quad (j=1,\dots,m)$ such that

$$\Big\| u - \sum_{i=1}^n u_i \Big\| < \varepsilon, \qquad \Big\| \tilde V u - \sum_{j=1}^m u'_j \Big\| < \varepsilon.$$

By A. and B. there exists a function $f \in L^\infty(\mu)$ where $\mu = \bigvee_{i=1}^n \mu_{x_i} \vee \bigvee_{j=1}^m \mu_{x'_j}$ such that $\tilde S|Z(x_i) = f(\tilde T)|Z(x_i)$ and $\tilde S|Z(x'_j) = f(\tilde T)|Z(x'_j)$ for all i,j. Then

$$\begin{aligned}
\|\tilde V\tilde S u - \tilde S\tilde V u\| &\le \Big\|\tilde V\tilde S u - \tilde V\tilde S\sum_{i=1}^n u_i\Big\| + \Big\|\tilde V\tilde S\sum_{i=1}^n u_i - \tilde S\sum_{j=1}^m u'_j\Big\| + \Big\|\tilde S\sum_{j=1}^m u'_j - \tilde S\tilde V u\Big\| \\
&\le \|\tilde V\|\|\tilde S\|\Big\|u - \sum_{i=1}^n u_i\Big\| + \Big\|\tilde V f(\tilde T)\sum_{i=1}^n u_i - f(\tilde T)\sum_{j=1}^m u'_j\Big\| + \|\tilde S\|\Big\|\sum_{j=1}^m u'_j - \tilde V u\Big\| \\
&\le \|\tilde V\|\|S\|\varepsilon + \|f(\tilde T)\|\Big(\Big\|\tilde V\sum_{i=1}^n u_i - \tilde V u\Big\| + \Big\|\tilde V u - \sum_{j=1}^m u'_j\Big\|\Big) + \|S\|\varepsilon \\
&\le 2\|\tilde V\|\|S\|\varepsilon + 2\|S\|\varepsilon.
\end{aligned}$$

Since ε was arbitrary, we have $\tilde V\tilde S u = \tilde S\tilde V u$ so that $\tilde S \in (\tilde T)''$.

Recall from [HN] that an algebra $\mathcal W \subset L(H)$ closed in the weak operator topology has property D if every weakly continuous functional φ on $\mathcal W$ can be written in the form $\varphi(A) = \langle Ax, y\rangle \quad (A \in \mathcal W)$ for certain vectors $x, y \in H$. Now we will show

Lemma 4. *Let T be spherical isometry. Then $AlgLatT \cap T'$ has property D.*

Proof. Let φ be a weakly continuous functional on $\mathcal B := AlgLatT \cap T'$. Then there are $x_i, y_i \in H, i = 1,\dots,n$ such that $\varphi(S) = \sum_{i=1}^n (Sx_i, y_i)$ for $S \in \mathcal B$. Let K_0 be the smallest closed subspace containing $x_i,\dots,x_n,y_1,\dots,y_n$, which reduces $\tilde T_i, i = 1,\dots,N$. Using the spectral theory there exist a finite measure $\mu = \bigvee_{i=1}^n(\mu_{x_i} \vee \mu_{y_i})$ on $\partial\mathbb B_N$, sets $\partial\mathbb B_N = \sigma_1 \supset \sigma_2 \supset \cdots \supset \sigma_{2n}$ and a unitary operator

$$U : K_0 \to \bigoplus_{i=1}^{2n} L^2(\mu|_{\sigma_i})$$

such that $U\,v(\tilde T|_{K_0})\,U^{-1} = M_v$ for every function $v \in L^\infty(\mu)$, where M_v is the operator of multiplication by v. Note that the space $\bigoplus_{i=1}^{2n} L^2(\mu|_{\sigma_i})$ can be identified with a subspace of $L^2(\mu,\mathcal F)$, where $\mathcal F$ is a Hilbert space with dim $\mathcal F = 2n$.

Let $S \in \mathcal B$. Since $S \in AlgLatT$, the space $H_0 := K_0 \cap H$ is invariant with respect to S. Since $\tilde S \in \tilde T'$, we have also $\tilde S K_0 \subset K_0$. Moreover, by Proposition 3, we have $\tilde S \in \tilde T''$ and one can easily show that $\tilde S|_{K_0} \in (\tilde T|_{K_0})''$. Thus there is a function $u_S \in L^\infty(\mu)$ such that $\tilde S|_{K_0} = u_S(\tilde T)|_{K_0}$, and consequently $S|_{H_0} = u_S(\tilde T)|_{H_0}$.

Now we have

$$\begin{aligned}\varphi(S) &= \sum_{i=1}^{n}\langle Sx_i, y_i\rangle = \sum_{i=1}^{n}\langle USx_i, Uy_i\rangle = \sum_{i=1}^{n}\langle Uu_S(\tilde{T})x_i, Uy_i\rangle \\ &= \sum_{i=1}^{n}\langle M_{u_S}Ux_i, Uy_i\rangle = \sum_{i=1}^{n}\int \langle u_S(z)(Ux_i)(z), (Uy_i)(z)\rangle_{\mathcal{F}} d\mu(z) \\ &= \int u_S(z)\sum_{i=1}^{n}\big\langle (Ux_i)(z), (Uy_i)(z)\big\rangle_{\mathcal{F}} d\mu(z) = \int u_S(z)f(z)d\mu(z),\end{aligned}$$

where $f(z) = \sum_{i=1}^{n}\big\langle (Ux_i)(z), (Uy_i)(z)\big\rangle_{\mathcal{F}} \in L^1(\mu)$.

By Lemma 4 of [HM] there is a linear combination w of $x_1,\dots,x_n,y_1,\dots,y_n$ such that μ_w and μ are absolutely continuous with respect to each other. Hence $d\mu = v d\mu_w$ for some function $v \in L^1(\mu_w)$ and $v > 0$ a.e. μ_w. Thus $fv \in L^1(\mu_w)$ and, by Corollary 2, there exist $g \in H^2(\mu_w)$ and $h \in L^2(\mu_w)$ such that $f\,v = g\,\overline{h}$ a.e. μ_w.

Denote by K_1 the smallest subspace of K containing w, which reduces $\tilde{T}_i$, $i = 1,\dots,N$. Let $V : K_1 \to L^2(\mu_w)$ be the unitary operator such that $V\,r(\tilde{T})w = r$ for all $r \in L^\infty(\mu_w)$. Then we have also $M_r V = V\,r(\tilde{T})$ for $r \in L^\infty(\mu_w) = L^\infty(\mu)$. Denote $x = V^{-1}g \in V^{-1}H^2(\mu_w) \subset H$ and $\tilde{y} = V^{-1}h \in K_1$. Then, for $S \in \mathcal{B}$, we have

$$\begin{aligned}\varphi(S) &= \int u_S\, f\, d\mu = \int u_S\, f\, v\, d\mu_w = \int u_S\, g\,\overline{h}\, d\mu_w = \langle M_{u_S}\, g, h\rangle_{L^2(\mu_w)} \\ &= \langle V^{-1}M_{u_S}\, g, V^{-1}h\rangle = \langle u_S(\tilde{T})\, V^{-1}\, g, V^{-1}h\rangle = \langle u_S(\tilde{T})\, x, \tilde{y}\rangle \\ &= \langle Sx, \tilde{y}\rangle = \langle Sx, P_H\tilde{y}\rangle = \langle Sx, y\rangle\ ,\end{aligned}$$

where $y = P_H\tilde{y}$. Hence $\mathcal{B}$ has property D.

Recall that, by Theorem 6.2(3) of [HN], a unital commutative weakly closed algebra $\mathcal{W} \subset L(H)$ is hyporeflexive if $AlgLat\mathcal{W} \cap \mathcal{W}'$ has property D. Hence we have proved the following main result of the paper.

Theorem 5. *Let T be a spherical isometry. Then T is hyporeflexive, i.e. $\mathcal{W}(T) = AlgLatT \cap T'$.*

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