QUASISIMILARITY OF POWER BOUNDED OPERATORS AND BLUM-HANSON PROPERTY

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Abstract. We construct a power bounded operator on a Hilbert space which is not quasisimilar to a contraction. To this aim, we solve an open problem from operator ergodic theory showing that there are power bounded Hilbert space operators without the Blum-Hanson property. We also find an example of a power bounded operator quasisimilar to a unitary operator which is not similar to a contraction, thus answering negatively open questions raised by Kérchy and Cassier. On the positive side, we prove that contractions on ℓ_p spaces $(1 \leq p < \infty)$ possess the Blum-Hanson property.

1. Introduction

One of the most challenging problems in operator theory was to decide whether every polynomially bounded operator on a Hilbert space is similar to a contraction. The problem was posed by Halmos in 1970 as a refined version of a B. Sz.-Nagy question (1959) on similarity to contractions of power bounded operators on Hilbert spaces. While the B. Sz.-Nagy problem was answered in the negative quite soon by S. R. Foguel [F], see also [H], the Halmos problem remained open for a long time. It was solved in the negative by G. Pisier in 1996 [P], following substantial contributions in [Pe], [Bo] and [AP], see also [DP].

The present paper deals with the following refined version of the Sz.-Nagy problem.

Quasisimilarity Problem *Is every power bounded operator on a Hilbert space quasisimilar to a contraction?*

The problem was implicitly considered in a number of papers.

Date: October 16, 2006.

1991 *Mathematics Subject Classification.* Primary 47A20; Secondary 47A35.

Key words and phrases. power bounded operator, contraction, quasisimilarity, Blum-Hanson property.

This work was completed with the support of Marie Curie "Transfer of Knowledge" program, project "TODEQ". The first author was also partially supported by grant No. 201/06/0128 of GA ČR and by the Institutional Research Plan AV 0Z 10190503. The second author was partially supported by KBN grant 1P03A 03826. The second author would also like to thank M.Lin and S. Bermudo for stimulating discussions.

Note that by [BP] each polynomially bounded operator T on H is "almost" quasisimilar to a contraction in the following sense: there exist Hilbert spaces H_1, H_2, contractions $T_1 \in B(H_1)$, $T_2 \in B(H_2)$ and injective linear operators $X_1 : H_1 \to H$, $X_2 : H \to H_2$ with dense ranges such that $X_1T_1 = TX_1$ and $T_2X_2 = X_2T$. It is still unknown whether one can choose $T_1 = T_2$ so that to make T quasisimilar to a contraction.

In the present paper we show that the quasisimilarity problem has a negative solution. Moreover, we construct a power bounded operator T on a Hilbert space H such that $T^n x \not\to 0$ for every nonzero vector $x \in H$ and such that even no contraction is a quasiaffine transform of T. To this aim we solve another open problem from operator ergodic theory concerning also power bounded operators.

The well-known mean ergodic theorem asserts that if T is a power bounded operator on a reflexive Banach space X, then $M_n(T) := \frac{1}{N}\sum_{n=1}^N T^n$ converge in the strong operator topology. From the point of view of ergodic theory, it is natural to ask which property of T would guarantee the convergence not only of the conventional Cesáro averages $M_n(T)$, but also the convergence of Cesáro averages along any subsequence of (T^n).

The following theorem proved in [AS2], [JK] and [Li] answers the question in the case when T is a Hilbert space contraction.

Theorem 1.1. *Let T be a contraction acting on a Hilbert space H. Then the following two properties are equivalent:*

(i) *the sequence $(T^n x)$ converges weakly for every $x \in H$;*
(ii) *for each $x \in H$ and every increasing sequence (k_n) of positive integers, the limit*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} T^{k_n} x$$

exists in the norm topology.

The equivalence of the properties (i) and (ii) was first noted by Blum and Hanson [BH] for unitary operators induced by measure preserving transformations.

Let now T be a bounded linear operator (not necessarily a contraction) on a Banach space.

Definition 1.2. We say that T has the Blum-Hanson property if T satisfies the condition (ii) of Theorem 1.1.

Note that the Blum-Hanson property implies condition (i) (convergence of the sequence (T^n) in the weak operator topology), see e.g. [Kr, p. 253], which in turns implies that T is power bounded. It is also worth to note that the limits in (i) and (ii) are equal for each x. Furthermore, condition (i) is equivalent to the Blum-Hanson property for all subsequences (k_n) of positive lower density, see [J2].

The Blum-Hanson property was thoroughly studied in the 60'th and 70's in relation with mixing in ergodic theory. Apart from the class of Hilbert space contractions, the equivalence of (i) and (ii) in Theorem 1.1 was proved

a) for contractions on L^1-spaces [AS2],
b) for positive contractions on L^p-spaces, $1 < p < \infty$ [AS1], see also [Bel],
c) for a class of positive power bounded operators on L^1 [M],
d) for some power bounded operators on Hilbert spaces [K].

The proofs relied either on some dilation theorems or on certain specific inequalities in L^p spaces. The Blum-Hanson property for sequences of elements in Hilbert spaces was treated in [BB], see also [Z].

On the other hand, it was shown in [AHR] that Theorem 1.1 does not hold for a certain positive contraction on a space $C(K)$, where K is a compact Hausdorff space. As far as we know, this was the only known example of an operator such that the sequence (T^n) is converging in the weak operator topology but T has not the Blum-Hanson property.

The problem whether Theorem 1.1 holds for all power bounded operators on Hilbert spaces was left open.

Ergodic Problem *Does every Hilbert space operator T such that the sequence (T^n) is convergent in the weak operator topology possess the Blum-Hanson property?*

We give a negative answer to this problem. On the other hand, we prove that contractions on ℓ_p spaces, $1 \leq p < \infty$, do possess the Blum-Hanson property. On ℓ_2 the result is known but our proof seems to be new.

Our approach to the Quasimilarity Problem is to link it to the Ergodic Problem stated above. Note that the Blum-Hanson property is preserved under quasisimilarity, see Lemma 3.3. Since Theorem 1.1 holds for contractions, a power bounded operator for which Theorem 1.1 is not true cannot be quasisimilar to a contraction. This fact is exploited in Section 3 to produce a negative answer to the Quasimilarity problem.

Another main result of the paper deals with similarity to a contraction of special classes of power bounded operators. Recall that, by classical result due to B. Sz.-Nagy and C. Foiąs, a power bounded operator T on a Hilbert space such that $T^n x \not\to 0$ and $T^{*n} x \not\to 0$ for every $x \in H$ is quasisimilar to a unitary operator [NF]. (It is also not difficult to show that the class of such power bounded operators is exactly the class of quasisimilarities of unitary operators.) Thus it is natural to ask whether such operators are, in fact, similar to contractions [Ke1, Question 1], [Ke2, Question 3]. This problem was studied intensively in the last years, see e.g. [C], [CF], [Ke2], [Ke1], [Ku]. Using technique developed in the present paper, we show that the answer to this question is also negative.

2. Blum-Hanson property

Let T be a power bounded operator on a Hilbert space H. We say that T is of class $C_{1,\cdot}$ if $\inf_n \|T^n x\| > 0$ for each nonzero $x \in H$. We say that T is of class $C_{\cdot,1}$ if T^* is of class $C_{1,\cdot}$. We say that T is of class $C_{1,1}$ if it is both of class $C_{1,\cdot}$ and $C_{\cdot,1}$.

Example 2.1. There exists a bounded linear operator T of class $C_{1\cdot}$ acting on a Hilbert space H, $x \in H$ and an increasing sequence (k_n) of positive integers such that $T^n \to 0$ in the weak operator topology and $\frac{1}{N}\sum_{n=1}^N T^{k_n}x$ does not converge as $N \to \infty$.

Construction. Let H be the Hilbert space with an orthonormal basis formed by the vectors $e_i \quad (i \ge 0)$ and $f_{i,j} \quad (i \ge 1, j \in \mathbb{Z})$.

Define function $r : \mathbb{N} \to \mathbb{N}$ by $r(k) = [\log_2 k] + 1$, where $[\cdot]$ denotes the integer part, i.e., $r(k) = s$ whenever $2^{s-1} \le k < 2^s \quad (k \ge 1, s \ge 1)$.

Define $T \in B(H)$ by

$$
\begin{aligned}
Tf_{i,j} &= f_{i,j-1} \qquad (i \ge 1, j \ne 0),\\
Tf_{i,0} &= 4^{-i} f_{i,-1} \qquad (i \ge 1),\\
Te_j &= e_{j+1} \qquad (j \notin \{3^k : k = 1, 2, \dots\}),\\
Te_{3^k} &= e_{3^k+1} + f_{r(k),3^k} \qquad (k = 1, 2, \dots).
\end{aligned}
$$

Let $H_0 = \bigvee\{e_j : j \ge 0\}$. For $i = 1, 2, \dots$ let $H_i = \bigvee\{f_{i,j} : j \in \mathbb{Z}\}$. Then $H = \bigoplus_{i=0}^\infty H_i$. In this decomposition T can be written in the matrix form as

$$
T = \begin{pmatrix} S_0 & 0 & 0 & \dots \\ Q_1 & S_1 & 0 & \dots \\ Q_2 & 0 & S_2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

where S_0 is the unilateral isometrical shift and $S_i \quad (i \ge 1)$ is a bilateral weighted shift. Note that all weights of $S_i \quad (i \ge 1)$ but one are equal to 1. Note also that S_0 is a "forward" shift and $S_i \quad (i \ge 1)$ are "backward" shifts.

We show first that T is power bounded. Fix $n \in \mathbb{N}$. We have

$$
T^n = \begin{pmatrix} S_0^n & 0 & 0 & \dots \\ Q_1' & S_1^n & 0 & \dots \\ Q_2' & 0 & S_2^n & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

Clearly the diagonal part of T^n,

$$
D = \begin{pmatrix} S_0^n & 0 & 0 & \dots \\ 0 & S_1^n & 0 & \dots \\ 0 & 0 & S_2^n & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}
$$

is a contraction.

Let $H_+ = \bigvee\{f_{i,j} : i \geq 1, j \geq 0\} \vee H_0$ and let P_+ be the orthogonal projection onto H_+.

Let $Q' : H_0 \to \bigoplus_{i=1}^\infty H_i$ be defined by $Q'e_j = \sum_{i=1}^\infty Q_i' e_j$. For $j \geq 0$ we have

$$\begin{aligned} Q'e_j &= \sum_{i=1}^\infty \sum_{0\leq a\leq n-1} S_i^{n-a-1} Q_i S_0^a e_j = \sum_{i=1}^\infty \sum_{k:j\leq 3^k<j+n} S_i^{n-3^k+j-1} Q_i S_0^{3^k-j} e_j \\ &= \sum_{i=1}^\infty \sum_{k:j\leq 3^k<j+n} S_i^{n-3^k+j-1} Q_i e_{3^k} = \sum_{k:j\leq 3^k<j+n} S_{r(k)}^{n-3^k+j-1} f_{r(k),3^k}. \end{aligned}$$

Clearly $P_+ S_{r(k)}^{n-3^k+j-1} f_{r(k),3^k} \neq 0$ if and only if $n - 3^k + j - 1 \leq 3^k$, i.e., if $n + j - 1 \leq 2\cdot 3^k$. Note that this happens for at most one k satisfying $j \leq 3^k < j + n$. Indeed, suppose on the contrary that there are $k < k'$ satisfying these conditions. Then $n + j - 1 \geq 3^{k'} > 2 \cdot 3^k$, a contradiction.

Moreover, if $j \neq j'$ then $P_+Q'e_j \perp P_+Q'e_{j'}$. Suppose on the contrary that there are $j \neq j'$ and $k, k' \in \mathbb{N}$ such that $n + j - 1 \leq 2 \cdot 3^k$, $j \leq 3^k < j + n$, $n + j' - 1 \leq 2 \cdot 3^{k'}$, $j' \leq 3^{k'} < j' + n$ and $2\cdot 3^k - n - j + 1 = 2 \cdot 3^{k'} - n - j' + 1$, i.e., $2\cdot 3^k - j = 2 \cdot 3^{k'} - j'$. Since $j \neq j'$ we have $k \neq k'$. Suppose without loss of generality that $k < k'$. Then $j' - j = 2\cdot 3^{k'} - 2\cdot 3^k > 3^{k'}$, a contradiction with the assumption that $j' \leq 3^{k'}$.

Hence P_+Q' is a partial isometry and $\|P_+Q'\| \leq 1$.

It remains to estimate $\|(I - P_+)Q'\|$. We have

$$(I - P_+)Q'e_j = \sum_k S_{r(k)}^{n-3^k+j-1} f_{r(k),3^k} = \sum_k 4^{-r(k)} f_{r(k),2\cdot 3^k-n-j+1},$$

where the sums are taken over all k satisfying $j \leq 3^k < j + n$ and $2 \cdot 3^k - n - j + 1 < 0$. Thus $(I - P_+)Q' = \sum_{k=1}^\infty V_k$, where

$$V_k e_j = 4^{-r(k)} f_{r(k),2\cdot 3^k-n-j+1}$$

if $j \leq 3^k < j + n$ and $2 \cdot 3^k - n - j + 1 < 0$ and $V_k e_j = 0$ otherwise. Clearly every V_k is a scalar multiple of a partial isometry and $\|V_k\| \leq 4^{-r(k)}$.

Hence

$$\begin{aligned} \Big\|\sum_{k=1}^\infty V_k\Big\| &\leq \sum_{k=1}^\infty 4^{-r(k)} = \sum_{s=1}^\infty 4^{-s} \cdot \operatorname{card}\{k \in \mathbb{N} : r(k) = s\} \\ &= \sum_{s=1}^\infty 4^{-s} 2^{s-1} = \sum_{s=1}^\infty 2^{-s-1} = 1/2. \end{aligned}$$

Hence $\|T^n\| = \big\|D + P_+Q' + \sum_{k=1}^\infty V_k\big\| \leq 5/2$ for all n and T is power bounded.

We show now that $T^n e_0 \to 0$ weakly. Let $t \geq 1$. For n sufficiently large ($n > 2 \cdot 3^{2^{t-1}} + t + 1$) we have

$$T^n e_0 \perp \bigvee\{f_{i,j} : 1 \leq i \leq t, j \geq -t\} \vee \bigvee\{e_0, \ldots, e_{n-1}\}.$$

Since T is power bounded, this implies that $T^n e_0 \to 0$ weakly.

Let $M = \{x \in H : T^n x \to 0 \text{ weakly}\}$. Since T is power bounded, M is a closed T-invariant subspace. Clearly $f_{i,j} \in M$ for all $i \geq 1, j \in \mathbb{Z}$. Hence $M \supset \bigoplus_{i=1}^\infty H_i$. Also, we have $e_0 \in M$. By induction we show that $e_j \in M$ for each j. Indeed, if $e_j \in M$, then $Te_j \in M$ and also $P_0 T e_j \in M$, where P_0 is the orthogonal projection onto $H_0 = \left(\bigoplus_{i=1}^\infty H_i\right)^\perp$. Since $P_0 T e_j = e_{j+1}$, we have $e_{j+1} \in M$. Thus $H_0 \subset M$. Hence $M = H$ and $T^n \to 0$ in the weak operator topology.

We show that T is of class $C_{1\cdot}$. Let $x \in H$ be a nonzero vector. Write $x = \bigoplus_{i=0}^\infty x_i$ where $x_i \in H_i$. If $x_0 \neq 0$ then $\inf_n \|T^n x\| \geq \inf_n \|S_0^n x_n\| = \|x_0\| > 0$.

Suppose that $x_0 = 0$. Then there exists $i \geq 1$ with $x_i \neq 0$. We have

$$\inf_n \|T^n x\| \geq \inf_n \|S_i^n x_i\| \geq 4^{-i} \|x_i\| > 0.$$

Hence T is of class $C_{1\cdot}$.

Let $k_n = 2 \cdot 3^n + 1 \quad (n = 1, 2, \dots)$. Then

$$T^{k_n} e_0 = e_{k_n} + f_{r(n),0} + \sum_{j=1}^{n-1} 4^{-r(j)} f_{r(j), 2\cdot 3^j - 2\cdot 3^k}.$$

Thus

$$\frac{1}{2^s - 1} \Big\| \sum_{n=1}^{2^s-1} T^{k_n} e_0 \Big\| \geq \frac{1}{2^s - 1} \Big| \sum_{n=1}^{2^s-1} \langle T^{k_n} e_0, f_{s,0} \rangle \Big| = \frac{2^{s-1}}{2^s - 1} \geq \frac{1}{2}$$

for each $s \in \mathbb{N}$. Hence the sequence $\frac{1}{N} \Big\| \sum_{n=1}^N T^{k_n} e_0 \Big\|$ does not converge to 0 as $N \to \infty$. Since the $\frac{1}{N} \sum_{n=1}^N T^{k_n} e_0 \to 0$ weakly, the sequence $\frac{1}{N} \sum_{n=1}^N T^{k_n} e_0$ does not converge in the norm topology.

Remark 2.2. The construction becomes simpler if we do not require the property $C_{1\cdot}$. Indeed, then it is sufficient to consider the operator $P_+ T | H_+$ acting in the Hilbert space H_+. The proof of the power boundedness of this operator becomes simpler.

Remark 2.3. Let $T \in B(H)$ be the operator constructed in Example 2.1. We can introduce on H a new norm $|||\cdot|||$ by $|||x||| = \sup_n \|T^n x\| \quad (x \in H)$. Then $|||\cdot|||$ is equivalent to the original norm, and so the space $(H, |||\cdot|||)$ is reflexive (even superreflexive). Furthermore, T becomes a contraction on this space which does not satisfy Theorem 1.1.

Remark 2.4. Example 2.1 can be also used to produce a *positive* contraction on the space $C(K)$, where K is a compact Hausdorff space, not having the Blum-Hanson property. This provides an alternative construction to that in [AHR].

As in the previous remark we can assume that T is a contraction on a reflexive Banach space X such that $T^n \to 0$ in the weak operator topology (for short $T^n \to 0 \quad (WOT)$) and such that T has not the Blum-Hanson property. Note that in this case, $T^{*n} \to 0 \quad (WOT)$. Let B^* be the unit ball in X^*. Then B^* with the w^* topology is a compact Hausdorff space. Define a linear operator U on the Banach space $C(B^*)$ by

$$(Uf)(x^*) = f(T^*x^*) \qquad (f \in C(B^*), x^* \in B^*).$$

Observe that U is a positive contraction on $C(B^*)$. Let

$$X_0 := \{f \in C(B^*) : (U^n f)(x^*) \to 0 \text{ for all } x^* \in B^*\}.$$

Clearly X_0 is a closed subspace of $C(B^*)$. For all $x \in X$ the functions $f_x(x^*) := \langle x, x^* \rangle$ belong to X_0 and separate points in B. Moreover, if $f, g \in X_0$ then $fg \in X_0$. Indeed, we have

$$(U^n(fg))(x^*) = (fg)(T^{*n}x^*) = f(T^{*n}x^*) \cdot g(T^{*n}x^*),$$

where $g(T^{*n}x^*) \to 0$ and $\sup_n |f(T^{*n}x^*)| \leq \|f\|_{C(B^*)} < \infty$. Hence $fg \in X_0$. Furthermore, if $f \in X_0$, then its complex conjugate $\bar{f}$ also belongs to X_0. Thus, X_0 is a closed selfadjoint algebra and by the Stone-Weierstrass theorem, $X_0 \oplus \{\text{constants}\} = C(B^*)$. By the Lebesgue bounded convergence theorem, U^n converges in the weak operator topology on $C(B^*)$.

On the other hand, there exist (k_n) and $x \in X$ such that the limit $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{k_n} x$ does not exist. Since

$$\begin{aligned} \frac{1}{N}\Big\|\sum_{n=1}^{N} T^{k_n}x\Big\| &= \sup_{x^* \in B^*} \left| x^* \left(\frac{1}{N} \sum_{n=1}^{N} T^{k_n} x \right) \right| \\ &= \left\| \frac{1}{N} \sum_{n=1}^{N} U^{k_n} f_x \right\|_{C(B^*)} \end{aligned}$$

and $U^n f_x \to 0$ weakly we conclude that $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^{k_n} f_x$ does not exist.

On the other hand, we prove that contractions on ℓ_p spaces have the Blum-Hanson property.

Theorem 2.5. *Let $1 \leq p < \infty$, $T : \ell_p \to \ell_p$ a contraction and $x \in \ell_p$. Suppose that the sequence $(T^n x)$ is weakly convergent and let (n_i) be an increasing sequence. Then the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} T^{n_i} x$$

exists in the norm topology.

Proof. Let $\bar{x}$ be the weak limit of $T^n x$. Clearly $T\bar{x} = \bar{x}$. Replacing x by $x - \bar{x}$ if necessary, we may assume without loss of generality that $T^n x \to 0$ weakly.

The statement is clear for $p = 1$ since the weak convergence in ℓ_1 implies the convergence in the norm.

Let $1 < p < \infty$. Let $e_1, e_2, \dots$ be the standard basis in ℓ_p. Denote by P_r the canonical projection onto the span of $e_1, \dots, e_r$.

Since T is a contraction, the limit $\lim_{n\to\infty} \|T^n x\|$ exists. Since the statement is clear if this limit is equal to 0, we may assume without loss of generality that $\lim_{n\to\infty} \|T^n x\| = 1$.

Let $\delta > 0$. Find a positive integer t such that $t^{\frac{1}{p}-1} < \delta/2$. Since $1+2^p s < 2^p(s+1)$ for all s, there exists $\varepsilon \in (0,1)$ such that

$$((1+\varepsilon)^p + 2^p s)^{1/p} < 2(s+1)^{1/p} - (s+1)\varepsilon$$

for $s = 1, \dots, t-1$.

Find k such that $\|T^k x\| < 1+\varepsilon$. Find r such that $\|(I-P_r)T^k x\| < \varepsilon$. Find d such that $\|P_r T^{k+j}\| < \varepsilon$ for all $j \geq d$.

We show that

$$\big\|T^{m_1}x + \cdots + T^{m_s}x\big\| \leq 2s^{1/p} \tag{1}$$

whenever $k \leq m_1 < m_2 < \cdots < m_s$, $s \leq t$ and $m_{i+1} - m_i \geq d$ for all i.

We prove (1) by induction on s. Clearly (1) is true for $s = 1$. Suppose that (1) is true for $s < t$ and that $m_1, \dots, m_{s+1}$ satisfy the required conditions. We have

$$\begin{aligned}
& \big\|T^{m_1}x + \cdots + T^{m_{s+1}}x\big\| \leq \big\|T^k x + T^{m_2-m_1+k}x + \cdots + T^{m_{s+1}-m_1+k}x\big\| \\
\leq \; & \big\|P_r T^k x + (I-P_r)(T^{m_2-m_1+k}x + \cdots + T^{m_{s+1}x-m_1+k}x)\big\| \\
& \quad + \|(I-P_r)T^k x\| + \big\|P_r(T^{m_2-m_1+k}x + \cdots + T^{m_{s+1}-m_1+k}x)\big\| \\
\leq \; & \Big(\|P_r T^k x\|^p + \big\|(I-P_r)(T^{m_2-m_1+k}x + \cdots + T^{m_{s+1}-m_1+k}x)\big\|^p\Big)^{1/p} \\
& \quad + (s+1)\varepsilon \\
\leq \; & ((1+\varepsilon)^p + 2^p s)^{1/p} + (s+1)\varepsilon < 2(s+1)^{1/p}.
\end{aligned}$$

This proves (1) for $s \leq t$.

Let (n_i) be an increasing sequence and let N be large enough. Write $N = k + mt + r$, where $1 \leq r \leq t$ and m is a positive integer, $m \geq d$. Then

$$\Big\|\sum_{i=1}^N T^{n_i}x\Big\| \leq \Big\|\sum_{i=1}^{k+r} T^{n_i}x\Big\| + \sum_{s=1}^m \Big\|\sum_{i=0}^{t-1} T^{n_{k+r+s+im}}x\Big\| \leq (k+r)\|x\| + m \cdot 2t^{1/p}.$$

Thus

$$\frac{1}{N}\Big\|\sum_{i=1}^N T^{n_i}x\Big\| \leq \frac{(k+r)\|x\|}{N} + \frac{2mt^{1/p}}{tm} = \frac{(k+r)\|x\|}{N} + 2t^{-1+1/p},$$

and

$$\limsup_{N\to\infty} \frac{1}{N}\Big\|\sum_{i=1}^N T^{n_i}x\Big\| \leq 2t^{-1+1/p} < \delta.$$

Since $\delta > 0$ was arbitrary, we have

$$\lim_{N\to\infty} \frac{1}{N}\Big\|\sum_{i=1}^N T^{n_i}x\Big\| = 0.$$

Hence T has the Blum-Hanson property. □

Problem 2.6. It is an interesting open problem whether Theorem 2.5 remains valid for contractions in L_p spaces $(1 < p < \infty)$, or more generally, in uniformly smooth spaces.

It was claimed in [Br] that Theorem 1.1 holds for invertible isometries on uniformly convex Banach spaces. However, the proof given there seems to be false.

3. Quasisimilarity

Definition 3.1. Let $T \in B(H)$ and $S \in B(K)$ be Hilbert space operators. We write $T \prec S$ if there exists an injective operator $A : H \to K$ with dense range such that $AT = SA$. In this case, T is called a quasiaffine transform of S. We say that T is quasisimilar to S if both $T \prec S$ and $S \prec T$.

The following two simple lemmas allow us to transfer the weak convergence and Blum-Hanson property via intertwinning relations.

Lemma 3.2. *Let H, K be Hilbert spaces, let $T \in B(H)$ and $S \in B(K)$ be power bounded operators. Suppose that $T \prec S$. Then*

(i) $T^n \to 0 \quad (WOT)$ *if and only if* $S^n \to 0 \quad (WOT)$;

(ii) $(T^n h)$ *is weakly convergent for each* $h \in H$ *if and only if* $(S^n k)$ *is weakly convergent for each* $k \in K$.

Proof. (i) Let $A : H \to K$ be an operator with dense range satisfying $AT = SA$.

We have $\langle T^n h, h' \rangle \to 0$ for all $h, h' \in H$. Thus for all $h \in H, k \in K$ we have

$$\langle S^n Ah, k \rangle = \langle AT^n h, k \rangle = \langle T^n h, A^* k \rangle \to 0$$

as $n \to \infty$. Hence $S^n x \to 0$ weakly for all $x \in AH$. Since S is power bounded and AH is dense in K, we have $S^n \to 0 \quad (WOT)$.

Conversely, suppose that $S^n \to 0 \quad (WOT)$. Then $S^{*n} \to 0 \quad (WOT)$ and $S^* \prec T^*$. Hence $T^{*n} \to 0 \quad (WOT)$ and so $T^n \to 0 \quad (WOT)$.

(ii) Let $h \in H$. Note first that the sequence $(T^n h)$ converges weakly if and only if $\langle T^n h, h' \rangle$ is convergent for each $h' \in H$. Indeed, suppose that this condition is satisfied and define $f(h') = \lim_{n\to\infty} \langle T^n h, h' \rangle$. Then f is a bounded antilinear functional, and so there is an $\bar{h} \in H$ such that $\langle \bar{h}, h' \rangle = f(h')$ for all $h' \in H$. Hence $T^n h \to \bar{h}$ weakly.

¿From this it follows easily, that $(T^n h)$ is weakly convergent for each $h \in H$ if and only if $(T^{*n} h)$ is weakly convergent for each $h \in H$. Furthermore, $\{h \in H : (T^n h) \text{ converges weakly}\}$ is a closed subspace of H.

Suppose now that $(T^n h)$ converges weakly for each $h \in H$, and let $A : H \to K$ be an injective operator with dense range such that $AT = SA$.

Let $h \in H$ and $T^n h \to \bar{h}$ weakly. Then $T\bar{h} = \bar{h}$ and $T^n(h - \bar{h}) \to 0$ weakly. Thus $H = \ker(I - T) + H_0$, where $H_0 = \{h \in H : T^n h \to 0 \text{ weakly}\}$. It is easy to see that $A\ker(I-T)\subset\ker(I-S)$ and

$$AH_0\subset\{k\in K: S^nk\to 0 \text{ weakly}\}.$$

Thus (S^nk) converges weakly for each $k\in AH$, and therefore for each $k\in K$.

Conversely, suppose that (S^nk) converges weakly for each $k\in K$. Then $S^*\prec T^*$ and $(S^{*n}k)$ converges weakly for each $k\in K$. By the previous case, $(T^{*n}h)$ converges weakly for each $h\in H$, and so (T^nh) converges weakly for each $h\in H$. □

Lemma 3.3. *Let H,K be Hilbert spaces, let $T\in B(H)$ and $S\in B(K)$ be power bounded operators. Suppose that $T\prec S$ and that T has the Blum-Hanson property. Then S has the Blum-Hanson property.*

Proof. Let $A:H\to K$ be an injective operator with dense range satisfying $AT=SA$.

Since T has the Blum-Hanson property, for each increasing subsequence of positive integers (n_j) and every $h\in H$ the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}T^{n_j}h$$

exists (in the norm topology). Thus

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}S^{n_j}Ah=\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}AT^{n_j}h$$

exists for each $h\in H$. Since AH is dense in K and the sequence $\frac{1}{N}\sum_{j=1}^{N}T^{n_j}$ is bounded, the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}S^{n_j}k$$

exists for all $k\in K$. Hence S has the Blum-Hanson property. □

Our main result is now a direct consequence of Example 2.1.

Theorem 3.4. *Let $T\in B(H)$ be the operator of class $C_{1,\cdot}$ which was constructed in Example 2.1. Then there is no Hilbert space contraction $C\in B(K)$ such that $C\prec T$.*

Proof. Suppose on the contrary that there are a Hilbert space contraction $C\in B(K)$ and an injective operator $A:K\to H$ with dense range satisfying $AC=TA$.

Since $T^n\to 0\quad(WOT)$ we have $C^n\to 0\quad(WOT)$ by Lemma 3.2. By Theorem 1.1, C has the Blum-Hanson property. Then Lemma 3.3 implies that T has the Blum-Hanson property, a contradiction. □

The operator without the Blum-Hanson property constructed in Example 2.1 was of class $C_{1,\cdot}$. It is interesting to note that a similar example of class $C_{\cdot,1}$ is not possible.

Corollary 3.5. *Let T be a power bounded operator of class $C_{\cdot,1}$ acting in a Hilbert space H. Then the following two properties are equivalent:*

(i) *the sequence $(T^n x)$ converges weakly for every $x \in H$;*
(ii) *for every $x \in H$ and an increasing sequence (k_n) of positive integers, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{k_n} x$$

exists in the norm topology.

Proof. Let $T \in B(H)$ be a power bounded operator of class $C_{\cdot,1}$. It is well known that there exist a Hilbert space K and an isometry $V \in B(K)$ such that $T^* \prec V$, see e.g. [Ke1, p. 174]. Then $V^* \prec T$.

Suppose that T^n is convergent in the weak operator topology. Then V^{*n} is convergent in the weak operator topology by Lemma 3.2, and since V^* is a contraction, it has the Blum-Hanson property. Consequently T has the Blum-Hanson property. □

Remark 3.6. Let $T \in B(H)$ be the operator of class $C_{1,\cdot}$ without the Blum-Hanson property constructed in Example 2.1. Then T^* is of class $C_{\cdot,1}$ and by the previous theorem, it has the Blum-Hanson property. Hence the Blum-Hanson property is not preserved by taking adjoints.

Corollary 3.7. *There exists a power bounded operator S such that*

(i) *there is no contraction C with $C \prec S$;*
(ii) *there is no contraction C' with $S \prec C'$.*

Proof. Let T be the operator constructed in Example 2.1. Let $S = T \oplus T^* \in B(H \oplus H)$. Suppose that there is a contraction $C' \in B(K)$ such that $S \prec C'$. Then there is an injective operator $A : H \oplus H \to K$ with $A(T \oplus T^*) = C'A$. Consider the restriction of A to the subspace $\{0\} \oplus H$. Thus there exists an injective operator $A_0 : H \to K$ such that $A_0 T^* = C' A_0$. Let $K_0 = \overline{A_0 H}$. Clearly K_0 is invariant for C'. Thus $T^* \prec C'_0$ where C'_0 is the restriction $C'|K_0$. Hence $C_0'^* \prec T$, a contradiction with Lemma 3.4.

If C is a contraction and $C \prec (T \oplus T^*)$ then $(T^* \oplus T) \prec C^*$, a contradiction with the previous case.

4. Similarity and final remarks

We prove that there are power bounded operators of class $C_{1,1}$ which are not similar to contractions, thus answering negatively Kerchy's question from [Ke1, Question 1] (see also [Ke2, Question 3]). It is instructive to recall from Introduction that each power bounded operator of class $C_{1,1}$ is *quasisimilar* to a contraction.

Example 4.1. There exists a power bounded operator of class $C_{1,1}$ which is not similar to a contraction.

Proof. Recall the operator constructed in [F], see also [H]. Let $e_i, f_i \quad (i \geq 0)$ be an orthonormal basis in a Hilbert space K. Define $T \in B(K)$ by $Tf_j = f_{j-1} \quad (j \geq 1)$, $Tf_0 = 0$, $Te_j = e_{j+1} \quad (j \neq 3^k)$, $Te_{3^k} = e_{3^k+1} + f_{3^k}$. It is known that T is power bounded but not polynomially bounded, see [L]. Thus there exists a sequence of polynomials p_n such that $\|p_n\| = 1$ for all n and $\|p_n(T)\| > n$.

Let $n \in \mathbb{N}$. There exists $x \in K$ such that $\|x\| = 1$ and $\|p_n(T)x_n\| > n$. Without loss of generality we may assume that x_n is a finite linear combination of the basis vectors e_j, f_j, i.e., there is an $N \in \mathbb{N}$ such that $\deg p_n \leq 3^N$ and $x_n \in \bigvee\{e_j, f_j : 0 \leq j \leq 3^N\}$.

Fix $n \in \mathbb{N}$ and consider x_n, p_n and N as above. We construct an operator V_n of class $C_{1,1}$ acting in a Hilbert space $H_n \supset K$ such that $\sup_k \|V_n^k\| \leq 3$ and $\|p_n(V_n)x_n\| > n$. The required non-polynomially bounded operator of class $C_{1,1}$ will be then the direct sum $\bigoplus_{n=1}^\infty V_n$.

Let H_n be the Hilbert space with an orthonormal basis $e_j, f_j \quad (j \in \mathbb{Z})$. Thus $H_n \supset K$.

Define $V_n \in B(H_n)$ by

$$\begin{aligned}
V_n f_j &= f_{j-1} \qquad (j \neq 0); \\
V_n f_0 &= \frac{1}{N} f_{-1}; \\
V_n e_j &= e_{j+1} \qquad (j \neq -1, j \neq 3^k \text{ with } 1 \leq k \leq N); \\
V_n e_{-1} &= \frac{1}{N} e_0; \\
V_n e_{3^k} &= e_{3^k+1} + f_{3^k} \qquad (k = 1, \dots, N).
\end{aligned}$$

Note that we have $x_n \in K \subset H_n$ and $p_n(T)x_n = P_K p(V_n)x_n$, where P_K is the orthogonal projection onto K. Thus $\|p_n(V_n)\| \geq \|P_n(V_n)x_n\| > n$.

It is easy to see that $\inf_k \|V_n^k u\| > 0$ for each nonzero vector $u \in H_n$. Note also that V_n^* is unitarily equivalent to V_n (the unitary equivalence is given by the operator interchanging e_j and f_j). So V_n is of class $C_{1,1}$.

It remains to show the power-boundedness of V_n. The argument is similar to the argument in Example 2.1.

Let $E = \bigvee\{e_j : j \in \mathbb{Z}\}$, $E_+ = \bigvee\{e_j : j \geq 0\}$, $F = \bigvee\{f_j : j \in \mathbb{Z}\}$ and $F_+ = \bigvee\{f_j : j \geq 0\}$. In the decomposition $H_n = E \oplus F$ we have

$$V_n = \begin{pmatrix} S_E & 0 \\ Q & S_F \end{pmatrix}$$

where S_E, S_F are weighted bilateral shifts.

Fix $m \in \mathbb{N}$. Then

$$V_n^m = \begin{pmatrix} S_E^m & 0 \\ Q' & S_F^m \end{pmatrix}$$

where the diagonal part

$$D = \begin{pmatrix} S_E^m & 0 \\ 0 & S_F^m \end{pmatrix}$$

is a contraction.

For $j \in \mathbb{Z}$ we have

$$\begin{aligned} Q'e_j &= \sum_{0\le a<m} S_F^{m-a-1} Q S_E^a e_j = \sum_{\substack{1\le k\le N\\ j\le 3^k<j+m}} S_F^{m-3^k+j-1} Q S_E^{3^k-j} e_j \\ &= \sum_{\substack{1\le k\le N\\ j\le 3^k<j+m}} S_F^{m-3^k+j-1} f_{3^k}. \end{aligned}$$

Thus $Q' = \sum_{k=1}^N W_k$ where $W_k e_j = S_F^{m-3^k+j-1} f_{3^k}$ if $j \le 3^k < j+m$ and $W_k e_j = 0$ otherwise.

Note that $W_k e_j$ is a multiple of f_s, where $s = 2\cdot 3^k - m - j + 1$. We have

$$\begin{aligned} W_k e_j &= \frac{1}{N^2} f_s \qquad (j<0, s<0),\\ W_k e_j &= \frac{1}{N} f_s \qquad (j<0, s\ge 0),\\ W_k e_j &= \frac{1}{N} f_s \qquad (j\ge 0, s<0),\\ W_k e_j &= f_s \qquad (j\ge 0, s\ge 0), \end{aligned}$$

Thus $\|W_k - P_{F_+} W_k P_{E_+}\| \le \frac{1}{N}$ and

$$\|Q' - P_{F_+} Q' P_{E_+}\| = \Big\|\sum_{k=1}^N (W_k - P_{F_+} W_k P_{E_+})\Big\| \le 1.$$

It remains to estimate $\|P_{F_+} Q' P_{E_+}\|$. However, as in [F] or in the argument in Example 2.1, one can show that $P_{F_+} Q' P_{E_+}$ is a partial isometry, and so $\|P_{F_+} Q' P_{E_+}\| \le 1$.

Thus

$$\|V_n^m\| = \big\|D + P_{F_+} Q' P_{E_+} + (Q' - P_{F_+} Q' P_{E_+})\big\| \le 3.$$

Consider now the operator $V = \bigoplus_{n=1}^\infty V_n$ acting in the Hilbert space $H = \bigoplus_{n=1}^\infty H_n$. Clearly $\|V^m\| = \sup_n \|V_n^m\| \le 3$ for every $m \in \mathbb{N}$, and so V is power bounded. Clearly V is of class $C_{1,1}$ since it is a direct sum of $C_{1,1}$ operators. Finally, for every n we have $\|p_n(V)\| \ge \|p_n(V_n)\| > n$, and so V is not polynomially bounded.

Consequently, V is not similar to a contraction. □

Using a similar idea, it was shown in [ER] that there is a power bounded operator T on a Hilbert space H such that

$$\lim_{n\to\infty} \|T^n x\| = \lim_{n\to\infty} \|T^{*n} x\| = 0$$

for every $x \in H$, and T is not similar to a contraction.

Note finally that there also exists a power bounded operator S on a Hilbert space K such that

$$\inf_{n\ge 0} \|S^n x\| > 0 \qquad \text{and} \qquad \lim_{n\to\infty} \|S^{*n} x\| = 0$$

for every $x \in K \setminus \{0\}$, and S is not similar to a contraction. Indeed, let $T \in B(H)$ be the operator constructed in Example 2.1. Consider the weighted forward shift S defined by

$$S(x_0, x_1 \dots) = (0, Tx_0, Tx_1, \dots)$$

for all $(x_0, x_1, \dots) \in K := \ell_2(\mathbb{N}, H)$. As T is of class $C_{1,\cdot}$, the operator S is of the same class. Moreover, the power boundedness of T implies that $\lim_{n\to\infty} \|S^{*n}x\| = 0$. By [Ku, Proposition 8.9], since T is not similar to a contraction, S is not similar to a contraction, either.

Thus, in general, asymptotic properties of T^n in the strong operator topology are too weak to imply similarity of T to a contraction.

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