On the joint spectral radius

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Abstract. We prove the ℓ_p-spectral radius formula for n-tuples of commuting Banach algebra elements. This generalizes results of [6], [7] and [10].

Let A be a Banach algebra with the unit element denoted by 1. Let $a = (a_1, \dots, a_n)$ be an n-tuple of elements of A. Denote by $\sigma(a)$ the Harte spectrum of a, i.e. $\lambda = (\lambda_1, \dots, \lambda_n) \notin \sigma(a)$ if and only if there exist $u_1, \dots, u_n, v_1, \dots, v_n \in A$ such that

$$\sum_{j=1}^{n}(a_j - \lambda_j)u_j = \sum_{j=1}^{n} v_j(a_j - \lambda_j) = 1.$$

Let $1 \le p \le \infty$. The (geometric) spectral radius of a is defined by

$$r_p(a) = \max\{\|\lambda\|_p : \lambda \in \sigma(a)\},$$

where

$$\|\lambda\|_p = \begin{cases} \max_{1\le j\le n} |\lambda_j| & (p = \infty), \\ \left(\sum_{j=1}^{n} |\lambda_j|^p\right)^{1/p} & (1 \le p < \infty), \end{cases}$$

see [10], cf. also [4].

If $\sigma(a)$ is empty we put formally $r_p(a) = -\infty$.

Clearly $r_p(a)$ depends on p. On the other hand instead of the Harte spectrum we can take any other reasonable spectrum (e.g. the left, right, approximate point, defect, Taylor etc.) without changing the value of $r_p(a)$, see [4], [9].

For a single Banach algebra element the just defined spectral radius $r_p(a)$ does not depend on p and coincides with the ordinary spectral radius $r(a_1) = \max\{|\lambda_1| : \lambda_1 \in \sigma(a_1)\}$. By the well-known spectral radius formula we have in this case

$$r(a_1) = \lim_{k\to\infty} \|a_1^k\|^{1/k} = \inf_k \|a_1^k\|^{1/k}.$$

The spectral radius formula for n-tuples of Banach algebra elements was studied by a number of authors, see e.g. [1], [2], [6], [7], [8]. In this paper we generalize results [6], [7] and [10].

Let $a = (a_1, \dots, a_n)$ be an n-tuple of elements of a Banach algebra A. Instead of powers of a single element it is natural to consider all possible products of $a_1, \dots, a_n$.

Denote by $F(k, n)$ the set of all functions from $\{1, \dots, k\}$ to $\{1, \dots, n\}$. Denote further

$$s_{k,p}(a) = \Big(\sum_{f\in F(k,n)} \|a_{f(1)} \cdots a_{f(k)}\|^p\Big)^{1/p} \qquad (1 \le p < \infty)$$

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and

$$s_{k,\infty}(a)=\max_{f\in F(k,n)}\|a_{f(1)}\cdots a_{f(k)}\|.$$

Lemma 1. $s_{k+l,p}\le s_{k,p}(a)\cdot s_{l,p}(a)$.

Proof. The statement is obvious for $p=\infty$. For $p<\infty$ we have

$$\begin{aligned}\bigl[s_{k,p}(a)\cdot s_{l,p}(a)\bigr]^p&=\sum_{f\in F(k,n)}\|a_{f(1)}\cdots a_{f(k)}\|^p\cdot\sum_{g\in F(l,n)}\|a_{g(1)}\cdots a_{g(l)}\|^p\\&\ge\sum_{f,g}\|a_{f(1)}\cdots a_{f(k)}a_{g(1)}\cdots a_{g(l)}\|^p=\bigl[s_{k+l,p}(a)\bigr]^p.\end{aligned}$$

It is well-known that the previous lemma implies that the limit $\lim_{k\to\infty}\bigl(s_{k,p}(a)\bigr)^{1/k}$ exists and it is equal to $\inf_k\bigl(s_{k,p}(a)\bigr)^{1/k}$.

Thus we may define

$$r''_p(a)=\lim_{k\to\infty}\Bigl(\sum_{f\in F(k,n)}\|a_{f(1)}\cdots a_{f(k)}\|^p\Bigr)^{1/pk}.$$

Similarly we define

$$r'_p(a)=\limsup_{k\to\infty}\Bigl(\sum_{f\in F(k,n)}r^p(a_{f(1)}\cdots a_{f(k)})\Bigr)^{1/pk}\tag{1}$$

(we write shortly $r^p(x)$ instead of $(r(x))^p$).

In general the limit in (1) does not exist. The limit exists if $a_1,\dots,a_n$ are mutually commuting. This can be proved analogously as in Lemma 1 by using the submultiplicativity of the spectral radius.

Theorem 2. Let $a=(a_1,\dots,a_n)$ be an n-tuple of elements of a Banach algebra A. Let $1\le p\le\infty$. Then

$$r_p(a)\le r'_p(a)\le r''_p(a).$$

Proof. The case $p=\infty$ was proved in [7], Theorem 1.

Let $p<\infty$. The second inequality is clear.

Let $\lambda=(\lambda_1,\dots,\lambda_n)\in\sigma(a)$. Denote by A_0 the closed subalgebra of A generated by the unit 1 and the elements $a_1,\dots,a_n$. By [5], Proposition 2 there exists a multiplicative functional $h:A_0\to\mathbf{C}$ such that $h(a_j)=\lambda_j\quad(j=1,\dots,n)$. Then

$$\begin{aligned}&\sum_{f\in F(k,n)}r^p(a_{f(1)}\cdots a_{f(k)})\ge\sum_{f\in F(k,n)}|h(a_{f(1)}\cdots a_{f(k)})|^p\\&=\sum_{f\in F(k,n)}|\lambda_{f(1)}|^p\cdots|\lambda_{f(k)}|^p=(|\lambda_1|^p+\cdots|\lambda_n|^p)^k=\|\lambda\|_p^{pk}.\end{aligned}$$

Thus
$$\sum_{f\in F(k,n)} r^p(a_{f(1)}\cdots a_{f(k)}) \geq r_p^{pk}(a)$$
and $r_p'(a) \geq r_p(a)$.

If $a = (a_1,\dots,a_n)$ is an n-tuple of mutually commuting elements then a better result can be proved.

We use the standard multiindex notation. Denote by $\mathbf{Z}_+$ the set of all non-negative integers. For $\alpha = (\alpha_1,\dots,\alpha_n) \in \mathbf{Z}_+^n$ and $m \in \mathbf{Z}_+$ denote $|\alpha| = \alpha_1+\cdots+\alpha_n$, $\alpha! = \alpha_1!\cdots\alpha_n!$, $a^\alpha = a_1^{\alpha_1}\cdots a_n^{\alpha_n}$ and $m\alpha = (m\alpha_1,\dots,m\alpha_n)$. If k is an integer, $k \geq |\alpha|$ then denote
$$\binom{k}{\alpha} = \frac{k!}{\alpha!(k-|\alpha|)!}$$
(for $n=1$ this definition coincides with the classical binomial coefficients).

We shall use frequently the following formula (for commuting variables x_i):
$$(x_1+\cdots+x_n)^k = \sum_{|\alpha|=k}\binom{k}{\alpha}x^\alpha.$$
In particular, for $x_1=\cdots=x_n=1$ we have $\sum_{|\alpha|=k}\binom{k}{\alpha} = n^k$.

If $a=(a_1,\dots,a_n)$ is a commuting n-tuple of elements of a Banach algebra A, then the definitions of $r_p'(a)$ and $r_p''(a)$ assume a simpler form:
$$r_p'(a) = \lim_{k\to\infty}\Bigl[\sum_{|\alpha|=k}\binom{k}{\alpha}r^p(a^\alpha)\Bigr]^{1/pk},$$
$$r_p''(a) = \lim_{k\to\infty}\Bigl[\sum_{|\alpha|=k}\binom{k}{\alpha}\|a^\alpha\|^p\Bigr]^{1/pk},$$
$(1\leq p<\infty)$.

Theorem 3. Let $a=(a_1,\dots,a_n)$ be an n-tuple of mutually commuting elements of a Banach algebra A. Let $1\leq p\leq\infty$. Then
$$r_p(a) = r_p'(a) = r_p''(a).$$

Proof. For $p=\infty$ the first equality was proved in [10] and the second in [7], Theorem 2.

We assume in the following $p<\infty$.

Recall that the number of all partitions of the set $\{1,\dots,k\}$ into n parts is equal to $\binom{k+n-1}{n-1} \leq (k+n-1)^{n-1}$.

We have
$$\max_{|\alpha|=k}\binom{k}{\alpha}\|a^\alpha\|^p \leq \sum_{|\alpha|=k}\binom{k}{\alpha}\|a^\alpha\|^p \leq \binom{k+n-1}{n-1}\max_{|\alpha|=k}\binom{k}{\alpha}\|a^\alpha\|^p.$$

Note that

$$\lim_{k\to\infty} \binom{k+n-1}{n-1}^{1/k} = 1.$$

Thus

$$r_p''(a) = \lim_{k\to\infty} \left[\sum_{|\alpha|=k} \binom{k}{\alpha} \|a^\alpha\|^p\right]^{1/kp} = \lim_{k\to\infty} \max_{|\alpha|=k} \left[\binom{k}{\alpha} \|a^\alpha\|^p\right]^{1/kp}.$$

Similarly

$$r_p'(a) = \lim_{k\to\infty} \max_{|\alpha|=k} \left[\binom{k}{\alpha} r^p(a^\alpha)\right]^{1/kp}.$$

We prove now the inequality $r_p'(a) \le r_p(a)$:

Choose k and $\alpha \in \mathbf{Z}_+^n$, $|\alpha| = k$. Let $\mu \in \sigma(a^\alpha)$ satisfy $|\mu| = r(a^\alpha)$. By the spectral mapping property there exists $\lambda = (\lambda_1, \ldots, \lambda_n) \in \sigma(a)$ such that $\mu = \lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n}$. Then

$$\begin{aligned}\binom{k}{\alpha} r_p^p(a) =& \binom{k}{\alpha} \|\mu\|^p = \binom{k}{\alpha} |\lambda_1|^{\alpha_1 p} \cdots |\lambda_n|^{\alpha_n p} \le \sum_{|\beta|=k} \binom{k}{\beta} |\lambda_1|^{\beta_1 p} \cdots |\lambda_n|^{\beta_n p} \\ =& \big(|\lambda_1|^p + \cdots |\lambda_n|^p\big)^k = \|\lambda\|_p^{pk} \le r_p^{pk}(a).\end{aligned}$$

Thus

$$r_p'(a) = \lim_{k\to\infty} \max_{|\alpha|=k} \left[\binom{k}{\alpha} r^p(a^\alpha)\right]^{1/kp} \le r_p(a).$$

The remaining inequality $r_p''(a) \le r_p'(a)$ will be proved by induction on n.

For $n = 1$ Theorem 3 reduces to the well-known spectral radius formula for a single element.

Let $n \ge 2$ and suppose that the inequality $r_p'' \le r_p'$ is true for all commuting $(n-1)$-tuples.

For each k there is $\alpha \in \mathbf{Z}_+^n$, $|\alpha| = k$ such that

$$\binom{k}{\alpha} \|a^\alpha\|^p = \max_{|\beta|=k} \binom{k}{\beta} \|a^\beta\|^p.$$

Using the compactness of $< 0, 1 >^n$ we can choose a sequence

$$\{\alpha(i)\}_{i=1}^\infty = \big\{(\alpha_1(i), \ldots, \alpha_n(i)\big\}_{i=1}^n \subset \mathbf{Z}_+^n$$

such that $\lim_{i\to\infty} |\alpha(i)| = \infty$,

$$\binom{|\alpha(i)|}{\alpha(i)} \|a^\alpha(i)\|^p = \max_{|\beta|=|\alpha(i)|} \binom{|\alpha(i)|}{\beta} \|a^\beta\|^p \qquad (i = 1, 2, \ldots) \tag{2}$$

and the sequences $\left\{\frac{\alpha_j(i)}{|\alpha(i)|}\right\}_{i=1}^\infty$ are convergent for $j = 1, \ldots, n$. Denote $k(i) = |\alpha(i)|$ and

$$t_j = \lim_{i\to\infty} \frac{\alpha_j(i)}{k(i)} \in < 0, 1 > \qquad (j = 1, \ldots, n).$$

By (2) we have

$$r_p''^p(a) = \lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k(i)p}.$$

We distinguish two cases:

(a) $t_j = 0$ for some j, $1 \le j \le n$. Without the loss of generality we may assume that $t_n = 0$.

Denote $a' = (a_1, \dots, a_{n-1})$, $\alpha'(i) = (\alpha_1(i), \dots, \alpha_{n-1}(i)) \in \mathbf{Z}_+^{n-1}$ and $k'(i) = |\alpha'(i)| = k(i) - \alpha_n(i)$. Clearly $\lim_{i\to\infty}\frac{k'(i)}{k(i)} = 1$. We have $\|a^{\alpha(i)}\| \le \|a'^{\alpha'(i)}\| \cdot \|a_n\|^{\alpha_n(i)}$. Then

$$r_p''^p(a') \ge \limsup_{i\to\infty}\left[\binom{k'(i)}{\alpha'(i)}\|a'^{\alpha'(i)}\|^p\right]^{1/k'(i)} \ge L_1 \cdot L_2 \cdot L_3,$$

where

$$L_1 = \limsup_{i\to\infty}\left[\frac{\binom{k'(i)}{\alpha'(i)}}{\binom{k(i)}{\alpha(i)}}\right]^{1/k'(i)}, \qquad L_2 = \lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k'(i)}$$

and

$$L_3 = \lim_{i\to\infty}\|a_n\|^{-\alpha_n(i)p/k'(i)}.$$

Since $\lim_{i\to\infty}\frac{\alpha_n(i)}{k'(i)} = 0$, we have $L_3 = 1$.

Further

$$L_2 = \lim_{i\to\infty}\left[\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k(i)}\right]^{k(i)/k'(i)} = r_p''^p(a).$$

Finally,

$$\begin{aligned} L_1 &= \limsup_{i\to\infty}\left[\frac{k'(i)!\cdot\alpha_n(i)!}{k(i)!}\right]^{1/k'(i)} \ge \limsup_{i\to\infty}\left[\frac{\left(\frac{\alpha_n(i)}{3}\right)^{\alpha_n(i)}}{k(i)^{\alpha_n(i)}}\right]^{1/k'(i)} \\ &= \limsup_{i\to\infty}\left(\frac{\alpha_n(i)}{3k(i)}\right)^{\frac{\alpha_n(i)}{k(i)}\cdot\frac{k(i)}{k'(i)}} = 1 \end{aligned}$$

since $\lim_{i\to\infty}\frac{k(i)}{k'(i)} = 1$ and

$$\lim_{i\to\infty}\left(\frac{\alpha_n(i)}{3k(i)}\right)^{\frac{\alpha_n(i)}{k(i)}} = \lim_{x\to 0_+}\left(\frac{x}{3}\right)^x = \lim_{x\to 0_+} x^x = \lim_{x\to 0_+} e^{x\ln x} = 1.$$

Thus $r_p''(a') \ge r_p''(a)$.

By the induction assumption $r_p''(a') = r_p'(a') = r_p(a')$ and by the definition $r_p(a') \le r_p(a) = r_p'(a)$. Hence $r_p''(a) \le r_p'(a)$.

(b) It remains the case $t_j > 0$ $(j = 1, \dots, n)$, where $t_j = \lim_{i\to\infty} \frac{\alpha_j(i)}{k(i)}$. Choose $\varepsilon > 0$, $\varepsilon < \min_{1\le j\le n} \frac{t_j}{n}$. For i sufficiently large we have

$$t_j - \frac{\varepsilon}{4} \le \frac{\alpha_j(i)}{k(i)} \le t_j + \frac{\varepsilon}{4}.$$

We approximate $t_1, \dots, t_n$ by rational numbers. Fix positive integers $c_1, \dots, c_n, d$ such that

$$t_j - \frac{\varepsilon}{2} \le \frac{c_j}{d} \le t_j - \frac{\varepsilon}{4} \qquad (j = 1, \dots, n).$$

Denote $\gamma = (c_1, \dots, c_n) \in \mathbf{Z}_+^n$ and $u = a^\gamma = a_1^{c_1} \cdots a_n^{c_n}$. For each i write $k(i) = m(i)d + z(i)$, where $0 \le z(i) \le d - 1$. So, for i sufficiently large, we have

$$\frac{c_j}{d} \le \frac{\alpha_j(i)}{k(i)}, \qquad \frac{\alpha_j(i)}{k(i)} - \frac{c_j}{d} \le \frac{3\varepsilon}{4}$$

and

$$\alpha_j(i) - m(i)c_j = \alpha_j(i) - \frac{k(i) - z(i)}{d} \cdot c_j = k(i)\left[\frac{\alpha_j(i)}{k(i)} - \frac{c_j}{d}\right] + \frac{z(i)c_j}{d}.$$

Thus $\alpha_j(i) - m(i)c_j \ge 0 \qquad (1 \le j \le n)$ and

$$k(i) - m(i)|\gamma| = \sum_{j=1}^{n} \big(\alpha_j(i) - m(i)c_j\big) \le k(i) \cdot \frac{3\varepsilon n}{4} + \sum_{j=1}^{m} \frac{z(i)c_j}{d} \le \varepsilon n k(i)$$

for i large enough. We have

$$\|a^{\alpha(i)}\| \le \|a_1^{m(i)c_1} \cdots a_n^{m(i)c_n}\| \cdot \|a_1\|^{\alpha_1(i) - m(i)c_1} \cdots \|a_n\|^{\alpha_n(i) - m(i)c_n} \le \|u^{m(i)}\| \cdot K^{n\varepsilon k(i)}$$

where $K = \max\{1, \|a_1\|, \dots, \|a_n\|\}$. Then, since $\binom{m(i)|\gamma|}{m(i)\gamma}^{1/m(i)|\gamma|} \le n$, we have

$$\begin{aligned} r_p'^p(a) &\ge \limsup_{i\to\infty} \left[\binom{m(i)|\gamma|}{m(i)\gamma} r^p(a^{m(i)\gamma})\right]^{1/m(i)|\gamma|} = \limsup_{i\to\infty} \binom{m(i)|\gamma|}{m(i)\gamma}^{1/m(i)|\gamma|} \cdot r(u)^{p/|\gamma|} \\ &= \limsup_{i\to\infty} \left[\binom{m(i)|\gamma|}{m(i)\gamma} \|u^{m(i)}\|^p\right]^{1/m(i)|\gamma|} \ge L_1 \cdot L_2 \cdot L_3, \end{aligned}$$

where

$$L_1 = \liminf_{i\to\infty} \left[\frac{\binom{m(i)|\gamma|}{m(i)\gamma}}{\binom{k(i)}{\alpha(i)}}\right]^{1/m(i)|\gamma|},$$

$$L_2 = \liminf_{i\to\infty} \left[\binom{k(i)}{\alpha(i)} \|a^{\alpha(i)}\|^p\right]^{1/m(i)|\gamma|}$$

and

$$L_3 = \liminf_{i\to\infty} K^{-n\varepsilon p k(i)/m(i)|\gamma|}.$$

Since $1 \le \frac{k(i)}{m(i)|\gamma|} \le \frac{1}{1-n\varepsilon}$ for i sufficiently large, we have $L_3 \ge K^{-\frac{n\varepsilon p}{1-n\varepsilon}}$.

Since

$$\lim_{i\to\infty}\left[\binom{k(i)}{\alpha(i)}\|a^{\alpha(i)}\|^p\right]^{1/k(i)}=r_p''^p(a),$$

we have $L_2\ge\min\{r_p''^p(a),(r_p''^p(a))^{1/1-n\varepsilon}\}$.

To estimate L_1, we use the well-known Stirling formula

$$l!=l^le^{-l}\sqrt{2\pi l}(1+o(l)).$$

Consequently,

$$(1-\varepsilon)\Big(\frac{\alpha_j(i)}{e}\Big)^{\alpha_j(i)/m(i)|\gamma|}\le\big(\alpha_j(i)!\big)^{1/m(i)|\gamma|}\le(1+\varepsilon)\Big(\frac{\alpha_j(i)}{e}\Big)^{\alpha_j(i)/m(i)|\gamma|}$$

for $j=1,\dots,n$ and for i sufficiently large. Similar estimates we can use for $(m(i)c_j)!$, $(m(i)|\gamma|)!$ and $|\alpha(i)|!$. Thus, for i sufficiently large, we have (to simplify the expressions we write m, k and α instead of $m(i)$, $k(i)$ and $\alpha(i)$)

$$\begin{aligned}\left[\frac{\binom{m|\gamma|}{m\gamma}}{\binom{k}{\alpha}}\right]^{1/m|\gamma|}&=\Big[\frac{(m|\gamma|)!\alpha_1!\cdots\alpha_n!}{k!(mc_1)!\cdots(mc_n)!}\Big]^{1/m|\gamma|}\\
\ge&\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{n+1}\frac{m|\gamma|\cdot\alpha_1^{\alpha_1/m|\gamma|}\cdots\alpha_n^{\alpha_n/m|\gamma|}\cdot e^{k/m|\gamma|}\cdot e^{c_1/|\gamma|}\cdots e^{c_n/|\gamma|}}{e\cdot e^{\alpha_1/m|\gamma|}\cdots e^{\alpha_n/m|\gamma|}\cdot k^{k/m|\gamma|}\cdot(mc_1)^{c_1/|\gamma|}\cdots(mc_n)^{c_n/|\gamma|}}\\
=&\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{n+1}\Big(\frac{\alpha_1}{mc_1}\Big)^{c_1/|\gamma|}\cdots\Big(\frac{\alpha_n}{mc_n}\Big)^{c_n/|\gamma|}\alpha_1^{(\alpha_1-mc_1)/m|\gamma|}\cdots\alpha_n^{(\alpha_n-mc_n)/m|\gamma|}\cdot\frac{m|\gamma|}{k^{k/m|\gamma|}}\\
\ge&\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{n+1}\cdot\Big(\frac{\alpha_1}{k}\Big)^{(\alpha_1-mc_1)/m|\gamma|}\cdots\Big(\frac{\alpha_n}{k}\Big)^{(\alpha_n-mc_n)/m|\gamma|}\cdot\frac{m|\gamma|}{k}.\end{aligned}$$

Then

$$L_1\ge\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{n+1}(1-n\varepsilon)(t_1\cdots t_n)^{\frac{\varepsilon}{1-n\varepsilon}}.$$

Hence

$$r_p'^p(a)\ge\Big(\frac{1-\varepsilon}{1+\varepsilon}\Big)^{n+1}(1-n\varepsilon)(t_1\cdots t_n)^{\frac{\varepsilon}{1-n\varepsilon}}\cdot K^{-\frac{n\varepsilon p}{1-n\varepsilon}}\cdot\min\{r_p''^p(a),(r_p''^p(a))^{1/1-n\varepsilon}\}.$$

Since ε was an arbitrary positive number, we conclude that $r_p'(a)\ge r_p''(a)$.

We apply now the previous result to the case of n-tuples of operators.

Let $T=(T_1,\dots,T_n)$ be an n-tuple of bounded operators in a Banach space X. Denote

$$\|T\|_p=\sup_{\substack{x\in X\\ \|x\|=1}}\Big(\sum_{j=1}^n\|T_jx\|^p\Big)^{1/p}.$$

Equivalently, $\|T\|_p$ is the norm of the operator $\tilde T:X\to X_p^n$, where X_p^n is the direct sum of n copies of X endowed with the ℓ_p-norm, $\|x_1\oplus\cdots\oplus x_n\|=\big(\sum_{j=1}^n\|x_j\|^p\big)^{1/p}$ and $\tilde Tx=T_1x\oplus\cdots\oplus T_nx$ (for $p=\infty$ the definitions are changed in the obvious way).

Let $T=(T_1,\dots,T_n)\in B(X)^n$ and $S=(S_1,\dots,S_m)\in B(X)^m$. Denote by TS the mn-tuple $TS=(T_1S_1,\dots,T_1S_m,T_2S_1,\dots,T_2S_m,\dots,T_nS_1,\dots,T_nS_m)$. Further, let $T^2=TT$ and $T^{k+1}=T\cdot T^k$. With this notation we can state the spectral radius formula in the familiar way:

Theorem 4. Let $T=(T_1,\dots,T_n)$ be an n-tuple of mutually commuting operators in a Banach space X, let $1\le p\le\infty$. Then $r_p(T)=\lim_{k\to\infty}\|T^k\|_p^{1/k}$.

Proof. We have

$$\|T^k\|_p=\sup_{\|x\|=1}\left[\sum_{|\alpha|=k}\binom{k}{\alpha}\|T^\alpha x\|^p\right]^{1/p}$$

and

$$\begin{aligned}
r_p(T)&=\lim_{k\to\infty}\left[\sum_{|\alpha|=k}\binom{k}{\alpha}\|T^\alpha\|^p\right]^{1/kp}=\lim_{k\to\infty}\max_{|\alpha|=k}\left[\binom{k}{\alpha}\|T^\alpha\|^p\right]^{1/kp}\\
&=\lim_{k\to\infty}\max_{|\alpha|=k}\sup_{\|x\|=1}\left[\binom{k}{\alpha}\|T^\alpha x\|^p\right]^{1/kp}=\lim_{k\to\infty}\sup_{\|x\|=1}\max_{|\alpha|=k}\left[\binom{k}{\alpha}\|T^\alpha x\|^p\right]^{1/kp}\\
&=\lim_{k\to\infty}\sup_{\|x\|=1}\left[\sum_{|\alpha|=k}\binom{k}{\alpha}\|T^\alpha x\|^p\right]^{1/kp}=\lim_{k\to\infty}\|T^k\|_p^{1/k}.
\end{aligned}$$

Remark. For $p=2$ and Hilbert space operators the previous result was proved in [6], cf. also [3].

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