Spectra and localization in pseudo-random networks

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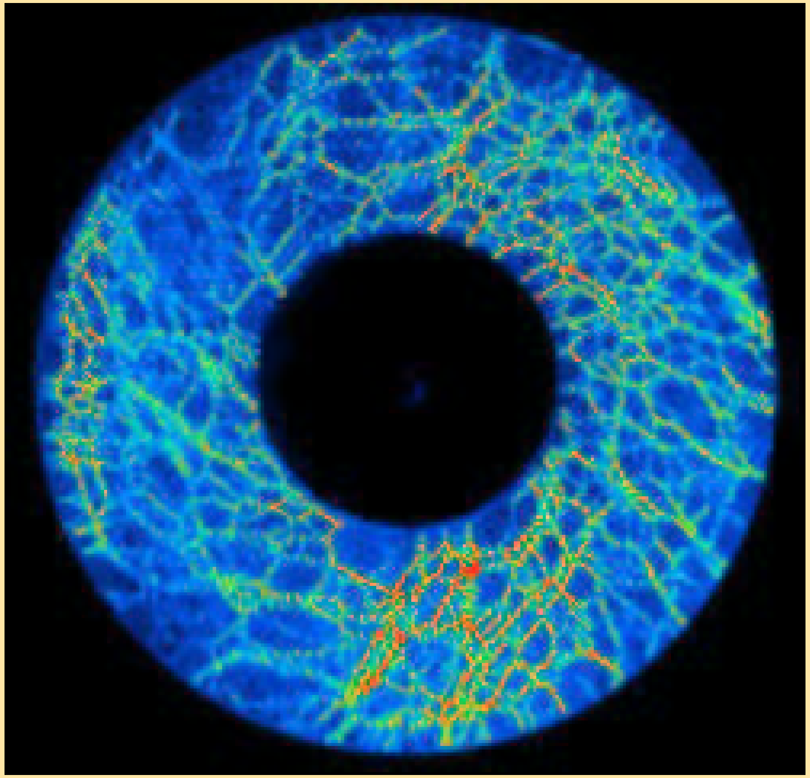
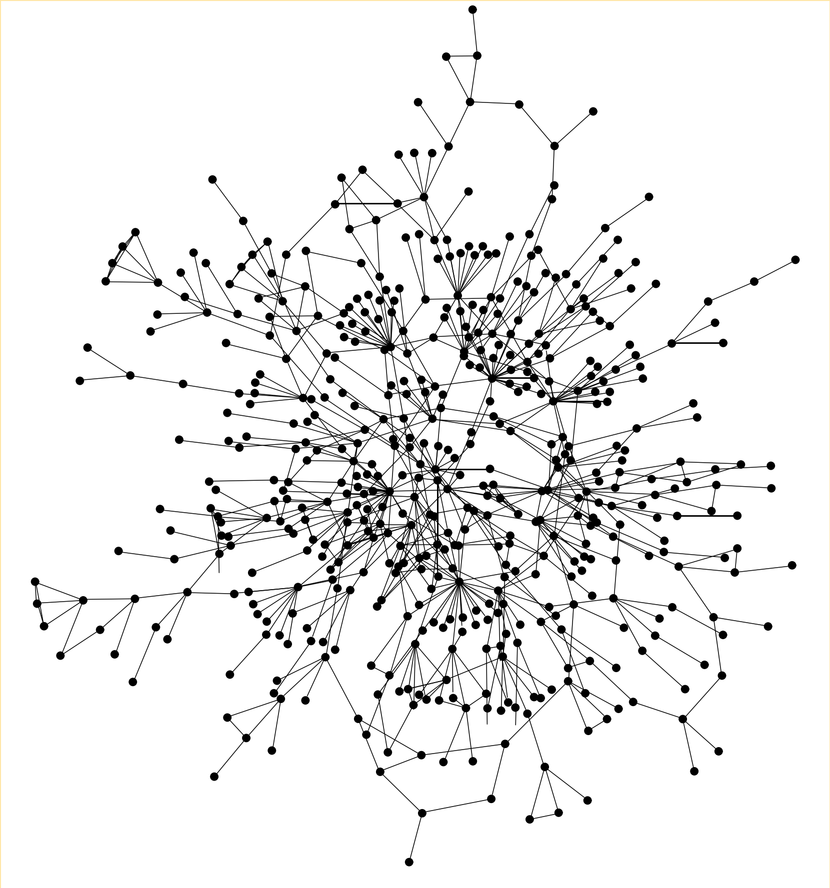
- Random networks: realizations and models; Spectra
- Hierarchical construction of scale-free network
- Green functions for all system sizes
- Asymptotic power-law tail
- Localization

Random networks

Granular matter*, Acoustics†, Scattering on graphs‡



Force chains in sheared sand.



Yeast *Saccharomyces Cerevisiae* proteome network.

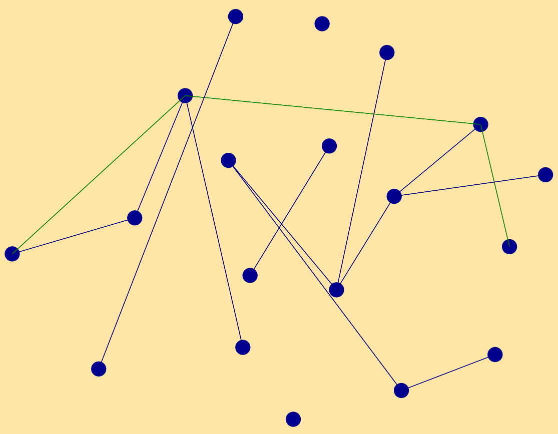
*X. Jia, C. Caroli, and B. Velicky *Phys. Rev. Lett.* **82**, 1863 (1999).

†B. Gutkin and U. Smilansky, Can one hear the shape of a graph? *J. Phys. A: Math. Gen.* **34**, 6061 (2001).

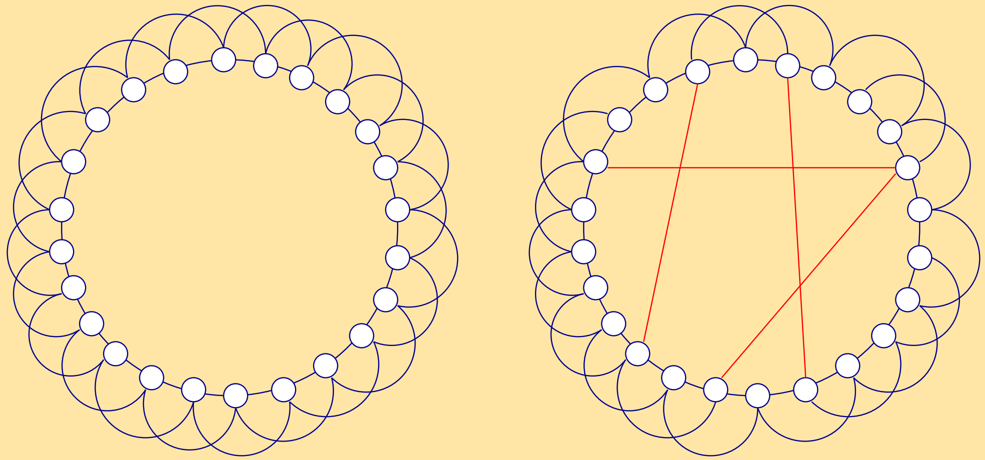
‡T. Kottos and U. Smilansky, *Phys. Rev. Lett.* **85**, 968 (2000); C. Texier and G. Montambaux *J. Phys. A: Math. Gen.* **34**, 10307, (2001).

Models of random graphs*

Erdös-Rényi: each edge is placed at random with probability p.



Small worlds of Watts and Strogatz†



Scale-free network: A.L. Barabási and R. Albert‡ (1) growth **and** (2) preferential attachment. New node with m edges, prob. attachment $W_s \propto am + k_s$.

Exact solution Dorogovtsev *et al.*¶.

$$P(k) = \frac{(1+a)\Gamma((m+1)a+1)\Gamma(q+ma)}{\Gamma(ma)\Gamma(k+2+(m+1)a)} \sim k^{-2-a}, \quad k \to \infty$$

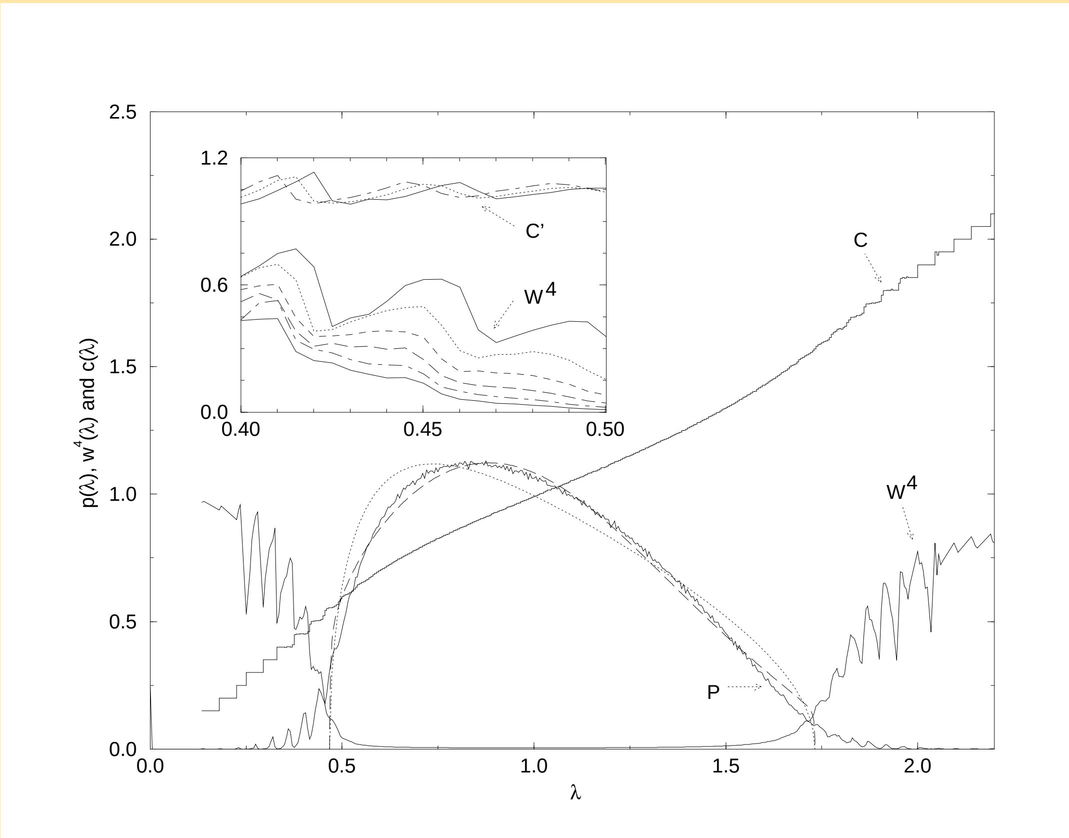
*B. Bolobás, *Random Graphs*, Academic Press, 1985.

†Watts and Strogatz, Nature 393, 440 (1998); D. J. Watts, *Small Worlds*, Princeton Univ. Press, 1999.

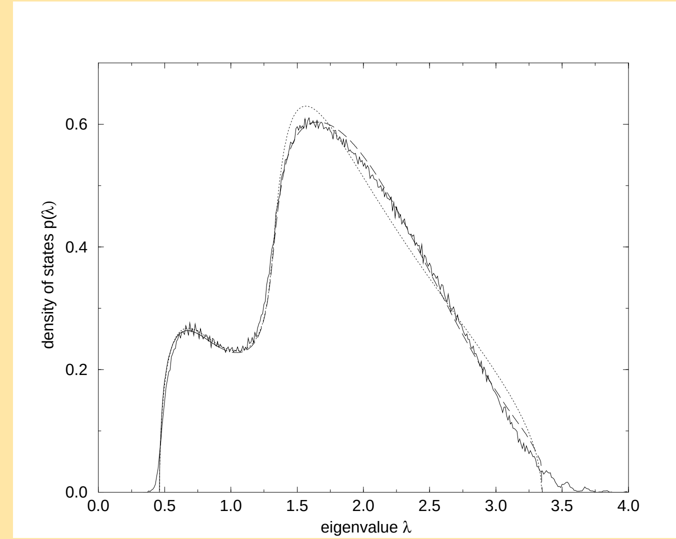
‡A.L. Barabási and R. Albert, Science 286, 509 (1999). R. Albert, H. Jeong, A.-L. Barabási, Nature 401, 130 (1999).

¶S. N. Dorogovtsev, J. F. F. Mendes, and A. N. Samukhin, Phys. Rev. Lett. 85, 4633 (2000).

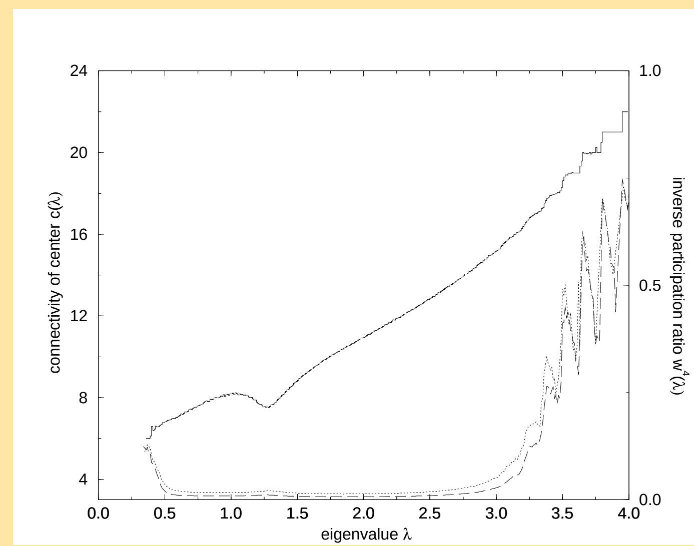
Spectra*



Random network.
Density of states $p(\lambda)$, inverse participation ratio $w^4(\lambda)$ and connectivity of the centers $c(\lambda)$ (divided by q) averaged over 2000 samples for $q = 20$, $N = 800$.



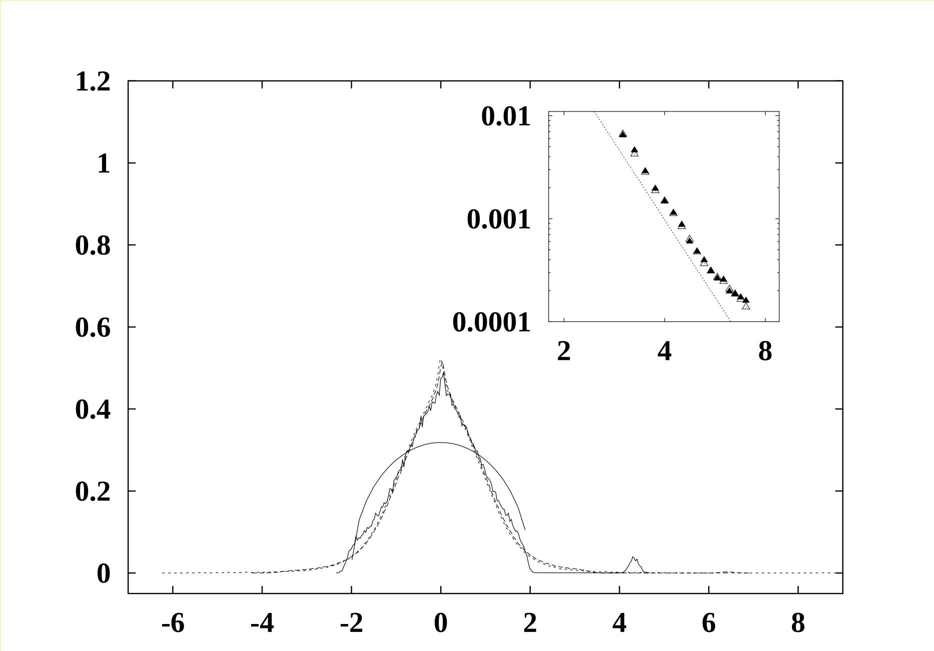
Small-world network $K = 3$, $q = 5$.
Density of states from numerics
, EMA and SDA approximation.



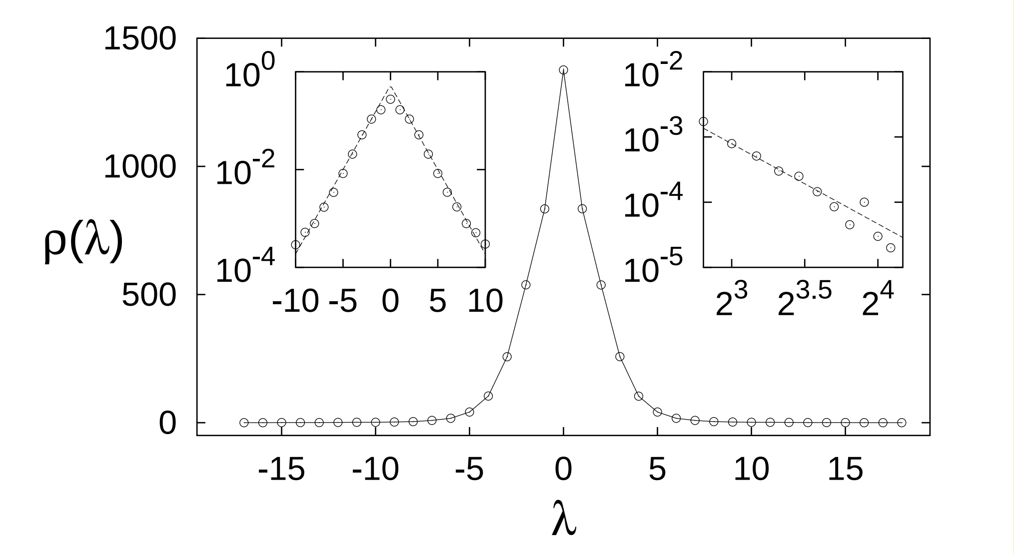
Inverse participation ratio for N = 256
and N = 512 averaged over 1000 samples.

*G. Biroli, R. Monasson, *J. Phys. A: Math. Gen.* **32**, L255 (1999); R. Monasson, *Eur. Phys. J. B* **12**, 555 (1999).

Scale-free networks*



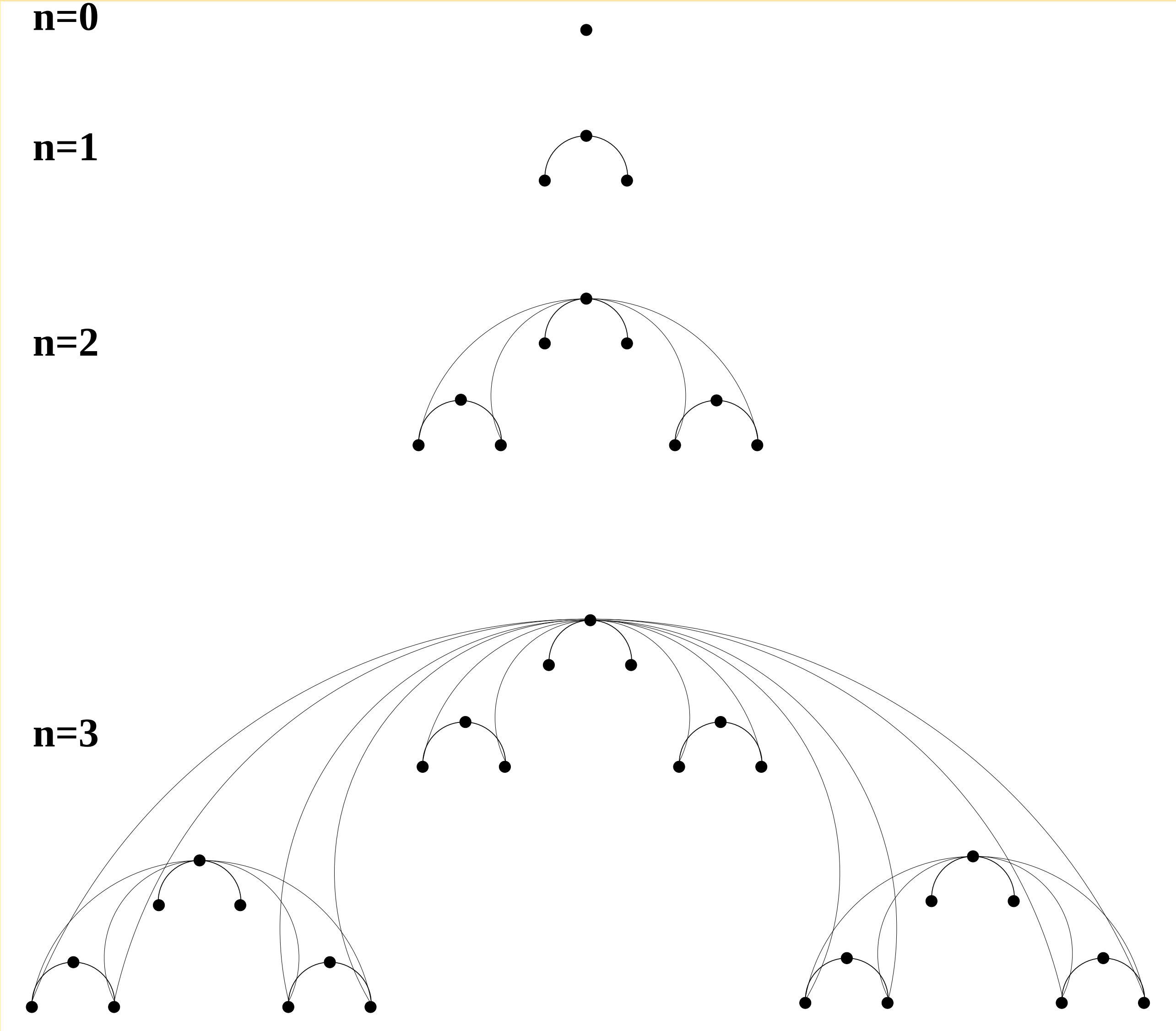
Density of states. Power-law tail.



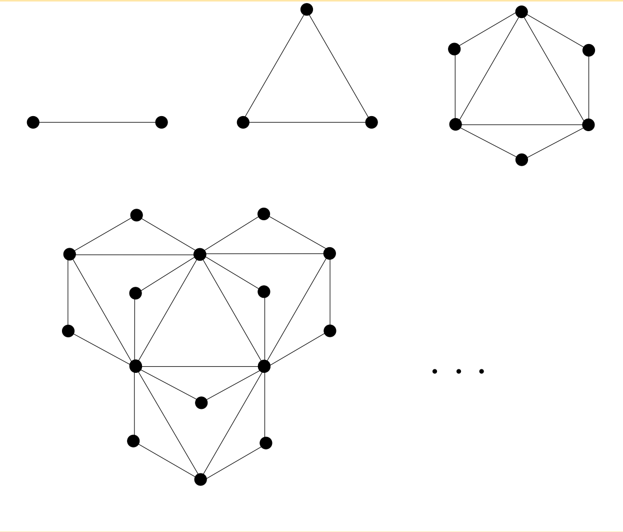
Density of states.

*Illes J. Farkas, Imre Derenyi, Albert-Laszlo Barabasi, Tamas Vicsek, *Physical Review E* **64**, 026704 (2001); K.-I. Goh, B. Kahng, and D. Kim, *Phys. Rev. E* **64**, 051903 (2001); S. Bilke and C. Peterson, *Phys. Rev. E* **64**, 036106 (2001).

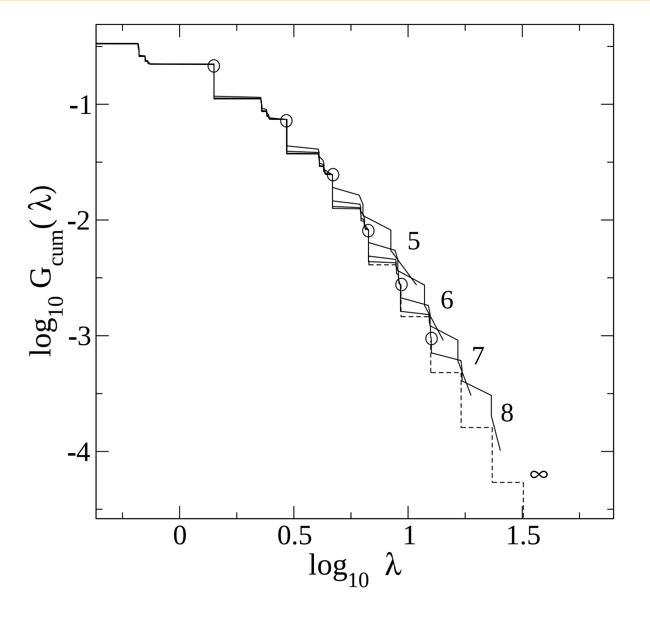
Hierarchical scale-free networks*



Hierarchical network construction.



Iterative construction



Cumulative density of states

*A.-L. Barabási, E. Ravasz, and T. Vicsek, *Physica A* **299**, 559 (2001) ; S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, *Phys. Rev. E* **65**, 066122 (2002); E. Ravasz, A.L. Somera, D.A. Mongru, Z.N. Oltvai, A.-L. Barabasi, *Science* **297**, 1551 (2002).

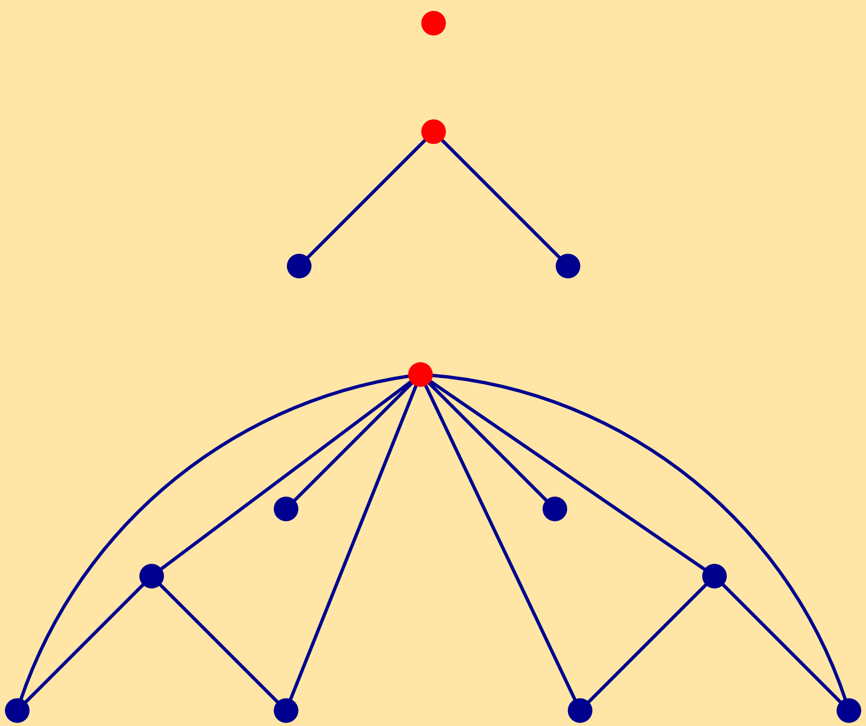
Iteration:

(i) make $p-1$ copies of the system;

(ii) connect all nodes of all the $p-1$ copies to first node of the old system

$$H_n = H_0 \oplus \underbrace{(H_1 \oplus \ldots \oplus H_1)}_{p-1 \text{ times}} \oplus \ldots \oplus \underbrace{(H_{n-1} \oplus \ldots \oplus H_{n-1})}_{p-1 \text{ times}} + V_n, \qquad [V_n]_{ij} = \delta_{i0} + \delta_{0j} - 2\delta_{i0}\delta_{0j}$$

example: p=3



$n = 0$

$$H_0 = 0$$

$n = 1$

$$H_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

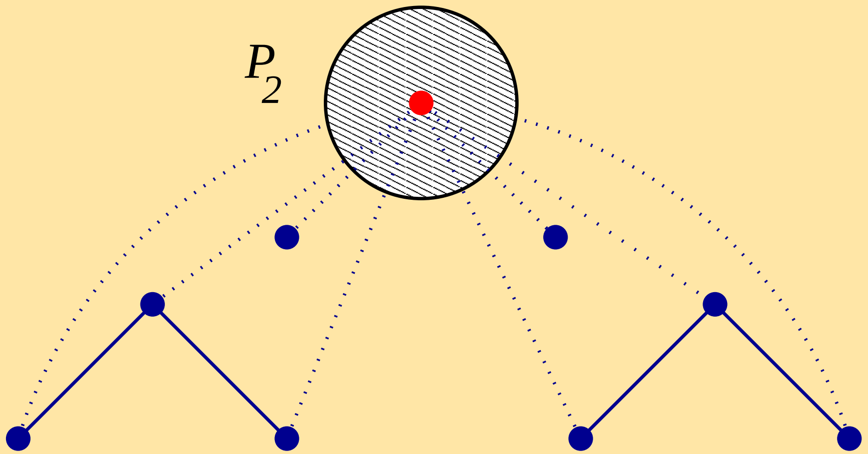
$n = 2$

$$H_2 = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Projection technique

Resolvent after n iterations $G_n(z) = (z - H_n)^{-1}$

Projector to first node: $[P_n]_{ij} = \delta_{i0}\delta_{j0}$, $Q_n = 1 - P_n$



Key quantity: $t_n(z) = \sum_i [G_n(z)]_{ii}$

$$g_n \equiv P_n G_n P_n = \frac{1}{z-\sigma_n}$$

Recurrence relation for self-energy: $$\sigma_{n+1} - \sigma_n = (p-1)\frac{1+(2+z)\sigma_n}{z-\sigma_n}$$

Auxiliary: $$u_{n+1} - u_n = (p-1)\frac{(1+\sigma_n)^2+(1+z)^2 u_n}{(z-\sigma_n)^2}$$

Trace: $$t_n = (p-1)\sum_{m=0}^{n-1} t_m + \frac{1+u_n}{z-\sigma_n}$$

Continuous limit

$p \to 1$, $n \to \infty$, $p^n = N$ fixed; $N \to 0$ at the end

$\xi = p^{-n}$, $\bar{z} = \xi^{1/2} z$, $\sigma_n(z) = \xi^{-1/2}\bar{\sigma}(\xi, \xi^{1/2} z)$, $u_n(z) = \bar{u}(\xi, \xi^{1/2} z)$,

recurrence relations become partial diffrential equations

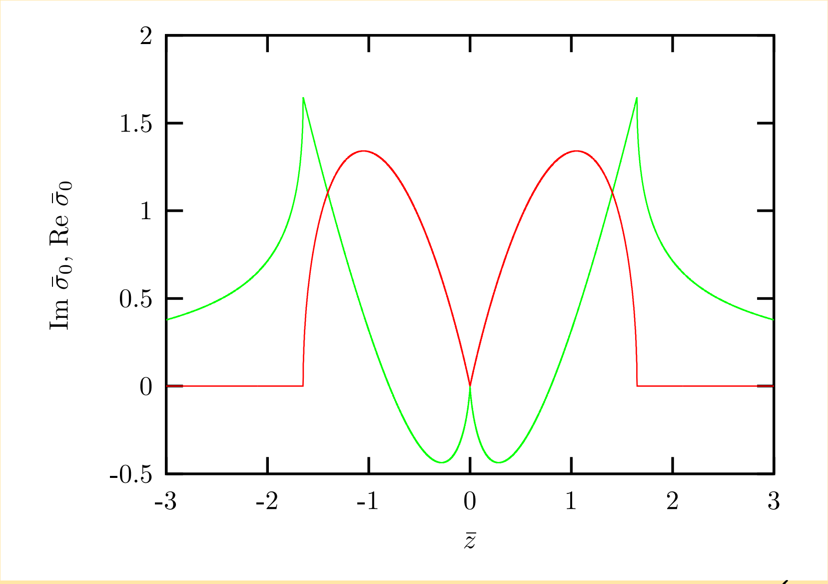
$$\xi \frac{\partial}{\partial \xi}\bar{\sigma} + \frac{1}{2}\bar{z}\frac{\partial}{\partial \bar{z}}\bar{\sigma} - \frac{1}{2}\bar{\sigma} + \frac{\xi + (2\sqrt{\xi} + \bar{z})\bar{\sigma}}{\bar{z} - \bar{\sigma}} = 0$$

$$\xi \frac{\partial}{\partial \xi}\bar{u} + \frac{1}{2}\bar{z}\frac{\partial}{\partial \bar{z}}\bar{u} + \frac{(\sqrt{\xi} + \bar{\sigma})^2 + (\sqrt{\xi} + \bar{z})^2 \bar{u}}{(\bar{z} - \bar{\sigma})^2} = 0$$

Density of states contain contributions from all sizes

$$\bar{t}(z) = \frac{2}{z^3}\int_0^z \frac{1 + \bar{u}\left(\left(\frac{\bar{z}}{z}\right)^2, \bar{z}\right)}{\bar{z} - \bar{\sigma}\left(\left(\frac{\bar{z}}{z}\right)^2, \bar{z}\right)} \bar{z}^2 \, \mathrm{d}\bar{z}$$

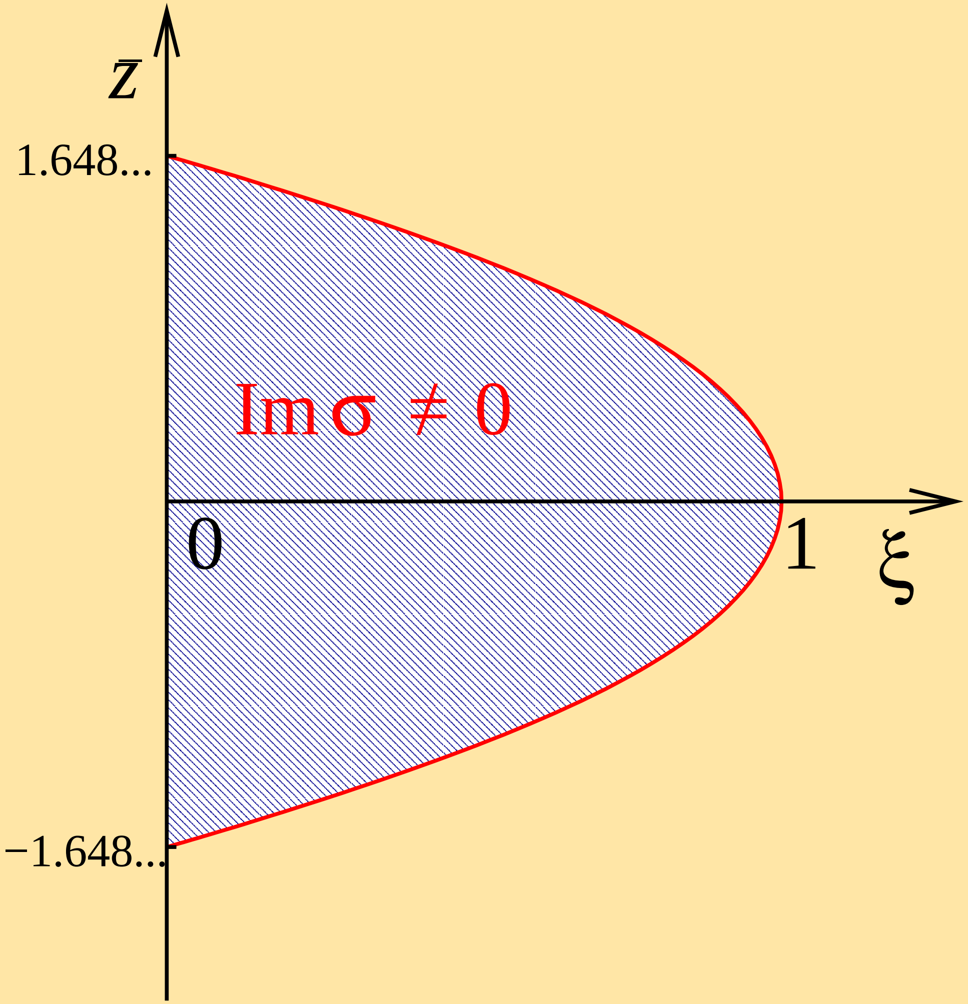
Solution*



Solution at $\xi = 0$: $\bar\sigma(0,\bar z) = -z W_L\left(-1/\bar z^2\right)$

$$\bar u(0,\bar z) = W_L\left(-1/\bar z^2\right)\frac{W_L\left(-1/\bar z^2\right)-1}{W_L\left(-1/\bar z^2\right)+1}$$

Band edge at $\bar z = \pm\sqrt{\mathrm{e}}$



Solution at arbitrary ξ: iterative integration: $\sigma(\xi,\bar z) = \sum_{l=0}^{\infty}\bar\sigma_l(\bar z)$

$$\bar\sigma_0(\bar z) = -z W_L\left(-1/\bar z^2\right)\ ,\ \bar\sigma_1(\bar z) = \frac{-\xi/\bar z + \sqrt{\xi}\,W_L(-1/\bar z^2)}{1+W_L(-1/\bar z^2)}\ ,\ \bar\sigma_2(\bar z)\ldots$$

*$W_L(x)$... Lambert function, solution of $W_L \exp(W_L) = x$.

Leading correction of order $\sqrt{\xi}$: express solution as series in $\xi^{1/2}$:

$\sigma(\xi,\bar{z}) = \bar{\sigma}_0(\bar{z}) + \sum_{k=1}^{\infty} \xi^{k/2} \tilde{\sigma}_k(\bar{z})$

ξ-expansion $\Longrightarrow$ $1/z$-expansion

$$\bar{t}(z) = \frac{2}{z^3}\int_0^z \frac{1+\bar{u}_0(\bar{z})}{\bar{z}-\bar{\sigma}_0(\bar{z})}\bar{z}^2\,\mathrm{d}\bar{z} + \frac{2}{z^4}\int_0^z \left[\frac{\tilde{u}_1(\bar{z})}{\bar{z}-\bar{\sigma}_0(\bar{z})} + \frac{(1+\bar{u}_0(\bar{z}))\tilde{\sigma}_1(\bar{z})}{(\bar{z}-\bar{\sigma}_0(\bar{z}))^2}\right]\bar{z}^3\,\mathrm{d}\bar{z} + \ldots$$

Density of states:

$$\boxed{\mathrm{Im}\,\bar{t}(\omega - \mathrm{i}\varepsilon) = \frac{2}{\omega^3}\int_0^{\sqrt{\mathrm{e}}} \mathrm{Im}\frac{1+W_L^2\left(-1/\bar{z}^2\right)}{\left(1+W_L\left(-1/\bar{z}^2\right)\right)^2}\,\bar{z}\,\mathrm{d}\bar{z} + O(\omega^{-4})}$$

Localization (towards...)

Key quantity: $\boxed{\Lambda(z,z') = \sum_i [G_n(z)]_{ii}[G_n(z')]_{ii}}$

4 functions $\Lambda_{\pm\pm}(\omega) = \Lambda(\omega \pm i\varepsilon, \omega \pm i\varepsilon)$

Inverse participation number:

$m(\omega) = -(2\pi)^{-2}\left(\Lambda_{--}(\omega) + \Lambda_{++}(\omega) - \Lambda_{-+}(\omega) - \Lambda_{+-}(\omega)\right)$

$$\Lambda_n = (p-1)\sum_{m=0}^{n-1}\Lambda_m + \frac{1+Y_n(z,z')}{(z-\sigma_n(z))(z'-\sigma_n(z'))} + \frac{X_n(z',z)}{z-\sigma_n(z)} + \frac{X_n(z,z')}{z'-\sigma_n(z')}$$

$$X_{n+1} - X_n = (p-1)\left(\frac{(1+\sigma_n(z'))^2}{(z-\sigma_n(z))(z'-\sigma_n(z'))^2} + \frac{(1+z')^2}{(z'-\sigma_n(z'))^2}X_n + \frac{(1+z')^2}{(z-\sigma_n(z))(z'-\sigma_n(z'))^2}Y_n\right)$$

$$Y_{n+1} - Y_n = (p-1)\left(\left(\frac{1+\sigma_n(z)}{z-\sigma_n(z)}\,\frac{1+\sigma_n(z')}{z'-\sigma_n(z')}\right)^2 + \left(\frac{1+z}{z-\sigma_n(z)}\,\frac{1+z'}{z'-\sigma_n(z')}\right)^2 Y_n\right)$$

After continuous limit (finite N, to be sent $\to \infty$ eventually):

for e. g. Λ_{+-}: (denote $\bar{\sigma}_{\pm}(\xi, \omega) = \bar{\sigma}_{\pm}(\xi, \omega \pm i\varepsilon)$)

$$\Lambda_{+-}(\omega) = \frac{2}{N\omega} \int_{\omega/N}^{\omega} \left[\frac{1 + \bar{Y}_{+-}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega})}{(\bar{\omega} - \bar{\sigma}_{+}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega}))(\bar{\omega} - \bar{\sigma}_{-}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega}))} \left(\frac{\bar{\omega}}{\omega}\right)^3 + \right.$$
$$\left. + \left(\frac{\bar{X}_{-+}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega})}{\bar{\omega} - \bar{\sigma}_{+}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega})} + \frac{\bar{X}_{+-}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega})}{\bar{\omega} - \bar{\sigma}_{-}((\frac{\bar{\omega}}{\omega})^2, \bar{\omega})} \right) \frac{\bar{\omega}}{\omega} \right] d\bar{\omega}$$

Asymptotic solution for $\bar{\omega} \to 0$: $\bar{Y} \sim \ln \bar{\omega}$, $\bar{X} \sim \bar{\omega}^{-1}$

$$\boxed{m(\omega) \sim \frac{\ln N}{N}}$$

Conclusions

- Spectrum of scale-free network calculated
- Power-law tail confirmed $\mathrm{Im}\,\overline{t}(\omega) \sim \omega^{-3}$
- Localization: “weak” — localized on small but infinite region $m(\omega) \sim \frac{\ln N}{N}$ (more precise analysis desirable)