A continuous semicharacter

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Abstract: *We exhibit an example of a continuous proper semicharacter on a Banach algebra. This gives an answer to the problem posed by Z. Słodkowski and W. Żelazko.*

A semicharacter on a Banach algebra A is a complex-valued function f defined on A such that, for every commutative subalgebra $A_0 \subset A$, the restriction $f|A_0$ is a multiplicative linear functional (=character) on A_0 (we do not assume continuity of f).

Multiplicative linear functionals play an important role in the theory of generalized spectra (see [3],[6],[2]) in commutative Banach algebras. As generalized spectra in non-commutative Banach algebras are defined only for commuting systems of elements, it is natural to replace multiplicative linear functionals in the non-commutative case by semicharacters.

However, usually it is rather difficult to find a proper semicharacter (i.e. a semicharacter which is not a character). Note that a linear semicharacter is clearly continuous and by [5] it is already multiplicative, so that it is a character. In [4] the problem was raised whether a continuous semicharacter is already a character.

The aim of this note is to give a negative answer to this question.

Theorem: *There exist a Banach algebra B and a continuous semicharacter $f : B \to \mathbb{C}$ which is not a multiplicative linear functional.*

Proof: Denote by $\mathbb{R}_+$ the set of all positive real numbers and by $D = \{z \in \mathbb{C}, |z| < 1\}$ the open unit disc in the complex plain. Let A be the disc algebra of all functions holomorphic in D and continuous in $\bar{D}$. For $a \in A$ denote $\|a\| = \max_{z \in \bar{D}} |a(z)|$. Set $B = A \times A$. We define the norm and the algebraic operations in B by

$$\begin{aligned}
&\|(a,b)\| = \|a\| + \|b\|,\\
&(a,b) + (a',b') = (a+a', b+b'),\\
&\alpha(a,b) = (\alpha a, b),\\
&(a,b)\cdot(a',b') = (aa', ab') \qquad (a,b,a',b' \in A, \quad \alpha \in \mathbb{C}).
\end{aligned}$$

In this way B becomes a Banach algebra.

Let $(a,b),(a',b') \in B$. Then $(a,b)\cdot(a',b') = (aa',ab')$ and $(a',b')\cdot(a,b) = (a'a, a'b)$ so that (a,b) and $(a',b') \in B$ commute if and only if $ab' = a'b$. Thus B has only few commutative subalgebras which are easy to describe.

For $n \in \mathbb{N}, \lambda = (\lambda_1, \dots, \lambda_n) \in D^n, r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ and $s > 0$ we denote

$$F_{\lambda,r,s} = \{z \in D, |z| \le 1-s, |z-\lambda_i| \ge r_i \quad (i = 1, \dots, n)\}.$$

Clearly $F_{\lambda,r,s}$ is a closed subset of D. Let $k > 0$ and $0 < s < 1/2$. Denote by $M_{k,s}$ the set of all pairs $(a,b) \in B$ for which there exist $n \in \mathbb{N}$,$\lambda = (\lambda_1, \dots, \lambda_n) \in D^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a(z) \neq 0 \quad \text{and} \quad \left|\frac{b(z)}{a(z)}\right| < k.$$

Clearly, if $\sum_{i=1}^n r_i < s < 1/2$ then $F_{\lambda,r,s}$ is a non-empty subset of D so that $(a,b) \in M_{k,s}$ implies $a \neq 0$. On the other hand, if $a \neq 0$ then $(a,0) \in M_{k,s}$ for every $k > 0$ and $0 < s < 1/2$. Indeed, a has only a finite number of zeros $\lambda_1, \ldots, \lambda_n$ in the disc $\{z \in \mathbb{C}, |z| \leq 1-s\}$ so that for any positive numbers $r_1, \ldots, r_n$ with $\sum_{i=1}^n r_i < s$ we have $z \in F_{\lambda,r,s} \Rightarrow a(z) \neq 0$.

Further, $M_{k,s} \subset M_{k',s'}$ if $k < k'$ and $s < s'$.

1) *If $k > 0$ and $0 < s < 1/2$ then $M_{k,s}$ is an open subset of B.*

Proof: Let $(a,b) \in M_{k,s}$. Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ satisfy $\sum_{i=1}^n r_i < s$ and $z \in F_{\lambda,r,s} \Rightarrow a(z) \neq 0$ and $\left|\frac{b(z)}{a(z)}\right| < k$. Denote by

$$\begin{aligned} k_0 &= \max_{z \in F_{\lambda,r,s}} \left|\frac{b(z)}{a(z)}\right| < k, \\ k_1 &= \max\{\|a\|, \|b\|\} \qquad \text{and} \\ k_2 &= \min_{z \in F_{\lambda,r,s}} |a(z)| > 0. \end{aligned}$$

Set $\delta = \min\{k_2/2, (k-k_0)k_2^2/2k_1\} > 0$. Let $(a',b') \in B, \|(a,b)-(a',b')\| < \delta$, i.e. $\|a-a'\| + \|b-b'\| < \delta$. Then, for $z \in F_{\lambda,r,s}$, we have

$$|a'(z)| \geq |a(z)| - \delta \geq k_2 - \frac{k_2}{2} = \frac{k_2}{2} > 0$$

and

$$\left|\frac{b'(z)}{a'(z)}\right| \leq \left|\frac{b(z)}{a(z)}\right| + \left|\frac{b'(z)}{a'(z)} - \frac{b(z)}{a(z)}\right| \leq k_0 + \left|\frac{a(z)(b'(z)-b(z)) + b(z)(a(z)-a'(z))}{a'(z)a(z)}\right| <$$

$$< k_0 + \frac{k_1\delta}{k_2(k_2-\delta)} \leq k_0 + \frac{2k_1\delta}{k_2^2} \leq k.$$

Thus $(a',b') \in M_{k,s}$ and $M_{k,s}$ is an open subset of B.

2) *Let $(a,b) \in M_{k,s}$ and let $(a',b') \in B$ satisfy $a' \neq 0$ and $a'b = b'a$. Then $(a',b') \in M_{k,s}$.*

Proof: Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$ satisfy $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a(z) \neq 0 \quad \text{and} \quad \left|\frac{b(z)}{a(z)}\right| < k.$$

The function a' has only a finite number of zeros $\lambda'_1, \ldots, \lambda'_m$ in the disc $\{z \in \mathbb{C}, |z| \leq 1-s\}$. Choose positive numbers $r'_1, \ldots, r'_m$ such that $\sum_{j=1}^m r'_j < s - \sum_{i=1}^n r_i$. Consider the set

$$F = \{z \in D, |z| \leq 1-s, |z-\lambda_i| \geq r_i \quad (i=1,\ldots,n), |z-\lambda'_j| \geq r'_j \quad (j=1,\ldots,m)\}.$$

Then $\sum_{i=1}^n r_i + \sum_{j=1}^m r'_j < s$ and

$$z \in F \quad \Rightarrow \quad a'(z) \neq 0 \quad \text{and} \quad \left|\frac{b'(z)}{a'(z)}\right| = \left|\frac{b(z)}{a(z)}\right| < k.$$

Hence $(a', b') \in M_{k,s}$.

3) *Let k, k', s, s' be positive numbers such that $k < k'$ and $s < s' < 1/2$. Then $\overline{M_{k,s}} \cap \{(a,b) \in B, a \neq 0\} \subset M_{k',s'}$.*

Proof: Let $(a,b) \in \overline{M_{k,s}}$ and $a \neq 0$. The function a has only a finite number of zeros $\lambda_1', \dots, \lambda_m'$ in the disc $\{z \in \mathbb{C}, |z| \leq 1 - s'\}$. Choose positive numbers $r_1', \dots, r_m'$ such that $\sum_{j=1}^n r_j' < s' - s$. Consider the set

$$F_{\lambda',r',s'} = \{z \in D, |z| \leq 1 - s', |z - \lambda_j'| \geq r_j' \quad (j = 1, \dots, m)\}.$$

Denote

$$k_1 = \max\{\|a\|, \|b\|\} \qquad \text{and}$$
$$k_2 = \min_{z \in F_{\lambda',r',s'}} |a(z)| > 0.$$

Let $\delta = \min\{k_2/2, (k'-k)k_2^2/2k_1\} > 0$. Then there exists $(a', b') \in M_{k,s}$ such that $\|(a',b') - (a,b)\| = \|a - a'\| + \|b - b'\| < \delta$. This means that there exist $n \in \mathbb{N}$, $\lambda = (\lambda_1, \dots, \lambda_n) \in D^n$ and $r = (r_1, \dots, r_n) \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n r_i < s$ and

$$z \in F_{\lambda,r,s} \quad \Rightarrow \quad a'(z) \neq 0 \quad \text{and} \quad \left|\frac{b'(z)}{a'(z)}\right| < k.$$

Then for $z \in F_{\lambda,r,s} \cap F_{\lambda',r',s'}$ we have $a(z) \neq 0$ and

$$\left|\frac{b(z)}{a(z)}\right| \leq \left|\frac{b'(z)}{a'(z)}\right| + \left|\frac{b(z)}{a(z)} - \frac{b'(z)}{a'(z)}\right| < k + \left|\frac{b(z)(a'(z) - a(z)) + a(z)((b(z) - b'(z))}{a(z)a'(z)}\right| <$$

$$< k + \frac{k_1\delta}{k_2(k_2 - \delta)} \leq k + \frac{2k_1\delta}{k_2^2} \leq k'.$$

Hence $(a,b) \in M_{k',s'}$.

Denote $B_0 = \{(a,b) \in B, a \neq 0\}$.

4) *There exists a non-constant continuous function $\varphi : B_0 \to \langle 0, 1/2 \rangle$ such that*

$$(a,b), (a',b') \in B_0, ab' = ba' \quad \Rightarrow \quad \varphi(a,b) = \varphi(a',b').$$

Proof: For $(a,b) \in B_0$ define

$$\varphi(a,b) = \begin{cases} \frac{1}{2} & \text{if } (a,b) \notin \cup_{0<s<1/2} M_{s,s} \\ \inf\{s, (a,b) \in M_{s,s}\} & \text{otherwise.} \end{cases}$$

Clearly, by 2), $\varphi(a,b) = \varphi(a',b')$ if $ab' = a'b$. The function φ is non-constant since $\varphi(1,0) = 0$ and $\varphi(1,1) = 1/2$. The proof of continuity of φ is standard. Let $s_0 \in (0, 1/2\rangle$. Then

$$\{(a,b) \in B_0, \varphi(a,b) < s_0\} = \bigcup_{s<s_0} M_{s,s},$$

which is an open subset of B_0. If $s_0 \in \langle 0, 1/2)$ then

$$\{(a,b) \in B_0, \varphi(a,b) \le s_0\} = \bigcap_{s>s_0} M_{s,s} = \bigcap_{s>s_0} \left(\overline{M_{s,s}} \cap B_0\right),$$

which is a closed subset of B_0. Thus φ is a continuous function.

Define a function $f : B \to \mathbb{C}$ by

$$f(a,b) = \begin{cases} 0 & \text{if } a = 0 \\ a(\varphi(a,b)) & \text{if } a \ne 0. \end{cases}$$

We show that f is a proper continuous semicharacter.

5) *Let $x = (a,b) \in B$ and $\alpha \in \mathbb{C}$. Then $f(\alpha x) = \alpha f(x)$.*

Proof: This is clear if $\alpha = 0$ or $a = 0$. If $a \ne 0$ and $\alpha \ne 0$, then $\varphi(x) = \varphi(\alpha x) = t_0$ so that $f(\alpha x) = f(\alpha a, \alpha b) = \alpha \cdot a(t_0) = \alpha f(x)$.

6) *Let $x = (a,b), x' = (a',b') \in B$ be commuting elements. Then $f(x+x') = f(x) + f(x')$ and $f(xx') = f(x) \cdot f(x')$.*

Proof: We have $ab' = a'b$. We distinguish several cases:

a) If $a = 0$ and $b = 0$, then $f(x) = 0 = f(xx')$ so that the statement is clear.
b) If $a = 0$ and $b \ne 0$, then $a' = 0$ so that $f(x) = f(x') = f(x+x') = f(xx') = 0$.
c) If $a' = 0$, then the statement can be proved analogously.
d) The remaining case is $a \ne 0, a' \ne 0$. Then

$$\varphi(a,b) = \varphi(a',b') = \varphi(aa', ab') = t_0,$$

so that

$$f(xx') = (aa')(t_0) = a(t_0)a'(t_0) = f(x) \cdot f(x').$$

Further either $a = -a'$ so that $b = -b'$ and $f(x+x') = f(x) + f(x') = 0$, or $a + a' \ne 0$ so that $\varphi(a+a', b+b') = t_0$ and

$$f(x+x') = (a+a')(t_0) = a(t_0) + a'(t_0) = f(x) + f(x').$$

Hence f is a semicharacter.

7) *f is a continuous semicharacter.*

Proof: Let $x = (0,b)$. Then $f(x) = 0$. If $x' = (a',b') \in B$ then either $a' = 0$ so that $f(x') = 0$, or $a' \ne 0$ so that $|f(x')| = |a'(\varphi(x'))| \le \|a'\|$. In both cases we have $|f(x') - f(x)| \le \|x' - x\|$, hence f is continuous at $x = (0,b)$.

Let $x = (a,b)$ where $a \ne 0$ and let $\epsilon > 0$. Find $\delta > 0$ such that $|t - \varphi(x)| < \delta \Rightarrow |a(t) - a(\varphi(x))| < \epsilon/2$. From the continuity of φ it is possible to find a positive number $\delta_1 < \epsilon/2$ such that

$$\|x' - x\| < \delta_1 \quad \Rightarrow \quad x' \in B_0 \quad \text{and} \quad |\varphi(x') - \varphi(x)| < \delta.$$

For $x' = (a', b') \in B, \|x' - x\| < \delta_1$ we have

$$|f(x') - f(x)| = |a'(\varphi(x')) - a(\varphi(x))| \leq |a'(\varphi(x')) - a(\varphi(x'))| + |a(\varphi(x')) - a(\varphi(x))| \leq$$

$$\leq \|a' - a\| + \epsilon/2 \leq \|x' - x\| + \epsilon/2 < \epsilon.$$

Hence f is a continuous semicharacter.

It remains to show that f is not a multiplicative linear functional.To this end consider $x = (1, 0)$ and $x' = (z, z)$. Then $x'x = (z, 0), \varphi(x) = 0, \varphi(x') = 1/2$ and $\varphi(x'x) = 0$ so that $f(x) = 1, f(x') = 1/2$ and $f(x'x) = 0 \neq f(x) \cdot f(x')$.

Remark 1: The above constructed algebra B has no unit element. If we consider its unital extension $B_1 = B \oplus \{\mathbb{C}e\}$ then $f : B \to \mathbb{C}$ can be extended to a proper continuous semicharacter $f_1 : B_1 \to \mathbb{C}$ by $f_1(x + \lambda e) = f(x) + \lambda \quad (x \in B, \lambda \in \mathbb{C})$.

Problem: *Suppose that f is a uniformly continuous semicharacter on a Banach algebra A, i.e., for some constant k we have $|f(x) - f(x')| \leq k \cdot \|x - x'\| \quad (x, x' \in A)$. Does it follow that f is a multiplicative linear functional?*

Remark 2: If f is a semicharacter on a Banach algebra A such that $z \to f(a+bz)$ is a holomorphic function for every $a, b \in A$, then f is already a multiplicative linear functional. Indeed, function $\varphi : z \to f(a + bz) - f(a) - z \cdot f(b)$ is holomorphic and $\varphi(0) = 0$ so that

$$\varphi_1 : z \to \frac{\varphi(z)}{z} = f\left(b + \frac{a}{z}\right) - \frac{f(a)}{z} - f(b) \quad (z \neq 0)$$

extends to an entire function and $\lim_{z\to\infty} \varphi_1(z) = 0$. Thus $\varphi_1(z) = 0$ for every $z \in \mathbb{C}$. In particular,

$$0 = \varphi_1(1) = f(a + b) - f(a) - f(b)$$

so that f is a linear functional, i.e. a semicharacter.

Remark 3: A notion analogous to semicharacters is that of a quasilinear functional on a Banach algebra A ($=$ a bounded function which is linear on each commutative subalgebra of A). This notion, which is motivated by quantum physics, has been studied intensively in the context of C^*-algebras, see [1].

References

[1] L. J. Bunce, J. D. Wright, The quasi–linearity problem for C^*-algebras, Pacific J. Math. (to appear).

[2] R. E. Curto, Applications of several complex variables to multiparameter spectral theory, Surveys of some recent results in operator theory, Res. Notes in Math., vol. 192, J. B. Conway and B. B. Morrel, London, 1982.

[3] Z. Słodkowski, W. Żelazko, On joint spectra of commuting families of operators, Studia Math. 50 (1979), 127–148.

[4] Z. Słodkowski, W. Żelazko, A note on semicharacters, Banach center publications, Vol. 8, Spectral Theory, Polish Scientific Publishers, Warszawa 1982, 397–402.

[5] W. Żelazko, A characterization of multiplicative linear functionals in complex Banach algebras, Studia Math. 30 (1968), 83–85.

[6] W. Żelazko, An axiomatic approach to joint spectra I, Studia Math. 64 (1979), 249–261.

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