Stability of index for semi-Fredholm chains

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Abstract. We extend the recent stability results of Ambrozie for Fredholm essential complexes to the semi-Fredholm case.

Let X, Y be Banach spaces. By an operator we always mean a bounded linear operator. The set of all operators from X to Y will be denoted by $\mathcal{L}(X, Y)$. Denote by $N(T)$ and $R(T)$ the kernel and range of an operator $T \in \mathcal{L}(X, Y)$.

Recall that an operator $T : X \to Y$ is called semi-Fredholm if it has closed range and at least one of the defect numbers $\alpha(T) = \dim N(T)$, $\beta(T) = \operatorname{codim} R(T)$ is finite. If both of them are finite then T is called Fredholm.

The index of a semi-Fredholm operator is defined by $\operatorname{ind}(T) = \alpha(T) - \beta(T)$.

We list the most important classical stability results for semi-Fredholm operators:

Let $T : X \to Y$ be a semi-Fredholm operator. Then

(1) There exists $\varepsilon > 0$ such that $\operatorname{ind} T' = \operatorname{ind} T$ for every (semi-Fredholm) operator $T' \in \mathcal{L}(X, Y)$ with $\|T' - T\| < \varepsilon$.

(2) There exists $\varepsilon > 0$ such that $\alpha(T') \le \alpha(T)$ and $\beta(T') \le \beta(T)$ for every (semi-Fredholm) operator $T' \in \mathcal{L}(X, Y)$ with $\|T' - T\| < \varepsilon$.

(3) $\operatorname{ind}(T') = \operatorname{ind}(T)$ for every (semi-Fredholm) operator $T' \in \mathcal{L}(X, Y)$ such that $T - T'$ is compact.

(the condition that T' is semi-Fredholm is satisfied automatically for operators close enough to T; this will not be the case in more general situations).

These results were generalized for Banach space complexes. By a complex it is meant an object of the following type:

$$\mathcal{K}: \qquad 0 \longrightarrow X_0 \xrightarrow{\delta_0} X_1 \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_{n-2}} X_{n-1} \xrightarrow{\delta_{n-1}} X_n \longrightarrow 0$$

where X_i are Banach spaces and δ_i operators such that $\delta_{i+i}\delta_i = 0$ for every i.

The complex $\mathcal{K}$ is semi-Fredholm if the operators δ_i have closed ranges and the index of $\mathcal{K}$,

$$\operatorname{ind}(\mathcal{K}) = \sum_{i=0}^{n} (-1)^i \alpha_i(\mathcal{K}), \quad \text{where} \quad \alpha_i(\mathcal{K}) = \dim(N(\delta_i)/R(\delta_{i-1}))$$

is well-defined.

It was shown in [1], [14] that the index and the defect numbers α_i of semi-Fredholm complexes exhibit properties (1) and (2). Property (3) proved to be surprisingly difficult. Some partial results were obtained in [11] and for Fredholm complexes (or better to say for Fredholm essential complexes) it was proved recently by Ambrozie [2], [3].

The aim of this paper is to extend the above mentioned results to semi-Fredholm chains (for the definition see below).

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We are going to use frequently the following elementary isomorphism result.

Lemma 1. Let U, V be subspaces of a Banach space X. Then

$$\dim(U+V)/V = \dim U/(U\cap V).$$

Proof. The required isomorphism $U/(U\cap V) \to (U+V)/V$ is induced by the natural embedding $U \to U+V$.

If U and V are subspaces of a Banach space X then we write for short $U \overset{e}{\subset} V$ (U is essentially contained in V) if $\dim U/(U\cap V) < \infty$. If $U \overset{e}{\subset} V$ and $V \overset{e}{\subset} U$ then we write $U \overset{e}{=} V$.

Let X be a Banach space. For closed subspaces M_1, M_2 of X denote

$$\delta(M_1, M_2) = \sup_{\substack{m\in M_1\\ \|m\|\le 1}} \operatorname{dist}\{m, M_2\}$$

and the gap between M_1 and M_2 by

$$\hat{\delta}(M_1, M_2) = \max\bigl\{\delta(M_1, M_2), \delta(M_2, M_1)\bigr\},$$

see [9]. Clearly $\delta(M_1, M_2) = 0$ if and only if $M_1 \subset M_2$.

For convenience we recall the following result of Fainshtein [7]:

Theorem 2. Let R, R_1, N, N_1 be closed subspaces of a Banach space X and let $R \subset N$.
(a) If $\delta(R, R_1) < 1/3$ and $\delta(N_1, N) < 1/3$ then

$$\dim N_1/(R_1\cap N_1) \le \dim N/R + \dim R_1/(R_1\cap N_1).$$

(b) If $\hat{\delta}(R, R_1) < 1/9$ and $\hat{\delta}(N_1, N) < 1/9$ then

$$\dim N_1/(R_1\cap N_1) = \dim N/R + \dim R_1/(R_1\cap N_1).$$

We start with the following generalization of the previous result:

Theorem 3. Let R, N be closed subspaces of a Banach space X, let $R \overset{e}{\subset} N$. Then there exists $\varepsilon > 0$ such that, for all closed subspaces R_1 and N_1 of X with $\delta(R, R_1) < \varepsilon$ and $\delta(N_1, N) < \varepsilon$, we have

$$\dim R/(R\cap N) + \dim N_1/(R_1\cap N_1) \le \dim R_1/(R_1\cap N_1) + \dim N/(R\cap N).$$

Proof. For $R \subset N$ this is the first statement of the previous theorem. We reduce the general situation to this case.

Choose a finite dimensional subspace $F \subset R$ such that $(R \cap N) \oplus F = R$. Let $\dim F = k < \infty$ and let $f_1, \dots, f_k$ be a basis in F with $\|f_1\| = \cdots = \|f_k\| = 1$. Clearly $F \cap N = \{0\}$.

For $f = \sum_{i=1}^k \alpha_i f_i \in F \quad (\alpha_i \in \mathbb{C})$ consider three norms: $\|f\|$, $\operatorname{dist}\{f, N\}$ and $\sum_{i=1}^k |\alpha_i|$. Since these three norms are equivalent, there exists $c > 0$ such that

$$c \cdot \sum_{i=1}^k |\alpha_i| \le \operatorname{dist}\Big\{\sum_{i=1}^k \alpha_i f_i, N\Big\} \le \Big\|\sum_{i=1}^k \alpha_i f_i\Big\| \le \sum_{i=1}^k |\alpha_i|$$

for all $\alpha_1, \dots, \alpha_k \in \mathbb{C}$. Clearly $c \le 1$.

Set $\varepsilon = \frac{c}{20}$. Let R_1 and N_1 be closed subspaces of X such that $\delta(R, R_1) < \varepsilon$ and $\delta(N_1, N) < \varepsilon$.

For $i = 1, \dots, k$ find elements $g_i \in R_1$ such that $\|f_i - g_i\| < \varepsilon$. Then $\|g_i\| < 1 + \varepsilon \quad (i = 1, \dots, k)$. Denote by G the subspace of R_1 generated by $g_1, \dots, g_k$.

We prove that $\dim G = k$. Indeed, if $\sum_{i=1}^k \alpha_i g_i = 0$ for some $\alpha_i \in \mathbb{C}$ then

$$0 = \Big\|\sum_{i=1}^k \alpha_i g_i\Big\| \ge \Big\|\sum_{i=1}^k \alpha_i f_i\Big\| - \Big\|\sum_{i=1}^k \alpha_i (g_i - f_i)\Big\| \ge c \sum_{i=1}^k |\alpha_i| - \varepsilon \sum_{i=1}^k |\alpha_i| = \frac{19c}{20} \sum_{i=1}^k |\alpha_i|$$

so that $\alpha_1 = \cdots = \alpha_k = 0$.

Further $G \cap N_1 = \{0\}$. Indeed, if $\sum_{i=1}^k \alpha_i g_i \in N_1$ for some $\alpha_i \in \mathbb{C}$ then

$$\sum_{i=1}^k |\alpha_i| \le c^{-1} \operatorname{dist}\Big\{\sum_{i=1}^k \alpha_i f_i, N\Big\} \le c^{-1}\Big[\sum_{i=1}^k \alpha_i \|f_i - g_i\| + \operatorname{dist}\Big\{\sum_{i=1}^k \alpha_i g_i, N\Big\}\Big]$$

$$\le c^{-1}\varepsilon \sum_{i=1}^k |\alpha_i| + c^{-1}\Big\|\sum_{i=1}^k \alpha_i g_i\Big\| \cdot \delta(N_1, N) \le \Big(\frac{\varepsilon}{c} + \frac{\varepsilon(1+\varepsilon)}{c}\Big) \cdot \sum_{i=1}^k |\alpha_i| \le \frac{3}{20} \sum_{i=1}^k |\alpha_i|$$

so that $\alpha_i = 0 \quad (i = 1, \dots, k)$.

Denote $N' = N + F$ and $N_1' = N_1 + G$. Clearly $N' = N + R \supset R$.

We prove that $\delta(N_1', N') < 1/3$. Let $n_1 + \sum_{i=1}^k \alpha_i g_i \in N_1'$ where $n_1 \in N_1$, $\alpha_i \in \mathbb{C} \ \ (i = 1, \dots, k)$ and $\|n_1 + \sum_{i=1}^k \alpha_i g_i\| = 1$. Then $\|n_1\| \le 1 + (1+\varepsilon)\sum_{i=1}^k |\alpha_i|$. There exists $n \in N$ such that $\|n_1 - n\| \le \varepsilon \|n_1\| \le \varepsilon + \varepsilon(1+\varepsilon)\sum_{i=1}^k |\alpha_i|$. We have

$$c \sum_{i=1}^k |\alpha_i| \le \operatorname{dist}\Big\{\sum_{i=1}^k \alpha_i f_i, N\Big\} \le \Big\|\sum_{i=1}^k \alpha_i f_i + n\Big\|$$

$$\le \Big\|\sum_{i=1}^k \alpha_i (f_i - g_i)\Big\| + \Big\|\sum_{i=1}^k \alpha_i g_i + n_1\Big\| + \|n - n_1\|$$

$$\le \varepsilon \sum_{i=1}^k |\alpha_i| + 1 + \varepsilon + \varepsilon(1+\varepsilon)\sum_{i=1}^k |\alpha_i| \le 1 + \varepsilon + 3\varepsilon \sum_{i=1}^k |\alpha_i|.$$

Thus

$$\sum_{i=1}^k |\alpha_i| \le \frac{1+\varepsilon}{c - 3\varepsilon} \le \frac{4}{3c}$$

and

$$\operatorname{dist}\Big\{n_1+\sum_{i=1}^k \alpha_i g_i, N'\Big\} \le \|n_1-n\| + \Big\|\sum_{i=1}^k \alpha_i(f_i-g_i)\Big\|$$
$$\le \varepsilon + \varepsilon(1+\varepsilon)\sum_{i=1}^k |\alpha_i| + \varepsilon \sum_{i=1}^k |\alpha_i| < 1/3.$$

Hence $\delta(N_1', N') < 1/3$ and, by Theorem 2,

$$\dim N_1'/(R_1\cap N_1') \le \dim N'/R + \dim R_1/(R_1 \cap N_1'). \tag{1}$$

We have

$$\begin{aligned}\dim N_1/(R_1\cap N_1) &= \dim(N_1+R_1)/R_1\\ &= \dim(N_1'+R_1)/R_1 = \dim N_1'/(R_1\cap N_1')\end{aligned} \tag{2}$$

and

$$\dim N/(R\cap N) = \dim(N+R)/R = \dim N'/R. \tag{3}$$

Further

$$\dim R/(R\cap N) = k \tag{4}$$

and

$$\begin{aligned}\dim R_1/(R_1\cap N_1) = \dim(N_1+R_1)/N_1 = \dim(N_1+R_1)/(N_1+G)\\ + \dim(N_1+G)/N_1 = \dim(N_1'+R_1)/N_1' + k = \dim R_1/(R_1\cap N_1') + k.\end{aligned} \tag{5}$$

Thus, by (1)–(5), we have

$$\dim R/(R\cap N) + \dim N_1/(R_1\cap N_1) = k + \dim N_1'/(R_1\cap N_1')$$
$$\le k + \dim N'/R + \dim R_1/(R_1\cap N_1') = \dim R_1/(R_1 \cap N_1) + \dim N/(R\cap N).$$

Let X, Y be Banach spaces and let $T \in \mathcal{L}(X,Y)$. Denote by $\gamma(T)$ the Kato reduced minimum modulus [9],

$$\gamma(T) = \inf\big\{\|Tx\| : \operatorname{dist}\{x, N(T)\} = 1\big\}$$

(if $T=0$ then $\gamma(T)=\infty$). It is well-known that T has closed range if and only if $\gamma(T)>0$. Further, if $0<s<\gamma(T)$ and $y\in R(T)$ then there exists $x\in X$ with $Tx=y$ and $\|x\|\le s^{-1}\|y\|$.

The following lemma is well-known, cf. [7]. For convenience we include the proof.

Lemma 4. Let X, Y be Banach spaces and let $T, T_1 \in \mathcal{L}(X,Y)$ be operators with closed ranges. Then
(a) $\delta(N(T_1), N(T)) \le \gamma(T)^{-1}\|T-T_1\|$,
(b) $\delta(R(T), R(T_1)) \le \gamma(T)^{-1}\|T-T_1\|$.

Proof. Let $0<s<\gamma(T)$.

(a) Suppose $x \in N(T_1)$ and $\|x\|\le 1$. Then $Tx\in R(T)$ and $\|Tx\| = \|(T-T_1)x\| \le \|T-T_1\|$ so that there exists $x'\in X$ with $Tx' = Tx$ and $\|x'\| \le s^{-1}\|T-T_1\|$. Since $x-x'\in N(T)$ we have $\operatorname{dist}\{x, N(T)\} \le \|x'\| \le s^{-1}\|T-T_1\|$.

Thus $\delta(N(T_1), N(T)) \leq s^{-1}\|T - T_1\|$. Since s was an arbitrary positive number, $s < \gamma(T)$, we have (a).

(b) Let $y \in R(T)$, $\|y\| \leq 1$. Then there exists $x \in X$ with $Tx = y$ and $\|x\| \leq s^{-1}$. Thus $\operatorname{dist}\{y, R(T_1)\} \leq \|y - T_1x\| = \|(T - T_1)x\| \leq s^{-1}\|T - T_1\|$. As in (a) we get the statement.

We are going to use the construction introduced by Sadoskii/Buoni, Harte and Wickstead [12], [5], [8]. For a Banach space X denote by $\ell^\infty(X)$ the Banach space of all bounded sequences of elements of X (with the sup-norm). Let $m(X)$ be the set of all sequences $\{x_i\}_{i=1}^\infty \in \ell^\infty(X)$ such that the closure of the set $\{x_i : i = 1, 2, \ldots\}$ is compact. Then $m(X)$ is a closed subspace of $\ell^\infty(X)$. Denote $\widetilde{X} = \ell^\infty(X)/m(X)$.

If $T \in \mathcal{L}(X, Y)$ then T defines pointwise an operator $T^\infty : \ell^\infty(X) \to \ell^\infty(Y)$ by $T^\infty(\{x_i\}_{i=1}^\infty) = \{Tx_i\}_{i=1}^\infty$. Clearly $T^\infty m(X) \subset m(Y)$. Denote by $\widetilde{T} : \widetilde{X} \to \widetilde{Y}$ the operator induced by T^∞.

We summarize the basic properties of the mappings $X \mapsto \widetilde{X}$ and $T \mapsto \widetilde{T}$, see [5], [6], [8], [10], [12].

Theorem 5. Let X, Y, Z be Banach spaces, let $S, S' \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Y, Z)$ and $\alpha \in \mathbb{C}$. Then
(1) $\widetilde{S} = 0 \Leftrightarrow S$ is compact,
(2) $\widetilde{S + S'} = \widetilde{S} + \widetilde{S'}, \widetilde{\alpha S} = \alpha\widetilde{S}$,
(3) $\widetilde{TS} = \widetilde{T}\widetilde{S}$,
(4) $\|\widetilde{S}\| \leq \|S\|$,
(5) if $M \subset X$ is a subspace of a finite codimension, then $\|\widetilde{S}\| \leq 2\|S|M\|$,
(6) if $R(T)$ is closed then $R(\widetilde{T})$ is closed,
(7) if S and T have closed ranges then

$$R(S)\overset{e}{\subset}N(T) \Leftrightarrow R(\widetilde{S})\overset{e}{\subset}N(\widetilde{T}) \Leftrightarrow R(\widetilde{S}) \subset N(\widetilde{T}),$$
$$N(T)\overset{e}{\subset}R(S) \Leftrightarrow N(\widetilde{T})\overset{e}{\subset}R(\widetilde{S}) \Leftrightarrow N(\widetilde{T}) \subset R(\widetilde{S}).$$

Theorem 6. Let X, Y, Z be Banach spaces, let Y_0 be a closed subspace of Y and let $S : X \to Y$ and $T : Y_0 \to Z$ be operators with closed ranges such that $R(S)\overset{e}{\subset}Y_0$. Then there exists $\eta > 0$ such that

$$\begin{aligned} &\dim R(S)/(R(S) \cap N(T)) + \dim N(T_1)/(R(S_1) \cap N(T_1)) \\ \leq &\dim R(S_1)/(R(S_1) \cap N(T_1)) + \dim N(T)/(R(S) \cap N(T)) \end{aligned} \tag{6}$$

for all operators $S_1 : X \to Y$, $T_1 : Y_0 \to Z$ with closed ranges such that $\|T_1 - T\| < \eta$ and $\|S_1 - S\| < \eta$.

Proof. (a) Suppose $\dim R(S)/(R(S) \cap N(T)) < \infty$. Set $R = R(T)$ and $N = N(T)$ and let ε be the number constructed in Theorem 3. Set $\eta = \varepsilon \cdot \min\{\gamma(T), \gamma(S)\}$. If $\|T_1 - T\| < \eta$ and $\|S_1 - S\| < \eta$ then $\delta(N(T_1), N(T)) < \varepsilon$ and $\delta(R(T), R(T_1)) < \varepsilon$ so that Theorem 3 for $N_1 = N(T_1)$ and $R_1 = R(S_1)$ gives the required inequality.

(b) If $\dim R(S)/(R(S)\cap N(T))=\infty$ and $\dim N(T)/(R(S)\cap N(T))=\infty$ then the statement is clearly true.

(c) Suppose $\dim R(S)/(R(S)\cap N(T))=\infty$ and $\dim N(T)/(R(S)\cap N(T))<\infty$, i.e. $N(T)\overset{e}{\subset}R(S)$. Denote $Y'=R(S)+Y_0$. Let T' be any extension of T to a bounded operator $T':Y'\to Z$ (since $Y'=Y_0\oplus M$ for some finite dimensional subspace M, we can define $T'|M=0$).

We show first that the range of $T'S$ is closed. We have $N(T')\overset{e}{=}N(T)\overset{e}{\subset}R(S)$. Let F be a finite dimensional subspace of $N(T')$ such that $N(T')\subset R(S)+F$. It is sufficient to show that $R(T'S)+T'F$ is closed.

Let $x_k\in X$, $f_k\in F$ $(k=1,2,\dots)$ and let $T'Sx_k+T'f_k\to z$ for some $z\in Z$. Since $R(T')$ is closed we have $z=T'y$ for some $y\in Y_0+R(S)$. Thus $T'(Sx_k+f_k-y)\to 0$. Consider the operator $\hat{T}':(Y_0+R(S))/N(T')\to Z$ induced by T'. Clearly $R(\hat{T}')=R(T')$ and $\hat{T}'$ is injective, hence bounded below. Thus $Sx_k+f_k-y+N(T')\to 0$ in $Y/N(T')$. So there are elements $y_k\in N(T')$ such that $Sx_k+f_k+y_k\to y$ (in Y). Thus $y\in R(S)+F$ and $z=T'y\in R(T'S)+T'F$. Consequently $R(T'S)$ is closed.

Further $\dim R(T'S)=\infty$ (otherwise $R(S)\overset{e}{\subset}N(T')\overset{e}{=}N(T)$ which contradicts to the assumption that $\dim R(S)/(R(S)\cap N(T))=\infty$), so that $T'S$ is not compact. If $\widetilde{S}:\widetilde{X}\to\widetilde{Y'}$ and $\widetilde{T'}:\widetilde{Y'}\to\widetilde{Z}$ are the operators defined above then $\widetilde{T'}\widetilde{S}\neq 0$.

Set $\eta=\min\left\{\|S\|,\frac{\|\widetilde{T'S}\|}{4\|S\|+2\|T\|}\right\}$. Let $S_1:X\to Y$ and $T_1:Y_0\to Z$ be operators with closed ranges such that $\|S_1-S\|<\eta$ and $\|T_1-T\|<\eta$. To prove (6) it is sufficient to show

$$\dim R(S_1)/(R(S_1)\cap N(T_1))=\infty. \tag{7}$$

We may assume $R(S_1)\overset{e}{\subset}Y_0$; otherwise

$$\dim R(S_1)/(R(S_1)\cap N(T_1))\geq \dim R(S_1)/(R(S_1)\cap Y_0)=\infty$$

and (7) is satisfied.

Denote $Y_1=Y'+R(S_1)=Y_0+R(S)+R(S_1)$. Then Y' is a subspace of Y_1 of a finite codimension. Let $J:Y'\to Y_1$ be the natural embedding and let $P:Y_1\to Y'$ be a projection onto Y'. Let T_1' be any extension of T_1 to an operator $T_1':Y_1\to Z$. Consider operators $\widetilde{S_1}:\widetilde{X}\to\widetilde{Y_1}$, $\widetilde{T_1'}:\widetilde{Y_1}\to\widetilde{Z}$, $\widetilde{J}:\widetilde{Y'}\to\widetilde{Y_1}$ and $\widetilde{P}:\widetilde{Y_1}\to\widetilde{Y'}$. We have

$$\begin{aligned}T_1'S_1=&(T'P)(JS)+(T'P)(S_1-JS)+(T_1'-T'P)S_1\\ =&T'S+(T'P)(S_1-JS)+(T_1'-T'P)S_1,\end{aligned}$$

$\|\widetilde{S_1}-\widetilde{JS}\|\leq\eta$, $\|\widetilde{T_1'}-\widetilde{T'P}\|\leq 2\|T_1-T\|\leq 2\eta$ and $\|\widetilde{T'P}\|\leq\|\widetilde{T'}\|\cdot\|\widetilde{P}\|\leq 2\|T\|$. Thus

$$\|\widetilde{T_1'S_1}\|\geq\|\widetilde{T'S}\|-2\eta\|\widetilde{T}\|-2\eta\|\widetilde{S_1}\|\geq\|\widetilde{T'S}\|-2\eta(\|S\|+\eta)-2\eta\|T\|>0$$

so that $T_1'S_1$ is not compact.

Consequently we have (7) (otherwise $R(S_1)\overset{e}{\subset}N(T_1)\overset{e}{=}N(T_1')$ and $\dim R(T_1'S_1)<\infty$). This finishes the proof of Theorem 6.

Fredholm pairs of operators were defined in [2].

Definition. A Fredholm pair in (X,Y) is a pair (S,T) of operators $S : X_0 \to Y$ and $T : Y_0 \to X$ where X_0 and Y_0 are closed subspaces of X and Y, respectively, such that $R(S)\overset{e}{=}N(T)$ and $R(T)\overset{e}{=}N(S)$. The index of a Fredholm pair is defined by

$$\begin{aligned}\operatorname{ind}(S,T) = &\dim N(S)/(R(T)\cap N(S)) - \dim R(T)/(R(T)\cap N(S)) \\ &- \dim N(T)/(R(S)\cap N(T)) + \dim R(S)/(R(S)\cap N(T)).\end{aligned} \tag{8}$$

Note that if (S,T) is a Fredholm pair then the ranges of S and T are closed.

This suggests the definition of semi-Fredholm pairs.

Definition. By a semi-Fredholm pair we mean a pair (S,T) of operators $S : X_0 \to Y$ and $T : Y_0 \to X$ where X_0 and Y_0 are closed subspaces of X and Y, respectively, such that

(1) $R(S)\overset{e}{\subset}Y_0$ and $R(T)\overset{e}{\subset}X_0$,
(2) S and T have closed ranges,
(3) either

$$\dim N(S)/(R(T)\cap N(S)) + \dim R(S)/(R(S)\cap N(T)) < \infty$$

or

$$\dim N(T)/(R(S)\cap N(T)) + \dim R(T)/(R(T)\cap N(S)) < \infty.$$

For a semi-Fredholm pair (S,T) we define the index of (S,T) by (8).

Lemma 7. Let X,Y be Banach spaces, let $S : X \to Y$ and $T : Y \to X$ be operators with closed ranges such that $R(S) = N(T)$ and $R(T) \subset N(S)$. Then there exists $\varepsilon > 0$ such that

$$\dim N(S)/R(T) + \dim R(T_1)/(R(T_1)\cap N(S_1)) = \dim N(S_1)/(R(T_1)\cap N(S_1))$$

for all operators $S_1 : X \to Y$ and $T_1 : Y \to X$ with closed ranges such that $\|S_1 - S\| < \varepsilon$, $\|T_1 - T\| < \varepsilon$ and $R(S_1) \subset N(T_1)$.

Proof. The sequence $X \xrightarrow{S} Y \xrightarrow{T} X$ is exact in the middle. By [14], Lemma 2.1 and [13], Corollary 2.2 there exist positive constants $\varepsilon_1 > 0$ and c such that $R(S_1) = N(T_1)$, $\gamma(S_1) \geq c$ and $\gamma(T_1) \geq c$ for all operators $S_1 : X \to Y$, $T_1 : Y \to X$ with closed ranges satisfying $\|S_1 - S\| < \varepsilon_1$, $\|T_1 - T\| < \varepsilon_1$ and $R(S_1) \subset N(T_1)$.

Set $\varepsilon = \min\{\varepsilon_1, \frac{c}{9}\}$. Let S_1 and T_1 be operators with closed ranges satisfying $\|S_1 - S\| < \varepsilon$, $\|T_1 - T\| < \varepsilon$ and $R(S_1) \subset N(T_1)$. Then, by Lemma 4, we have $\hat\delta(N(S), N(S_1)) \leq c^{-1}\|S_1 - S\| < 1/9$ and $\hat\delta(R(T), R(T_1)) \leq c^{-1}\|T_1 - T\| < 1/9$. By Theorem 2 (b), we have the required equality.

Theorem 8. Let X,Y be Banach spaces, $X_0 \subset X$, $Y_0 \subset Y$ closed subspaces, let $S : X_0 \to Y$ and $T : Y_0 \to X$ be operators and let (S,T) be a semi-Fredholm pair. Then there exists $\varepsilon > 0$ such that $\operatorname{ind}(S_1,T_1) = \operatorname{ind}(S,T)$ for every semi-Fredholm pair (S_1,T_1) of operators $S_1 : X_0 \to Y$ and $T_1 : Y_0 \to X$ satisfying $\|S_1 - S\| < \varepsilon$ and $\|T_1 - T\| < \varepsilon$.

Proof. Denote

$$\alpha(S,T) = \dim N(S)/(R(T)\cap N(S)) - \dim R(T)/(R(T)\cap N(S))$$

and

$$\beta(S,T) = \dim N(T)/(R(S)\cap N(T)) - \dim R(S)/(R(S)\cap N(T)).$$

Then $\operatorname{ind}(S,T) = \alpha(S,T) - \beta(S,T)$.

By Theorem 6, $\alpha(S_1,T_1) \le \alpha(S,T)$ and $\beta(S_1,T_1) \le \beta(S,T)$ if (S_1,T_1) is close enough to (S,T).

We distinguish three cases:

(a) Let $\alpha(S,T) = -\infty$. Then $\alpha(S_1,T_1) = -\infty$ for every semi-Fredholm pair (S_1,T_1) close enough to (S,T). In particular $\operatorname{ind}(S_1,T_1) = \operatorname{ind}(S,T) = -\infty$.

Similar considerations can be done if $\beta(S,T) = -\infty$.

In the rest of the proof we assume $\alpha(S,T) \ne -\infty$ and $\beta(S,T) \ne -\infty$ so that $R(S)\overset{e}{\subset}N(T)$ and $R(T)\overset{e}{\subset}N(S)$.

Denote $X' = X_0 + R(T)$ and $Y' = Y_0 + R(S)$ and fix any projections $P : X' \overset{\text{onto}}{\longrightarrow} X_0$ and $Q : Y' \overset{\text{onto}}{\longrightarrow} Y_0$. Consider operators $\widetilde{S} : \widetilde{X_0} \to \widetilde{Y'}$ and $\widetilde{T} : \widetilde{Y_0} \to \widetilde{X'}$ and denote $\hat{S} = \widetilde{Q}\widetilde{S} : \widetilde{X_0} \to \widetilde{Y_0}$ and $\hat{T} = \widetilde{P}\widetilde{T} : \widetilde{Y_0} \to \widetilde{X_0}$. Since $R(QS)\overset{e}{=}R(S)\overset{e}{\subset}N(T)\overset{e}{=}N(PT)$, we have $R(\hat{S}) \subset N(\hat{T})$ and similarly $R(\hat{T}) \subset N(\hat{S})$.

Analogously, for a semi-Fredholm pair of operators $S_1 : X_0 \to Y_0 + R(S_1)$ and $T_1 : Y_0 \to X_0 + R(T_1)$ denote $\hat{S_1} = \widetilde{Q_1}\widetilde{S_1} : \widetilde{X_0} \to \widetilde{Y_0}$ and $\hat{T_1} = \widetilde{P_1}\widetilde{T_1} : \widetilde{Y_0} \to \widetilde{X_0}$ where $P_1 : X_0 + R(T_1) \overset{\text{onto}}{\longrightarrow} X_0$ and $Q_1 : Y_0 + R(S_1) \overset{\text{onto}}{\longrightarrow} Y_0$ are any (fixed) projections. Since $S^{-1}(Y_0) \cap S_1^{-1}(Y_0)$ is a subspace of a finite codimension in X_0, by Theorem 5 (7) we have $\|\hat{S} - \hat{S_1}\| \le 2\|S - S_1\|$. Similarly $\|\hat{T} - \hat{T_1}\| \le 2\|T - T_1\|$.

(b) Let $\alpha(S,T) = \infty$. Since the pair (S,T) is semi-Fredholm and $\beta(S,T) \ne -\infty$, $\beta(S,T)$ is finite, so that $R(S)\overset{e}{=}N(T)$ and $R(\hat{S}) = N(\hat{T})$.

The equality $\operatorname{ind}(S_1,T_1) = \operatorname{ind}(S,T) = \infty$ is true for every semi-Fredholm pair (S_1,T_1) with $\beta(S_1,T_1) = -\infty$. If $\beta(S_1,T_1) \ne -\infty$ then $R(S_1)\overset{e}{\subset}N(T_1)$ so that $R(\hat{S_1}) \subset N(\hat{T_1})$. If (S_1,T_1) is close enough to (S,T) then, by the previous lemma,

$$\infty = \dim N(\hat{S})/R(\hat{T}) = \dim N(\hat{S_1})/(R(\hat{T_1}) \cap N(\hat{S_1})) - \dim R(\hat{T_1})/(R(\hat{T_1}) \cap N(\hat{S_1})).$$

Hence $\dim N(\hat{S_1})/(R(\hat{T_1}) \cap N(\hat{S_1})) = \infty$ so that $\dim N(S_1)/(R(T_1)\cap N(S_1)) = \infty$ and $\operatorname{ind}(S_1,T_1) = \operatorname{ind}(S,T) = \infty$.

Similar considerations can be done in case of $\beta(S,T) = \infty$.

(c) It remains the case $|\alpha(S,T)| < \infty$ and $|\beta(S,T)| < \infty$. Then (S,T) is a Fredholm pair, i.e. $R(\hat{S}) = N(\hat{T})$ and $R(\hat{T}) = N(\hat{S})$. Since (S_1,T_1) is semi-Fredholm, either $\alpha(S_1,T_1) \ne -\infty$ or $\beta(S_1,T_1) \ne -\infty$. Without loss of generality we can assume $\beta(S_1,T_1) \ne -\infty$ so that $R(\hat{S_1}) \subset N(\hat{T_1})$. By [13] or [14], for (S_1,T_1) close enough to (S,T), we have $R(\hat{S_1}) = N(\hat{T_1})$. Further $\alpha(S_1,T_1) \ne \infty$ so that $N(S_1)\overset{e}{\subset}R(T_1)$, i.e. $N(\hat{S_1}) \subset R(\hat{T_1})$. By Lemma 7 we have

$$0 = \dim N(\hat{S_1})/(R(\hat{T_1}) \cap N(\hat{S_1})) = \dim N(\hat{T_1})/(R(\hat{S_1}) \cap N(\hat{T_1})).$$

Consequently $N(\hat{S_1}) = R(\hat{T_1})$, i.e. $N(S_1)\stackrel{e}{=}R(T_1)$ and (S_1, T_1) is also a Fredholm pair.

The equality $\operatorname{ind}(S_1, T_1) = \operatorname{ind}(S, T)$ for Fredholm pairs (S_1, T_1) close enough to (S, T) was proved in [2] and [3].

The next result — the stability of index under finite dimensional perturbations — is an easy consequence of the corresponding result for Fredholm pairs, see [3], Theorem 3.10. We give a simpler proof.

Theorem 9. Let X, Y be Banach spaces, X_0, Y_0 their subspaces and $S, S_1 : X_0 \to Y$, $T, T_1 : Y_0 \to X$ operators. Suppose that (S, T) is a semi-Fredholm pair and that $S - S_1$ and $T - T_1$ are operators of finite rank. Then (S_1, T_1) is a semi-Fredholm pair and $\operatorname{ind}(S_1, T_1) = \operatorname{ind}(S, T)$.

Proof. Clearly $N(S)\stackrel{e}{=}N(S_1)$, $N(T)\stackrel{e}{=}N(T_1)$, $R(S)\stackrel{e}{=}R(S_1)$ and $R(T)\stackrel{e}{=}R(T_1)$. So $\dim N(S)/(R(T) \cap N(S)) = \infty$ if and only if $\dim N(S_1)/(R(T_1 \cap N(S_1)) = \infty$. Similar equivalences are true also for the remaining terms appearing in the definition of the index (8). Thus (S_1, T_1) is a semi-Fredholm pair. Further $\operatorname{ind}(S, T) = \pm\infty$ if and only if $\operatorname{ind}(S_1, T_1) = \pm\infty$.

Thus we can assume that $\operatorname{ind}(S, T)$ is finite, i.e., $N(S)\stackrel{e}{=}R(T)$ and $N(T)\stackrel{e}{=}R(S)$ and both (S, T) and (S_1, T_1) are Fredholm pairs.

It is sufficient to show that $\operatorname{ind}(S, T) = \operatorname{ind}(S_1, T)$. Indeed, from the symmetry we have also $\operatorname{ind}(S_1, T) = \operatorname{ind}(S_1, T_1)$.

Denote

$$\begin{aligned} M &= N(S) \cap N(S_1) \cap R(T), & M' &= N(S) + N(S_1) + R(T), \\ L &= R(S) \cap R(S_1) \cap N(T), & L' &= R(S) + R(S_1) + N(T). \end{aligned}$$

Clearly $M \subset X_0$, $L \subset Y_0$, $\dim M'/M < \infty$ and $\dim L'/L < \infty$. Then

$$\begin{aligned} \operatorname{ind}(S, T) =& \dim N(S)/(N(S) \cap R(T)) - \dim R(T)/(N(S) \cap R(T)) \\ &- \dim N(T)/(N(T) \cap R(S)) + \dim R(S)/(N(T) \cap R(S)) \\ =& \dim N(S)/M - \dim R(T)/M - \dim N(T)/L + \dim R(S)/L \end{aligned}$$

and similarly

$$\operatorname{ind}(S_1, T) = \dim N(S_1)/M - \dim R(T)/M - \dim N(T)/L + \dim R(S_1)/L.$$

Thus

$$\operatorname{ind}(S, T) - \operatorname{ind}(S_1, T) = \dim N(S)/M - \dim N(S_1)/M + \dim R(S)/L - \dim R(S_1)/L.$$

Define operators $\tilde{S}, \tilde{S}_1 : X_0/M \to L'$ by $\tilde{S}(x + M) = Sx$, $\tilde{S}_1(x + M) = S_1 x \quad (x + M \in X_0/M)$. Clearly $R(\tilde{S}) = R(S)$, $R(\tilde{S}_1) = R(S_1)$, $\dim N(\tilde{S}) = \dim N(S)/M < \infty$ and $\dim N(\tilde{S}_1) = \dim N(S_1)/M < \infty$. Thus $\tilde{S}, \tilde{S}_1$ are upper semi-Fredholm operators and $\tilde{S} - \tilde{S}_1$ has finite rank.

Further

$$\dim L'/L = \dim L'/R(S) + \dim R(S)/L = \dim L'/R(S_1) + \dim R(S_1)/L.$$

Hence

$$\begin{aligned}
&\operatorname{ind}(S,T)-\operatorname{ind}(S_1,T)\\
=&\dim N(S)/M-\dim N(S_1)/M-\dim L'/R(S)+\dim L'/R(S_1)\\
=&\dim N(\tilde S)-\operatorname{codim}R(\tilde S)-\dim N(\tilde S_1)+\operatorname{codim}R(\tilde S_1)\\
=&\operatorname{ind}(\tilde S)-\operatorname{ind}(\tilde S_1)=0.
\end{aligned}$$

Theorem 10. Let X,Y be Banach spaces, let $S,K:X\to Y$ and $T,L:Y\to X$ be operators, let K and L be compact and let (S,T) and $(S+K,T+L)$ be semi-Fredholm pairs. Then $\operatorname{ind}(S+K,T+L)=\operatorname{ind}(S,T)$.

Proof. We use the approach of Ambrozie, see [3] or [4]. Set $C=C\langle 0,1\rangle$. Since $\overline{R(K)}$ and $\overline{R(L)}$ are separable Banach spaces, there exist isometric embeddings $i:\overline{R(K)}\to C$ and $j:\overline{R(L)}\to C$. Consider the spaces $X\oplus C$ and $Y\oplus C$ with ℓ^1-norms and let $G(-i)=\{y\oplus(-iy),y\in\overline{R(K)}\}$ and $G(-j)=\{x\oplus(-jx),x\in\overline{R(L)}\}$ be the graphs of $-i$ and $-j$, respectively. Let $E=(X\oplus C)/G(-j)$ and $F=(Y\oplus C)/G(-i)$. Let $\alpha:X\to E$ and $\beta:Y\to F$ be defined by $\alpha x=(x\oplus 0)+G(-j)$ and $\beta y=(y\oplus 0)+G(-i)$. Since i and j are isometries, it is easy to check that α and β are isometries. Denote $X'=R(\alpha)\subset E$ and $Y'=R(\beta)\subset F$. Thus X' and Y' are "copies" of X and Y. Denote by S',T',K',L' copies of S,T,K,L. More precisely, let $S',K':X'\to Y'$ and $T',L':Y'\to X'$ be defined by $S'=\beta S\alpha^{-1}$, $K'=\beta K\alpha^{-1}$, $T'=\alpha T\beta^{-1}$ and $L'=\alpha L\beta^{-1}$.

Clearly $\operatorname{ind}(S',T')=\operatorname{ind}(S,T)$ and $\operatorname{ind}(S'+K',T'+L')=\operatorname{ind}(S+K,T+L)$. Since operators $iK:X\to C$ and $jL:Y\to C$ are compact and C has the approximation property, there exist finite dimensional operators $U_n:X\to C$ and $V_n:Y\to C$ $(n=1,2,\dots)$ such that $\|U_n-iK\|\to 0$ and $\|V_n-jL\|\to 0$.

Define operators $\gamma:C\to F$ and $\delta:C\to E$ by $\gamma c=(0\oplus c)+G(-i)$ and $\delta c=(0\oplus c)+G(-j)$ $(c\in C)$. It is easy to check that γ and δ are isometries. Define $U_n':X'\to F$ and $V_n':Y'\to E$ by $U_n'=\gamma U_n\alpha^{-1}$ and $V_n'=\delta V_n\beta^{-1}$ $(n=1,2,\dots)$.

Since $\operatorname{ind}(S',T')=\operatorname{ind}(S'+U_n',T'+V_n')$ for every n, by Theorem 8 it is sufficient to show that $\|K'-U_n'\|=\|(S'+K')-(S'+U_n')\|\to 0$ and $\|L'-V_n'\|\to 0$. Let x' be an element of X' with $\|x'\|=1$. Let $x'=\alpha x=(x\oplus 0)+G(-j)$ for some $x\in X$, $\|x\|=1$. Then

$$\begin{aligned}
&\|(K'-U_n')x'\|=\|(\beta K-\gamma U_n)x\|=\|[(Kx\oplus 0)+G(-i)]-[(0\oplus U_nx)+G(-i)]\|\\
=&\|(Kx\oplus(-U_nx))+G(-i)\|=\|0\oplus(iK-U_n)x+G(-i)\|=\|\gamma((iK-U_n)x)\|\\
=&\|((iK-U_n)x\|\le\|iK-U_n\|.
\end{aligned}$$

Thus $\|K'-U_n'\|\to 0$ and similarly $\|L'-V_n'\|\to 0$. This finishes the proof.

Definition. A chain is a sequence $\mathcal{K}=\{X_i,\delta_i\}_{i=0}^n$ where $X_0,X_1,\dots,X_n$ are Banach spaces and $\delta_i:X_i\to X_{i+1}$ operators. Formally we set $X_i=0$ for $i<0$ or $i>n$ and $\delta_i=0$ $(i<0$ or $i\ge n)$.

Thus a chain is an object of the following type:

$$\mathcal{K}:\qquad 0\longrightarrow X_0\xrightarrow{\delta_0}X_1\xrightarrow{\delta_1}\ \cdots\ \xrightarrow{\delta_{n-1}}X_n\longrightarrow 0.$$

We say that $\mathcal{K}$ is a semi-Fredholm chain if

(1) the operators $\delta_0, \dots, \delta_{n-1}$ have closed ranges,
(2) either

$$\sum_{i \text{ even}} \dim N(\delta_i)/(R(\delta_{i-1}) \cap N(\delta_i)) + \sum_{i \text{ odd}} \dim R(\delta_{i-1})/(R(\delta_{i-1}) \cap N(\delta_i))$$

or

$$\sum_{i \text{ odd}} \dim N(\delta_i)/(R(\delta_{i-1}) \cap N(\delta_i)) + \sum_{i \text{ even}} \dim R(\delta_{i-1})/(R(\delta_{i-1}) \cap N(\delta_i))$$

For a semi-Fredholm chain and for $0 \le i \le n$ define

$$\alpha_i(\mathcal{K}) = \dim N(\delta_i)/(R(\delta_{i-1}) \cap N(\delta_i)) - \dim R(\delta_{i-1})/(R(\delta_{i-1}) \cap N(\delta_i))$$

and the index of $\mathcal{K}$,

$$\operatorname{ind}(\mathcal{K}) = \sum_{i=0}^{n} (-1)^i \alpha_i(\mathcal{K}).$$

(Simply, a chain $\mathcal{K}$ is semi-Fredholm if the operators δ_i have closed ranges and the index is well-defined.)

Remark. A semi-Fredholm chain $\mathcal{K}$ with $|\operatorname{ind}(\mathcal{K})| < \infty$ was called a Fredholm essential complex in [4] and [11]. In the present notation it would be logical to call it a Fredholm chain.

For a chain $\mathcal{K} = \{X_i, \delta_i\}_{i=0}^n$ denote

$$X = \bigoplus_{i \text{ even}} X_i, \quad Y = \bigoplus_{i \text{ odd}} X_i, \quad S = \bigoplus_{i \text{ even}} \delta_i, \quad \text{and} \quad T = \bigoplus_{i \text{ even}} \delta_i.$$

It is easy to see that the chain $\mathcal{K}$ is semi-Fredholm if and only if the corresponding pair (S, T) is semi-Fredholm and $\operatorname{ind}(\mathcal{K}) = \operatorname{ind}(S, T)$. Thus we get the following perturbation properties of semi-Fredholm chains:

Theorem 11. Let $\mathcal{K} = \{X_i, \delta_i\}_{i=0}^n$ be a semi-Fredholm chain. Then there exists $\varepsilon > 0$ such that, for every semi-Fredholm chain $\mathcal{K}' = \{X_i, \delta_i'\}_{i=0}^n$ with $\|\delta_i' - \delta_i\| < \varepsilon \quad (i = 0, \dots, n-1)$ we have
(1) $\alpha_i(\mathcal{K}') \le \alpha_i(\mathcal{K}) \quad (i = 0, \dots, n)$,
(2) $\operatorname{ind}(\mathcal{K}') = \operatorname{ind}(\mathcal{K})$.

Theorem 12. Let $\mathcal{K} = \{X_i, \delta_i\}_{i=0}^n$ and $\mathcal{K}' = \{X_i', \delta_i'\}_{i=0}^n$ be semi-Fredholm complexes such that $\delta_i' - \delta_i$ are compact for $i = 0, \dots, n-1$. Then $\operatorname{ind}(\mathcal{K}') = \operatorname{ind}(\mathcal{K})$.

Remark. It is necessary to assume that $\mathcal{K}'$ is semi-Fredholm.

Let H be a separable infinite dimensional Hilbert space and consider the following complex:

$$\mathcal{K}: \qquad 0 \longrightarrow H \xrightarrow{\delta_0} H \oplus H \xrightarrow{\delta_1} H \oplus H \xrightarrow{\delta_2} H \longrightarrow 0$$

where the mappings δ_i are defined by $\delta_0 h = h \oplus 0$, $\delta_1(h \oplus g) = 0 \oplus g$, $\delta_2(h \oplus g) = h$. It is easy to check that $\mathcal{K}$ is exact.

(a) Let $A : H \to H$ be an operator with a small norm and non-closed range. Then $\delta_1' : H \oplus H \to H \oplus H$ defined by $\delta_1'(h \oplus g) = Ah \oplus g$ has not closed range.

(b) Let ε be a small positive number. Define $\delta_1'' : H \oplus H \to H \oplus H$ by $\delta_1'(h \oplus g) = \varepsilon h \oplus g$. Then δ_1'' has closed range but the chain

$$\mathcal{K}' : \qquad 0 \longrightarrow H \xrightarrow{\delta_0} H \oplus H \xrightarrow{\delta_1''} H \oplus H \xrightarrow{\delta_2} H \longrightarrow 0$$

is not semi-Fredholm.

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