On the topological boundary of the one-sided spectrum

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Abstract. It is well-known that the topological boundary of the spectrum of an operator is contained in the approximate point spectrum. We show that the one-sided version of this result is not true. This gives also a negative answer to a problem of Schmoeger.

Denote by $\mathcal{L}(X)$ the algebra of all bounded linear operators acting in a Banach space X. For $T \in \mathcal{L}(X)$ denote by $\sigma(T)$, $\sigma_l(T)$ and $\sigma_\pi(T)$ the spectrum, left spectrum and the approximate point spectrum of T, respectively:

$$\begin{aligned} \sigma(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not invertible}\}, \\ \sigma_l(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not left invertible}\}, \\ \sigma_\pi(T) &= \{\lambda \in \mathbf{C} : T - \lambda \text{ is not bounded below}\}. \end{aligned}$$

It is well-known that $\partial\sigma(T) \subset \sigma_\pi(T) \subset \sigma_l(T) \subset \sigma(T)$. This implies in particular that the outer topological boundaries (= the boundaries of the polynomially convex hull) of $\sigma(T), \sigma_l(T)$ and $\sigma_\pi(T)$ coincide.

The aim of this paper is to show that the inner topological boundaries of σ_l and σ_π can be different.

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We use the following notations. If X is a closed subspace of a Banach space Y then we denote $c(X,Y) = \inf\{\|P\| : P \in \mathcal{L}(Y) \text{ is a projection with range } X\}$ (if X is not complemented in Y then we set $c(X,Y) = \infty$).

For Banach spaces X and Y denote by $X \hat{\otimes} Y$ and $X \check{\otimes} Y$ the projective and injective tensor products (see [2]). Thus $X \hat{\otimes} Y$ and $X \check{\otimes} Y$ are the completions of the algebraic tensor product $X \otimes Y$ endowed with the projective (injective) norms

$$\|u\|_{X \hat{\otimes} Y} = \inf\Big\{\sum_i \|x_i\| \cdot \|y_i\| : u = \sum_i x_i \otimes y_i\Big\}$$

and

$$\|u\|_{X \check{\otimes} Y} = \sup\big\{|(x^* \otimes y^*)(u)| : x^* \in X^*, y^* \in Y^*, \|x^*\| \le 1, \|y^*\| \le 1\big\}.$$

Clearly elements of $Y \hat{\otimes} X^*$ can be identified with the trace class operators $X \to Y$ (with the trace norm).

If $\{Y_i\}$ is a family of Banach spaces then we denote by $\bigoplus_i Y_i$ the direct sum of Y_i's with the ℓ_1 norm, $\|\bigoplus y_i\| = \sum_i \|y_i\|$.

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Lemma 1. Let $X_i, Y_i \quad (i \in \mathbf{Z})$ be Banach spaces, $X_i \subset Y_i$. Then

$$c\Big(\bigoplus_i X_i, \bigoplus_i Y_i\Big) = \sup_i \{c(X_i, Y_i)\}.$$

Proof. Denote $X = \bigoplus_i X_i$ and $Y = \bigoplus_i Y_i$.

$\leq$: If $P_i \in \mathcal{L}(Y_i)$ are projections with ranges X_i and $\sup_i \|P_i\| < \infty$ then $P = \bigoplus_i P_i$ is a projection onto X with the norm $\|P\| = \sup_i \|P_i\|$.

$\geq$: Suppose $P \in \mathcal{L}(Y)$ is a projection with range X. Denote $P_k = Q_k P J_k \quad (k \in \mathbf{Z})$ where $J_k : Y_k \to Y$ is the natural embedding and $Q_k : X \to X_k$ the canonical projection. It is easy to check that P_k is a projection with range X_k and $\|P_k\| \leq \|P\|$ so that $c(X_k, Y_k) \leq c(X, Y)$.

Lemma 2. Let E be a finite dimensional subspace of a Banach space X. Then

$$c(E, X) = \sup\{|tr(S)| : S \in \mathcal{L}(E), \|JS\|_{X\hat{\otimes}E^*} \leq 1\}$$

where $J : E \to X$ is the natural embedding.

Proof. $\geq$: Let P be a projection from X onto E and let $S \in \mathcal{L}(E)$. Then

$$|tr(S)| = |tr(PJS)| \leq \|PJS\|_{E\hat{\otimes}E^*} \leq \|P\| \cdot \|JS\|_{X\hat{\otimes}E^*}.$$

$\leq$: Consider $\mathcal{M} = \{JS : S \in \mathcal{L}(E)\}$ as a subspace of $X\hat{\otimes}E^*$. Define $f \in \mathcal{M}^*$ by $f(JS) = tr(S)$. The norm of f is equal to $k = \sup\{|tr(S)| : S \in \mathcal{L}(E), \|JS\|_{X\hat{\otimes}E^*} \leq 1\}$. By the Hahn-Banach theorem there exists an extension $g \in (X\hat{\otimes}E^*)^*$ with the same norm k. Since $(X\hat{\otimes}E^*)^*$ is isometrically isometric to $\mathcal{L}(X, E)$ (see [2], p.230), there exists $P \in \mathcal{L}(X, E)$ with $\|P\| = k$ and, for all $x \in X$ and $e^* \in E^*$, $< Px, e^* >= g(x \otimes e^*)$. In particular, for $e \in E$ and $e^* \in E^*$,

$$< Pe, e^* >= g(e \otimes e^*) = f(e \otimes e^*) = tr(e \otimes e^*) =< e, e^* >$$

so that $Pe = e$ and P is a projection with range E. Hence $c(E, X) \leq k$.

Proposition 3. Let X_1 and X_2 be Banach spaces, let $E_1 \subset X_1$ and $E_2 \subset X_2$ be finite dimensional subspaces. Then

$$c(E_1 \check{\otimes} E_2, X_1 \check{\otimes} X_2) = c(E_1, X_1) \cdot c(E_2, X_2).$$

Proof. It is well-known that $E_1 \check{\otimes} E_2$ is a subspace of $X_1 \check{\otimes} X_2$ (see [2], p.225).

$\leq$: If $P_i \in \mathcal{L}(X_i)$ is a projection with range $E_i \quad (i = 1, 2)$ then it is easy to check that $P_1 \otimes P_2 \in \mathcal{L}(X_1 \check{\otimes} X_2)$ is a projection onto $E_1 \check{\otimes} E_2$ with $\|P_1 \otimes P_2\| \leq \|P_1\| \cdot \|P_2\|$.

$\geq$: Denote by $J_i : E_i \to X_i \quad (i = 1, 2)$ the natural embedding. Then $J = J_1 \otimes J_2$ is the natural embedding of $E_1 \check{\otimes} E_2$ into $X_1 \check{\otimes} X_2$. Let $\varepsilon > 0$. By Lemma 2 there exist $S_i \in \mathcal{L}(E_i) \quad (i = 1, 2)$ such that $\|J_i S_i\|_{X_i \hat{\otimes} E_i^*} = 1$ and $|tr(S_i)| > c(E_i, X_i) - \varepsilon \quad (i = 1, 2)$. Consider $S = S_1 \otimes S_2 \in \mathcal{L}(E_1 \check{\otimes} E_2)$. It is easy to check that

$$tr(S) = tr(S_1) \cdot tr(S_2) > (c(E_1, X_1) - \varepsilon) \cdot (c(E_2, X_2) - \varepsilon) \tag{1}$$

and

$$\|JS\|_{(X_1\check{\otimes}X_2)\hat{\otimes}(E_1\check{\otimes}E_2)^*} \le \|J_1S_1\|_{X_1\hat{\otimes}E_1^*}\|J_2S_2\|_{\check{\otimes}X_2\hat{\otimes}E_2^*} = 1. \tag{2}$$

To see (2), observe that if $\delta > 0$, $J_1S_1 = \sum_i x_{1i} \otimes e_{1i}^*$ and $J_2S_2 = \sum_j x_{2j} \otimes e_{2j}^*$ for some $x_{1i} \in X_1$, $x_{2j} \in X_2$, $e_{1i}^* \in E_1^*$, $e_{2j} \in E_2^*$, $\sum_i \|x_{1i}\| \cdot \|e_{1i}^*\| < 1+\delta$ and $\sum_i \|x_{2j}\| \cdot \|e_{2j}^*\| < 1+\delta$ then

$$JS = \sum_{i,j}(x_{1i} \otimes x_{2j}) \otimes (e_{1i}^* \otimes e_{2j}^*)$$

where $x_{1i} \otimes x_{2j} \in X_1\check{\otimes}X_2$, $e_{1i}^* \otimes e_{2j}^* \in (E_1\check{\otimes}E_2)^*$ and

$$\sum_{i,j} \|x_{1i} \otimes x_{2j}\|_{X_1\check{\otimes}X_2} \cdot \|e_{1i}^* \otimes e_{2j}^*\|_{(E_1\check{\otimes}E_2)^*} < (1+\delta)^2.$$

Thus we have (2) and together with (1) and Lemma 2 we obtain for $\varepsilon \to 0$ the required inequality

$$c(E_1\check{\otimes}E_2, X_1\check{\otimes}X_2) \ge c(E_1, X_1) \cdot c(E_2, X_2).$$

Theorem 4. There exists a Banach space Z and an operator $T \in \mathcal{L}(Z)$ such that $\operatorname{dist}\{0, \sigma_\pi(T)\} > \operatorname{dist}\{0, \sigma_l(T)\} > 0$.

Proof. Fix a Banach space X and a finite dimensional subspace $E \subset X$ such that $c(E,X) = a > 1$ (it is well-known that such a pair exists, see e.g. [11], $ 32). Set

$$\begin{aligned}
Y_0 &= X \oplus X\check{\otimes}X \oplus X\check{\otimes}X\check{\otimes}X \oplus \cdots,\\
Y_1 &= E \oplus E\check{\otimes}X \oplus E\check{\otimes}X\check{\otimes}X \oplus \cdots,\\
Y_2 &= E \oplus E\check{\otimes}E \oplus E\check{\otimes}E\check{\otimes}X \oplus \cdots,\\
&\ \vdots\\
Y_k &= \bigoplus_{i=1}^{\infty} \underbrace{E\check{\otimes}\cdots\check{\otimes}E}_{\min\{k,i\}} \check{\otimes} \underbrace{X\check{\otimes}\cdots\check{\otimes}X}_{\max\{i-k,0\}}\\
&\ \vdots
\end{aligned}$$

We can consider Y_{k+1} as a subspace of Y_k so that $Y_0 \supset Y_1 \supset Y_2 \supset \cdots$. By Lemma 1 and Proposition 3, $c(Y_j, Y_k) = a^{j-k} \quad (k<j)$. Set $Z = \cdots \oplus Y_0 \oplus \cdots \oplus Y_0 \oplus Y_1 \oplus Y_2 \oplus \cdots$ and let $T \in \mathcal{L}(Z)$ be the shift operator to the left,

$$T(\cdots y_{-2} \oplus y_{-1} \oplus \boxed{y_0} \oplus y_1 \oplus y_2 \cdots) = (\cdots y_{-2} \oplus y_{-1} \oplus y_0 \oplus \boxed{y_1} \oplus y_2 \cdots)$$

(the box denotes the zero position). Clearly T is an isometry so that $\sigma_\pi(T) = \{\lambda \in \mathbf{C} : |\lambda| = 1\}$ and $\operatorname{dist}\{0, \sigma_\pi(T)\} = 1$.

Further

$$c(T^kZ, Z) = c(\cdots Y_{k-1} \oplus \boxed{Y_k} \oplus Y_{k+1} \oplus \cdots, \cdots Y_0 \oplus \boxed{Y_0} \oplus Y_1 \oplus \cdots) = a^k.$$

In particular TZ is complemented in Z so that T is left invertible.

Denote $t = \operatorname{dist}\{0, \sigma_l(T)\}$ and $U = \{\lambda \in \mathbf{C} : |\lambda| < t\}$. By [1] there exists an analytic function $F : U \to \mathcal{L}(Z)$ such that $F(\lambda)(T-\lambda) = I \quad (\lambda \in U)$. Let

$$F(\lambda) = \sum_{i=0}^{\infty} F_i \lambda^i \quad (\lambda \in U)$$

be the Taylor expansion of F. Since $F(\lambda)(T-\lambda) = I$ we have $F_0 T = I, F_i T = F_{i-1} \quad (i \geq 1)$ so that $F_i T^{i+1} = I \quad (i = 0, 1, \ldots)$. It is easy to check that $T^{i+1}F_i$ is a projection onto $T^{i+1}Z$. Thus

$$a^i = c(T^i Z, Z) \leq \|T^i F_{i-1}\| = \|F_{i-1}\|$$

so that the radius of convergence of the function $F(\lambda) = \sum_{i=0}^{\infty} F_i \lambda^i$ is

$$t = \left(\limsup_{i \to \infty} \|F_i\|^{1/i}\right)^{-1} \leq a^{-1} < 1.$$

Hence $0 < \operatorname{dist}\{0, \sigma_l(T)\} < \operatorname{dist}\{0, \sigma_\pi(T)\}$.

Corollary 5. In general $\partial\sigma_l(T) \not\subset \sigma_\pi(T)$.

Remark 6. An operator $T \in \mathcal{L}(X)$ is called semiregular if T has closed range and $\ker(T) \subset \bigcap_{n \geq 0} T^n X$. A semiregular operator with a generalized inverse (i.e, with $\ker(T)$ and the range TX complemented) is called regular. Semiregular and regular operators have been studied by many authors, see e.g. [4], [6], [7], [8], [9], [10].

Denote by $\sigma_{sr}(T) = \{\lambda : T - \lambda \text{ is not semiregular}\}$ and $\sigma_{reg}(T) = \{\lambda : T - \lambda \text{ is not regular}\}$ the corresponding spectra. The sets $\sigma_{sr}(T)$ and $\sigma_{reg}(T)$ are nonempty compact sets and $\partial\sigma(T) \subset \sigma_{sr}(T) \subset \sigma_{reg}(T) \subset \sigma(T)$.

The previous example shows that in general $\partial\sigma_{reg}(T) \not\subset \sigma_{sr}(T)$. Indeed, let T be the operator constructed in Theorem 4. For $|\lambda| < 1$ the operator $T - \lambda$ is bounded below and so semiregular. Further T has a left inverse so that it is regular. On the other hand there exists $\mu \in \mathbf{C}$ with $|\mu| = a^{-1} < 1$ such that $T - \mu$ is not left invertible. This means that the range of $T - \mu$ is not complemented and so $T - \mu$ is not regular. Hence $\operatorname{dist}\{0, \sigma_{sr}\} > \operatorname{dist}\{0, \sigma_{reg}\} > 0$ and $\partial\sigma_{reg}(T) \not\subset \sigma_{sr}(T)$. This gives a negative answer to Question 1 of [11] (note that by [5], $\operatorname{dist}\{0, \sigma_{sr}(T)\} = \lim \gamma(T^n)^{1/n}$ where γ denotes the Kato reduced minimum modulus).

Remark 7. Let A be a unital Banach algebra and $a \in A$. Denote by

$$\sigma_l(a) = \{\lambda : A(a - \lambda) \not\ni 1\}$$

and

$$\tau_l(a) = \Big\{\lambda : \inf\{\|(a-\lambda)x\| : x \in A, \|x\| = 1\} = 0\Big\}$$

the left spectrum and the left approximate point spectrum of a, respectively. The right spectrum σ_r and the right approximate point spectrum τ_r can be defined analogously. For the algebra $\mathcal{L}(X)$ of operators in a Banach space X, τ_l coincides with σ_π and τ_r

coincides with σ_δ. Thus in general $\partial\sigma_l(a) \not\subset \tau_l(a)$ and $\partial\sigma_r(a) \not\subset \tau_r(a)$. In fact, it is much simpler to construct the corresponding example in the context of Banach algebras:

Let A be the Banach space of all formal power series $u = \sum_{i,j=0}^{\infty} \alpha_{ij} a^i b^j$ in two variables a, b with complex coefficients α_{ij} such that

$$\|u\| = \sum_{i,j=0}^{\infty} |\alpha_{ij}| 2^i < \infty.$$

The algebra multiplication in A is determined uniquely by setting $ba = 1_A$ so that

$$(a^i b^j) \cdot (a^k b^l) = \begin{cases} a^{i+k-j} b^l & (k \geq j), \\ a^i b^{l+j-k} & (k < j). \end{cases}$$

With this multiplication A becomes a unital Banach algebra.

Clearly $\|a\| = 2$, $\|b\| = 1$ and a is left invertible since $ba = 1$. Further $\|ax\| = 2\|x\|$ for every $x \in A$ so that $\operatorname{dist}\{0, \tau_l(a)\} = 2$.

We show that $\operatorname{dist}\{0, \sigma_l(a)\} = 1$. Since $ba = 1$ and $\|b\| = 1$ it is easy to check that $\operatorname{dist}\{0, \sigma_l(a)\} \geq 1$. On the other hand we show that $a - 1$ is not left invertible. Suppose on the contrary that

$$\Big(\sum_{i,j=0}^{\infty} \alpha_{ij} a^i b^j\Big)(a - 1) = 1 \tag{3}$$

for some α_{ij} with $\sum |\alpha_{ij}| 2^i < \infty$. This means

$$1 = \sum_{i,j=0}^{\infty} a^i b^j (\alpha_{i,j+1} - \alpha_{ij})$$

so that $\alpha_{i,j+1} = \alpha_{ij}$ if either i or j is nonzero. Since $\sum_{i,j} |\alpha_{ij}| 2^i < \infty$ we conclude that $\alpha_{ij} = 0$ for $(i, j) \neq (0, 0)$. This leads to a contradiction with (3).

On the other hand, the following "mixed" result can be proved in a standard way:

Theorem 8. Let a be an element of a unital Banach algebra A. Then $\partial\sigma_l(a) \subset \tau_r(a)$ and $\partial\sigma_r(a) \subset \tau_l(a)$.

Proof. Let $\lambda \in \partial\sigma_l(a)$, let $\lambda_n \notin \sigma_l(a)$ and $\lambda_n \to \lambda$. Then $b_n(a - \lambda_n) = 1$ for some $b_n \in A$. We distinguish two cases:

(a) Suppose $\sup \|b_n\| = \infty$. Then $c_n = \frac{b_n}{\|b_n\|}$ satisfies $\|c_n\| = 1$ and

$$\|c_n(a - \lambda)\| = \frac{\|b_n(a - \lambda)\|}{\|b_n\|} \leq \frac{\|b_n(a - \lambda_n)\|}{\|b_n\|} + \frac{\|b_n(\lambda_n - \lambda)\|}{\|b_n\|} \leq \frac{1}{\|b_n\|} + |\lambda_n - \lambda| \to 0$$

so that $\lambda \in \tau_r(a)$.

(b) Suppose $\sup \|b_n\| < \infty$. Then

$$b_n(a - \lambda) = b_n(a - \lambda_n) + b_n(\lambda_n - \lambda) = 1 + b_n(\lambda_n - \lambda)$$

and $b_n(\lambda_n - \lambda) \to 0$ so that $b_n(a - \lambda)$ is invertible for n big enough. Thus $a - \lambda$ has a left inverse, a contradiction with the assumption $\lambda \in \partial\sigma_l(a) \subset \sigma_l(a)$.

Corollary 9. Let a be a left invertible element of a unital Banach algebra A. Then

$$\operatorname{dist}\{0, \sigma_r(a)\} \le \operatorname{dist}\{0, \tau_r(a)\} \le \operatorname{dist}\{0, \sigma_l(a)\} \le \operatorname{dist}\{0, \tau_l(a)\}.$$

If a has a right inverse then

$$\operatorname{dist}\{0, \sigma_l(a)\} \le \operatorname{dist}\{0, \tau_l(a)\} \le \operatorname{dist}\{0, \sigma_r(a)\} \le \operatorname{dist}\{0, \tau_r(a)\}.$$

(if a is invertible then all these four numbers are equal).

Added in proofs. As another example of an operator T with $\partial\sigma_l(T) \not\subset \sigma_\pi(T)$ may serve the operator constructed by A. Pietsch, Zur Theorie der σ-Transformationen in lokalconvexen Vektorräumen, Math. Nachr. 21 (1960), 347-369, see p. 367-368. This operator is bounded below but not left invertible. Further (see L. Burlando, Continuity of spectrum and spectral radius in algebras of operators, Ann. Fac. Sci. Toulouse 9 (1988), 5–54, Example 1.11), $T - \lambda$ is left invertible for all λ in a punctured neighbourhood of 0.

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