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Geometrical and Topological Properties of Credal Set Operator

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Abstract

The credal set operator is studied as a set-valued mapping that assigns the set of dominating probabilities to a coherent lower prevision on some set of gambles. It is shown that this mapping is affine on certain classes of coherent lower previsions, which enables to find a decomposition of credal sets. Continuity of the credal set operator is investigated on finite universes with the aim of approximating credal sets.

1 Introduction

The main purpose of this paper is to investigate the geometrical-topological relations between the two important classes of imprecise probability models of Walley [12]: coherent lower previsions and credal sets of linear previsions. The credal set operator is studied as a set-valued mapping that sends every coherent lower prevision to the nonempty, weak*-compact and convex set of dominating linear previsions. Since the set of all coherent lower previsions is a convex subset of a linear topological space, the basic question is whether the credal set operator acts as a morphism between the corresponding mathematical objects. Precisely, the question is if the credal set operator is

(i) an *affine mapping*, that is, convex combinations of coherent lower previsions are mapped to the corresponding "convex combinations" of the credal sets,

(ii) *homeomorphism*, provided a topology is introduced on the set of all credal sets.

In Section 2 we introduce basic notions and notations. The main tool used in this paper are the elements of subdifferential (superdifferential) calculus developed for continuous convex (concave) functions [10]. Theorem 2 in Section 3 shows that every credal set can be represented as the superdifferential. This idea goes back to the solution of coalition games by core and appears already in Aubin's work [1]. Further, it is proven that the credal set operator is an affine mapping on the class of all coherent lower probabilities defined on the set of all subsets of some universe (Theorem 3) and on the class of all coherent lower previsions defined on the set of all the gambles (Corollary 1). It is demonstrated in section 3.1 how the former result can be used to obtain a decomposition of credal sets of belief measures.

Section 4 is devoted to the topological properties. The exposition is confined to the case of finite universes. If the Hausdorff metric is introduced on the set of all nonempty compact convex subset of the set of all linear previsions, then the credal set operator is a homeomorphism (Theorem 7). The consequence of this result mentioned in section 4.1 makes possible to approximate an arbitrary credal set by a "simple" credal set in the Hausdorff metric. The study of continuity of credal set operator need not be limited to finite universes. The principal difficulty in the general non-metrizable case is how to define a topology on the set of all nonempty, weak*-compact convex subsets of the dual of the Banach space of all gambles considered in its weak* topology. Only a brief discussion of this issue would, however, lead to introducing quite complicated mathematical apparatus such as uniformities defined on spaces of credal sets (cf. [2, Chapter II]). Since such considerations go far beyond the intended scope of the paper, the general case is left for separate future investigations.

2 Basic Notions

In this section we introduce the notation and repeat the notions and basic results from Walley's theory of imprecise probabilities [12]. Let Ω be a nonempty set. A *gamble* is a bounded function $\Omega \to \mathbb{R}$. If $a \in \mathbb{R}$, then we use the same symbol a to denote a constant gamble on Ω. By $\mathscr{L}$ we denote the Banach space of all gambles with the supremum norm $\|.\|_\infty$, that is,

$$\|f\|_\infty = \sup\{|f(\omega)| \mid \omega \in \Omega\}, \quad f \in \mathscr{L}.$$

Let $\mathscr{K} \subseteq \mathscr{L}$. A *lower prevision* $\underline{P}$ is a real function defined on $\mathscr{K}$. If the set $\mathscr{K}$ contains only characteristic functions of subsets of Ω, then $\underline{P}$ is called a *lower probability.* The *conjugate upper prevision* $\overline{P}$ is defined on $-\mathscr{K} = \{f \mid -f \in \mathscr{K}\}$ by letting $\overline{P}(f) = -\underline{P}(-f)$ for every $f \in -\mathscr{K}$. A *coherent lower prevision* on $\mathscr{L}$ is a lower prevision $\underline{P}$ defined on $\mathscr{L}$ that satisfies the following conditions for every $f, g \in \mathscr{L}$:

(i) $\underline{P}(f) \geq \inf\{f(\omega) \mid \omega \in \Omega\}$,

(ii) $\underline{P}(\lambda f) = \lambda \underline{P}(f)$, for every $\lambda \geq 0$,

(iii) $\underline{P}(f+g) \geq \underline{P}(f) + \underline{P}(g)$.

In particular, every coherent lower prevision on $\mathscr{L}$ is a continuous concave function on the Banach space $\mathscr{L}$. If $\underline{P}$ is a lower prevision defined on $\mathscr{K}$, then $\underline{P}$ is called *coherent* provided there exists a coherent lower prevision defined on $\mathscr{L}$ and coinciding with $\underline{P}$ on $\mathscr{K}$.

A *linear prevision* P on $\mathscr{L}$ is a self-conjugate coherent lower prevision on $\mathscr{L}$, that is, $P(-f) = -P(f)$ for every $f \in \mathscr{L}$. Every linear prevision P is a positive linear functional on $\mathscr{L}$ with $P(1) = 1$. A real functional defined on $\mathscr{K}$ is called a *linear prevision on* $\mathscr{K}$ if it can be extended to a linear prevision on $\mathscr{L}$. By $\mathscr{L}^*$ we denote the dual Banach space of $\mathscr{L}$: the elements of $\mathscr{L}^*$ are precisely the continuous linear functionals $\mathscr{L} \to \mathbb{R}$. Every linear prevision belongs to $\mathscr{L}^*$.

On the one hand, the sets of linear previsions appearing in the theory of imprecise probabilities are usually not compact in the norm topology of $\mathscr{L}^*$. On the other hand, if the Banach space $\mathscr{L}^*$ is considered with the weak* topology, then the set $\mathcal{P}$ of all linear previsions on $\mathscr{L}$ becomes a weak*-compact subset of $\mathscr{L}^*$ [12, p.610]. Let $\underline{P}$ be a coherent lower prevision on $\mathscr{K}$. The *credal set of* $\underline{P}$ is the set

$$\mathcal{M}(\underline{P}) = \{P \in \mathcal{P} \mid P(f) \geq \underline{P}(f), f \in \mathscr{K}\}.$$

The terminology is not unified so $\mathcal{M}(\underline{P})$ is called a *core* or a *structure* by some authors. The credal set $\mathcal{M}(\underline{P})$ is a nonempty, convex and weak*-compact subset of $\mathscr{L}^*$.

Given a coherent lower prevision $\underline{P}$ on $\mathscr{K}$, put

$$E_{\underline{P}}(f) = \inf\{P(f) \mid P \in \mathcal{M}(\underline{P})\}, \text{ for every } f \in \mathscr{L},$$

and call the function $E_{\underline{P}}$ the *natural extension of* $\underline{P}$. Every natural extension $E_{\underline{P}}$ is the (pointwise) smallest coherent lower prevision that extends $\underline{P}$ to the set $\mathscr{L}$.

3 Superdifferential of Coherent Lower Prevision

The notion of superdifferential of a continuous concave function is one of the generalizations of the classical concept of Gâteaux derivative of a differentiable function. In the next paragraph only basic definitions and results are needed. The reader is referred to [10] for details. Although the theory is developed for subdifferentials of convex functions in [10], the analogous definitions and theorems for superdifferentials of concave functions are derived straightforwardly.

Let X be a Banach space and E be a nonempty open convex subset of X. By X^* we denote the dual space of X. In this paragraph we always assume that φ is a *concave function* $E \to \mathbb{R}$: for every $x_1, x_2 \in E$ and every $\alpha \in [0,1]$, we have

$$\varphi(\alpha x_1 + (1-\alpha)x_2) \geq \alpha\varphi(x_1) + (1-\alpha)\varphi(x_2).$$

For every $x_0 \in E$ and $x \in X$, put

$$\mathrm{d}^+\varphi(x_0)(x) = \lim_{t \to 0_+} \frac{\varphi(x_0 + tx) - \varphi(x_0)}{t}$$

and call $\mathrm{d}^+\varphi(x_0)(x)$ the *right-hand directional derivative of* φ *at* x_0. If follows from [10, Lemma 1.2] that the limit defining $\mathrm{d}^+\varphi(x_0)(x)$ exists for every $x_0 \in E$ and every $x \in X$, and that $\mathrm{d}^+\varphi(x_0)$ is a positively homogeneous concave function on X. The function φ is said to be *Gâteaux differentiable at* x_0 if the functional $\mathrm{d}^+\varphi(x_0) : X \to \mathbb{R}$ is actually linear (not necessarily continuous). Equivalently, the function φ is Gâteaux differentiable at $x_0 \in E$ iff the limit

$$\mathrm{d}\varphi(x_0)(x) = \lim_{t \to 0} \frac{\varphi(x_0 + tx) - \varphi(x_0)}{t}$$

exists for each $x \in X$, and in this case $\mathrm{d}\varphi(x_0) = \mathrm{d}^+\varphi(x_0)$. The functional $\mathrm{d}\varphi(x_0)$ is the *Gâteaux derivative of* φ *at* x_0.

Definition 1. *Let* $x \in E$. *The* superdifferential of φ at x *is the set*

$$\partial\varphi(x) = \{\varphi^* \in X^* \mid \varphi^*(y) \geq \mathrm{d}^+\varphi(x)(y),\ y \in X\}.$$

The superdifferential of φ at x can be equivalently expressed as

$$\partial\varphi(x) = \{\varphi^* \in X^* \mid \varphi^*(y - x) \geq \varphi(y) - \varphi(x),\ y \in E\}. \tag{3.1}$$

The elements of $\partial\varphi(x)$ are called *supergradients of* φ *at* x. Each supergradient φ^* is viewed as a plausible candidate for a derivative of φ at x. The following existence result is well-known ([10, Proposition 1.11]).

Theorem 1. *Let* X *be a Banach space and* E *be a nonempty open convex subset of* X. *If the concave function* φ *is continuous at* $x \in E$, *then* $\partial\varphi(x)$ *is a nonempty, convex and weak*-compact subset of* X^*.

The next theorem enables to identify the credal set of $\underline{P}$ with the set of all supergradients at 1 of the natural extension of $\underline{P}$.

Theorem 2. *Let* $\mathscr{K} \subseteq \mathscr{L}$. *If* $\underline{P}$ *is a coherent lower prevision on* $\mathscr{K}$ *and* $E_{\underline{P}}$ *is the corresponding natural extension, then*

$$\mathcal{M}(\underline{P}) = \partial E_{\underline{P}}(1).$$

Moreover, if $E_{\underline{P}}$ *is Gâteaux differentiable at* 1, *then* $\underline{P}$ *is a linear prevision on* $\mathscr{K}$.

Proof. Let $P \in \mathcal{M}(\underline{P})$. Then $P \geq E_{\underline{P}}$ and $P(1) = 1 = E_{\underline{P}}(1)$, which implies for every gamble $f \in \mathscr{L}$ that

$$P(f-1) \geq E_{\underline{P}}(f) - E_{\underline{P}}(1).$$

Since every linear prevision is a norm continuous linear functional, the inequality above means that P is a supergradient of $E_{\underline{P}}$ at 1 by (3.1).

Suppose, on the other hand, that $P^* \in \partial E_{\underline{P}}(1)$. The equation (3.1) gives that for every gamble $f \in \mathscr{L}$ we have

$$P^*(f-1) \geq E_{\underline{P}}(f) - 1. \tag{3.2}$$

Hence for every real $\alpha > 0$,

$$P^*(\alpha f - 1) \geq E_{\underline{P}}(\alpha f) - 1,$$

and after dividing by α,

$$P^*(f) - \frac{P^*(1)}{\alpha} \geq E_{\underline{P}}(f) - \frac{1}{\alpha}.$$

Letting $\alpha \to 0$ leads to $P^*(f) \geq E_{\underline{P}}(f)$. If $f = 0$, then $P^*(-1) \geq -1$ from (3.2) so that $P^*(1) = 1$. The functional P^* is a linear prevision as P^* is self-conjugate and satisfies

$$P^*(f) \geq E_{\underline{P}}(f) \geq \inf\{f(\omega) \mid \omega \in \Omega\}$$

for every $f \in \mathscr{L}$. Since $E_{\underline{P}}(f) = \underline{P}(f)$ whenever $f \in \mathscr{K}$, we get $P^* \in \mathcal{M}(\underline{P})$.

To prove the second assertion, assume that $E_{\underline{P}}$ is Gâteaux differentiable at 1. It follows from [10, Proposition 1.8] that this is equivalent to

$$\partial E_{\underline{P}}(1) = \{\mathrm{d}E_{\underline{P}}(1)\}.$$

Since $\mathcal{M}(E_{\underline{P}}) = \partial E_{\underline{P}}(1)$, this means that the continuous concave function $E_{\underline{P}}$ is dominated by the unique continuous linear functional $\mathrm{d}E_{\underline{P}}(1)$. The Hahn-Banach theorem then implies that $E_{\underline{P}}$ itself must be linear and hence, a fortiori, $\underline{P}$ must be a linear prevision. $\square$

The second assertion of the previous theorem can not be reversed: if $\underline{P}$ is a linear prevision on $\mathscr{K}$, then the natural extension $E_{\underline{P}}$ is not in general Gâteaux differentiable at 1.

On the set $2^{\mathscr{L}^*}$ of all subsets of $\mathscr{L}^*$ we consider the multiplication of a set $\mathcal{A} \subseteq \mathscr{L}^*$ by a real number α and the (Minkowski) sum of sets $\mathcal{A}_1 \subseteq \mathscr{L}^*$ and $\mathcal{A}_2 \subseteq \mathscr{L}^*$:

$$\alpha\mathcal{A} = \{\alpha P^* \mid P^* \in \mathcal{A}\},$$
$$\mathcal{A}_1 \oplus \mathcal{A}_2 = \{P_1^* + P_2^* \mid P_1^* \in \mathcal{A}_1, P_2^* \in \mathcal{A}_2\}.$$

Let K_1, K_2 be convex subsets of linear spaces X_1, X_2, respectively. A mapping $a : K_1 \to K_2$ is *affine*, whenever for every convex combination $\sum_{i=1}^n \alpha_i x_i$ of elements $x_1, \ldots, x_n \in K_1$, we have

$$a\left(\sum_{i=1}^n \alpha_i x_i\right) = \sum_{i=1}^n \alpha_i a(x_i).$$

Let 2^Ω be the set of all subsets of Ω. A lower probability $\underline{P}$ on 2^Ω is *supermodular* if

$$\underline{P}(A \cup B) + \underline{P}(A \cap B) \geq \underline{P}(A) + \underline{P}(B)$$

for every $A, B \in 2^\Omega$.

Theorem 3. *If $\underline{P}^1, \ldots, \underline{P}^n$ are supermodular coherent lower probabilities on 2^Ω and $\alpha_i \in [0,1], i = 1, \ldots, n$, are such that $\sum_{i=1}^n \alpha_i = 1$, then*

$$\mathcal{M}\left(\sum_{i=1}^n \alpha_i \underline{P}^i\right) = \bigoplus_{i=1}^n \alpha_i \mathcal{M}(\underline{P}^i). \tag{3.3}$$

Proof. The lower probability $\sum_{i=1}^n \alpha_i \underline{P}^i$ is coherent [12, Theorem 2.6.4]. The coherent lower probability $\sum_{i=1}^n \alpha_i \underline{P}^i$ is supermodular since each $\underline{P}^i$ is supermodular, so the set of all supermodular coherent lower probabilities on 2^Ω is a convex subset of $\mathbb{R}^{2^\Omega}$. It follows from [8, Theorem 5.2] that the natural extension $E_{\underline{P}}$ of any supermodulat coherent lower probability $\underline{P}$ on 2^Ω coincides with the asymmetric Choquet integral $I^a_{\underline{P}} : \mathscr{L} \to \mathbb{R}$, where

$$\begin{aligned} I^a_{\underline{P}}(f) = &\int_{-\infty}^0 \underline{P}(f^{-1}((t,\infty))) - \underline{P}(\Omega)\ \mathrm{d}t \\ &+ \int_0^\infty \underline{P}(f^{-1}((t,\infty)))\ \mathrm{d}t, \end{aligned}$$

for every $f \in \mathscr{L}$. A routine verification shows that the mapping sending each supermodular coherent lower probability $\underline{P}$ to $I^a_{\underline{P}}$ is affine, hence

$$E_{\sum_{i=1}^n \alpha_i \underline{P}^i} = I^a_{\sum_{i=1}^n \alpha_i \underline{P}^i} = \sum_{i=1}^n \alpha_i I^a_{\underline{P}^i} = \sum_{i=1}^n \alpha_i E_{\underline{P}^i}$$

Theorem 2 together with the preceding equality give

$$\begin{aligned} \mathcal{M}\left(\sum_{i=1}^n \alpha_i \underline{P}^i\right) &= \partial\left(E_{\sum_{i=1}^n \alpha_i \underline{P}^i}\right)(1) = \\ &= \partial\left(\sum_{i=1}^n \alpha_i E_{\underline{P}^i}\right)(1). \end{aligned}$$

It follows directly from the definition of superdifferential that for every $i = 1, \ldots, n$,

$$\partial(\alpha_i E_{\underline{P}^i})(1) = \alpha_i \partial(E_{\underline{P}^i})(1). \tag{3.4}$$

By the Moreau-Rockafellar theorem [10, Theorem 3.23], the equality (3.4) and Theorem 2, we obtain

$$\begin{aligned} \partial\left(\sum_{i=1}^n \alpha_i E_{\underline{P}^i}\right)(1) &= \bigoplus_{i=1}^n \partial(\alpha_i E_{\underline{P}^i})(1) = \\ = \bigoplus_{i=1}^n \alpha_i \partial(E_{\underline{P}^i})(1) &= \bigoplus_{i=1}^n \alpha_i \mathcal{M}(\underline{P}^i). \end{aligned}$$

□

One of key ingredients in the above proof is the affinity of the natural extension operator $\underline{P} \mapsto E_{\underline{P}}$ derived from the representation of the natural extension by the assymetric Choquet integral [8, Theorem 5.2]. This suggests the following general result.

Theorem 4. *Let $\mathscr{K}$ be a set of gambles and $\underline{\mathcal{C}}_{\mathscr{K}}$ be the convex set of all coherent lower probabilities on $\mathscr{K}$. If the mapping*

$$\underline{P} \in \underline{\mathcal{C}}_{\mathscr{K}} \mapsto E_{\underline{P}}$$

is affine, then the equality (3.3) *holds true for every $\underline{P}^1, \ldots, \underline{P}^n \in \underline{\mathcal{C}}_{\mathscr{K}}$.*

Proof. Let $\underline{P}^1, \ldots, \underline{P}^n \in \underline{\mathcal{C}}_{\mathscr{K}}$ and $\alpha_i \in [0,1], i = 1, \ldots, n$, be such that $\sum_{i=1}^n \alpha_i = 1$. Then

$$E_{\sum_{i=1}^n \alpha_i \underline{P}^i} = \sum_{i=1}^n \alpha_i E_{\underline{P}^i},$$

so

$$\mathcal{M}\left(\sum_{i=1}^n \alpha_i \underline{P}^i\right) = \partial\left(\sum_{i=1}^n \alpha_i E_{\underline{P}^i}\right)(1),$$

and the equality (3) again follows from the Moreau-Rockafellar theorem [10, Theorem 3.23] together with (3.4) and Theorem 2. □

Let S be the set of all nonempty weak*-compact convex subsets of $\mathcal{P}$. In the sequel we will study the properties of the set-valued mapping

$$\mathcal{M}(.) : \underline{P} \mapsto \mathcal{M}(\underline{P})$$

that sends a coherent lower probability on some set of gambles $\mathscr{K}$ to a credal set from S. A superficial look at the equality (3.3) would then suggest that this mapping is affine on the class of coherent lower probabilities mentioned in Theorem 3. A necessary condition is that the codomain S of $\mathcal{M}$ is a convex set. But this notion of convexity is not even defined in the present framework since the set $2^{\mathscr{L}^*}$ endowed with the Minkowski sum of sets and the scalar multiplication of a set is not a linear space. The main difficulty is that the algebra $(2^{\mathscr{L}^*}, \oplus)$ is not a group but only a commutative monoid. The properties of the Minkowski sum and the scalar multiplication of sets defined above can be summarized as follows.

Proposition 1. *The set $2^{\mathscr{L}^*}$ together with the Minkowski sum $\oplus$ is a* real semilinear space*, that is:*

(i) *the algebra $(2^{\mathscr{L}^*}, \oplus)$ is a commutative monoid with the neutral element $\{0\}$,*
(ii) $\alpha(\beta\mathcal{A}) = (\alpha\beta)\mathcal{A}$*, for every $\alpha, \beta \in \mathbb{R}$ and every $\mathcal{A} \in 2^{\mathscr{L}^*}$,*
(iii) $1\mathcal{A} = \mathcal{A}$*,*
(iv) $0\mathcal{A} = \{0\}$*,*
(v) $\alpha(\mathcal{A}_1 \oplus \mathcal{A}_2) = (\alpha\mathcal{A}_1) \oplus (\alpha\mathcal{A}_2)$*, for every $\mathcal{A}_1, \mathcal{A}_2 \in 2^{\mathscr{L}^*}$.*

Semilinear spaces, which generalize linear spaces, are algebraic structures close to semirings [5]. The definitions of convexity and affine maps can be directly carried over to a more general framework of semilinear spaces. In that follows these generalized definitions are tacitly assumed. Thus we will say that S is *convex* (as a subset of $2^{\mathscr{L}^*}$) if

$$\alpha\mathcal{A}_1 \oplus (1-\alpha)\mathcal{A}_2 \in \mathsf{S}$$

holds true for every $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{S}$ and every $\alpha \in [0,1]$.

Proposition 2. *The set* S *is a convex subset of the real semilinear space* $2^{\mathscr{L}^*}$.

Proof. Consider any $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{S}$ and a real number $\alpha \in [0,1]$. Put $\mathcal{A} = \alpha\mathcal{A}_1 \oplus (1-\alpha)\mathcal{A}_2$. Then $\mathcal{A}$ is a nonempty convex subset of $\mathcal{P}$ since both $\mathcal{A}_1, \mathcal{A}_2$ are nonempty and convex. As both $\alpha\mathcal{A}_1$ and $(1-\alpha)\mathcal{A}_2$ are weak*-closed, their Minkowski sum $\mathcal{A}$ is a weak*-closed subset of $\mathcal{P}$, and thus weak*-compact. □

With these facts in mind, it is safe to interpret the conclusions of Theorem 3 and 4 as expressing the fact that "the mapping $\mathcal{M}$ is affine". We will show that the mapping $\mathcal{M}$ is an affine isomorphism[1] from the convex set $\underline{\mathcal{C}}$ of all coherent lower previsions on $\mathscr{L}$ to S. The essential result is the following theorem [12, Theorem 3.6.1].

Theorem 5 (Walley)**.** *The mapping* $\mathcal{M}$ *is a bijection from* $\underline{\mathcal{C}}$ *to* S. *The inverse mapping* $\mathcal{M}^{-1}$ *sends* $\mathcal{A} \in \mathsf{S}$ *to the coherent lower prevision*

$$\mathcal{M}^{-1}(\mathcal{A})(f) = \inf\{P(f) \mid P \in \mathcal{A}\}, \quad f \in \mathscr{L}.$$

Corollary 1. *The mapping* $\mathcal{M}$ *is an affine isomorphism of* $\underline{\mathcal{C}}$ *and* S.

Proof. The mapping $\mathcal{M}$ is one-to-one by Theorem 5. It suffices to show that $\underline{P} \in \underline{\mathcal{C}} \mapsto E_{\underline{P}}$ is affine since this gives the affinity of $\mathcal{M}$ by Theorem 4. However, this is trivial as $\underline{P} = E_{\underline{P}}$ for every $\underline{P} \in \underline{\mathcal{C}}$. □

Hence the mutual correspondence between the two different models of imprecise probabilities (coherent lower previsions and credal sets) introduced by Walley is retained also on the geometric level.

3.1 Decomposition of credal sets

Theorem 3 can be useful in situations in which a coherent lower probability $\underline{P}$ on 2^{Ω} is a convex combination of the coherent lower probabilities whose credal sets have a special shape (such as simplices). In this case, the credal set of $\underline{P}$ is decomposed into the convex combination of the respective "basic" credal sets. In particular, Theorem 3 is an infinite-dimensional generalization of Corollary 4 from [3], where a similar result is achieved for finite Ω and totally monotone set functions investigated in the framework of cooperative games. We will explicitly show how Theorem 3 can be applied to the credal sets of belief measures [11] by reformulating [3, Corollary 4] as a consequence of Theorem 3 in this paper.

Theorem 6. *Let* Ω *be finite,* $\underline{P}$ *be a belief measure on* 2^{Ω} *and* $\mu^{\underline{P}}$ *its Möbius transform. Then*

$$\mathcal{M}(\underline{P}) = \bigoplus_{A \subseteq \Omega} \mu^{\underline{P}}(A)\mathcal{S}_A,$$

where $\mathcal{S}_A$ *is the simplex of probabilities on* 2^{Ω} *supported by* A, *that is,* $\mathcal{S}_A = \{P \in \mathcal{P} \mid P(A) = 1\}$.

[1]An *affine isomorphism* is a bijective affine mapping between two convex subsets of real semilinear spaces. Its inverse is then necessarily an affine mapping too.

Proof. The set $\mathcal{S}_A$ is a simplex since it is a face of the simplex of all probabilities on 2^Ω. A belief measure $\underline{P}$ is a supermodular coherent lower probability on 2^Ω, so Theorem 3 can be employed. Since $\sum_{A\subseteq\Omega}\mu^{\underline{P}}(A)=1$, where $\mu^{\underline{P}}(A)\geq 0$ for each $A\subseteq\Omega$, and

$$\underline{P}=\sum_{A\subseteq\Omega}\mu^{\underline{P}}(A)\underline{P}_A,$$

where the set functions

$$\underline{P}_A(B)=\begin{cases}1, & A\subseteq B,\\ 0, & \text{otherwise},\end{cases}$$

are belief functions, it suffices to realize that $\mathcal{M}(\underline{P}_A)=\mathcal{S}_A$. $\square$

4 Continuity of Credal Set Mapping

The main purpose of this section is to study the topological properties of the credal set operator. We will confine the investigations to the case of finite Ω. The first necessary step is an introduction of topologies on both $\underline{\mathcal{C}}$ and S.

If $\Omega=\{1,\dots,n\}$, then the set of all gambles $\mathscr{L}$ can be identified with the Euclidean space $\mathbb{R}^n$. A gamble is then viewed as an n-dimensional vector $f=(f_1,\dots,f_n)\in\mathbb{R}^n$. The dual space $\mathscr{L}^*$ is identified with $\mathbb{R}^n$. If $\langle .,.\rangle$ denotes the usual scalar product on $\mathbb{R}^n$, then every linear prevision P on $\mathscr{L}$ canonically corresponds to the vector of reals $p=(p_1,\dots,p_n)$ such that $\langle p,1\rangle=1$ and $p_i\geq 0$ for each $i=1,\dots,n$. We have

$$P(f)=\langle p,f\rangle,\quad f\in\mathscr{L}. \tag{4.5}$$

The pointwise limit of coherent lower previsions on any set of gambles $\mathscr{K}$ is a coherent lower prevision on $\mathscr{K}$. Consequently, the set $\underline{\mathcal{C}}$ is a closed convex subset of the locally convex space $\mathbb{R}^{\mathscr{L}}$. Let $\|.\|$ be the Euclidean norm on $\mathbb{R}^n$. The topology of pointwise convergence on $\underline{\mathcal{C}}$ is described by the metric

$$\Delta(\underline{P}^1,\underline{P}^2)=\max\{|\underline{P}^1(f)-\underline{P}^2(f)|\mid \|f\|\leq 1\}.$$

Precisely, the sequence $(\underline{P}_n)$ in $\underline{\mathcal{C}}$ pointwise converges to $\underline{P}\in\underline{\mathcal{C}}$ iff $\Delta(\underline{P}_n,\underline{P})\to 0$ (see [7, Theorem 1.3.5,p.133]).

The set S contains all the nonempty compact convex subsets of

$$\mathcal{P}=\{p\in\mathbb{R}^n\mid \langle p,1\rangle=1, p_i\geq 0,\ i=1\dots,n\}.$$

The topology on S can be introduced by the Hausdorff metric [2, Chapter II]. For every $\mathcal{A}\in\mathsf{S}$ and every $p\in\mathcal{P}$, define

$$d_{\mathcal{A}}(p)=\min\{\|p-p'\|\mid p'\in\mathcal{A}\}. \tag{4.6}$$

If $\mathcal{A}_1,\mathcal{A}_2\in\mathsf{S}$, then denote

$$e_H(\mathcal{A}_1,\mathcal{A}_2)=\sup\{d_{\mathcal{A}_2}(p_1)\mid p_1\in\mathcal{A}_1\}.$$

The *Hausdorff metric* Δ_H on S is defined as

$$\Delta_H(\mathcal{A}_1, \mathcal{A}_2) = \max\{e_H(\mathcal{A}_1, \mathcal{A}_2), e_H(\mathcal{A}_2, \mathcal{A}_1)\},$$

for every $\mathcal{A}_1, \mathcal{A}_2 \in \mathsf{S}$.

The topology corresponding to the metric Δ_H is called the *Hausdorff metric topology.* The Hausdorff metric topology depends only on the topology of $\mathcal{P}$: if any metric equivalent to the Euclidean metric is used in place of $\|.\|$ in (4.6), the resulting metric topology on S would coincide with the Hausdorff metric topology. Indeed, it follows from [2, Theorem II-6] that the Hausdorff metric topology on the family K of nonempty compact subsets of $\mathcal{P}$ is generated by the sets

$$\{K \in \mathsf{K} \mid K \subseteq U,\ U \text{ open in } \mathcal{P}\}$$

and

$$\{K \in \mathsf{K} \mid K \cap V \neq \emptyset,\ V \text{ open in } \mathcal{P}\}.$$

The Hausdorff metric topology of S arises as a subspace topology from K. Hence it is immaterial if the Euclidean norm or the supremum norm originally defined on the space of gambles is used.

Theorem 7. *Let $\Omega = \{1, \ldots, n\}$. If S is endowed with the Hausdorff metric topology, then the mapping $\mathcal{M} : \underline{\mathcal{C}} \to \mathsf{S}$ is an affine isomorphism and homeomorphism.*

Proof. The mapping $\mathcal{M}$ is an affine isomorphism by Corollary 1 so that it remains to prove the continuity in both directions. To this end, we use the following convergence result, which can be easily deduced from [7, Corollary 3.3.8, p.156]: if $(\mathcal{A}_n)$ is a sequence in S and $\mathcal{A} \in \mathsf{S}$, then $\mathcal{A}_n \to \mathcal{A}$ in the Hausdorff metric iff the sequence of functions

$$((f \in \mathbb{R}^n \mapsto \inf\{\langle p, f\rangle \mid p \in \mathcal{A}_n\})_n)$$

pointwise converges to the function

$$f \in \mathbb{R}^n \mapsto \inf\{\langle p, f\rangle \mid p \in \mathcal{A}\}.$$

To show that the mapping $\mathcal{M}$ is continuous, consider a sequence $(\underline{P}_n)$ converging to $\underline{P}$ in $\underline{\mathcal{C}}$. Theorem 5 and (4.5) together yield

$$\underline{P}_n(f) = \mathcal{M}^{-1}(\mathcal{M}(\underline{P}_n))(f) = \inf\{\langle p, f\rangle \mid p \in \mathcal{M}(\underline{P}_n)\}$$

and

$$\underline{P}(f) = \mathcal{M}^{-1}(\mathcal{M}(\underline{P}))(f) = \inf\{\langle p, f\rangle \mid p \in \mathcal{M}(\underline{P})\}.$$

This implies $\mathcal{M}(\underline{P}_n) \to \mathcal{M}(\underline{P})$ in the Hausdorff metric. Continuity of the inverse mapping $\mathcal{M}^{-1}$ is shown similarly. □

4.1 Approximation of credal sets

By Theorem 5 of Walley every nonempty compact convex subset $\mathcal{A} \in \mathsf{S}$ is a credal set of the coherent lower prevision $\mathcal{M}^{-1}(\mathcal{A})$. Although every credal set is characterized by the Krein-Milman theorem as the closed convex hull of its vertices, it can be convenient to find a class of subsets of S whose members have a particular geometric structure and which is sufficient for an approximation of every credal set. The polytopes from S are natural candidates for such a task. A *polytope* is the convex hull of finitely-many points in $\mathbb{R}^n$. For our purposes it will be even enough to focus on so-called simple polytopes. A polytope is called *simple* if each of its vertices is contained in the same number of facets. For example, a cube or a simplex are simple polytopes, an Egyptian pyramid is not a simple polytope. It was proven in [9] that the credal set of every possibility measure is a simple polytope. The class of simple polytopes is considered to be computationally tractable: see [13] or the discussion in [9, p.243-244] and the references therein.

Theorem 8. *Let $\Omega = \{1, \ldots, n\}$ and S be endowed with the Hausdorff metric topology. If $\underline{P}$ is any coherent lower prevision on a set of gambles $\mathscr{K} \subseteq \mathbb{R}^n$, then there exists a sequence $(\mathcal{S}_n)$ of simple polytopes in S such that*

(i) $\mathcal{S}_n \to \mathcal{M}(\underline{P})$ in the Hausdorff metric,

(ii) $\mathcal{M}^{-1}(\mathcal{S}_n) \to \underline{P}$ pointwise on $\mathscr{K}$,

(iii) $\mathcal{M}^{-1}(\mathcal{S}_n) \to \underline{P}$ uniformly on each compact subset of $\mathscr{K}$.

Proof. (i) is basically Theorem 2.8 in [4], which says that simple polytopes form a dense set in S. The assertion (ii) follows from (i) in conjunction with Theorem 7: the sequence of coherent lower previsions $\mathcal{M}^{-1}(\mathcal{S}_n)$ pointwise converges to $\underline{P}$ on $\mathscr{K}$ as $\mathcal{M}^{-1}$ is continuous. The last assertion (iii) is a wellknown property of the convergence of concave functions $\mathbb{R}^n \to \mathbb{R}$ (see [7, Theorem B.3.1.4], for instance). $\square$

The proof of [4, Theorem 2.8] is based on a strong compactness argument: given any open cover of a polytope K by balls with a given diameter and with centers in the extreme boundary of K, there exists a finite refinement of this cover. The idea is analogous to inscribing a polygon into a circle. Hence the theorem does not give an algorithm for finding the convergent sequence of simple polytope. Nevertheless, at least in case that $\mathcal{M}(\underline{P})$ is a polytope, it is possible to explicitly find a simple polytope "sufficiently close to $\mathcal{M}(\underline{P})$" [13].

5 Conclusions

In this contribution we identified two main cases in which the credal set mapping is affine (Theorem 3 and Corollary 1). Yet none of them covers the whole variety of coherent lower previsions since "completeness" of their domains is required: the set of gambles is required to be the set of all events or the set of all the gambles. Theorem 4 then gives a sufficient condition enabling to get rid of those assumptions: it is the affinity of the natural extension operator. In general, the natural extension operator is not stable under the usual operations with imprecise probabilities: it need not preserve neither convex combinations nor limits of convergent sequences of coherent lower previsions [6, Section 5]. In future investigations our aim will be to single out the sets of gambles satisfying the assumption of Theorem 4 and to extend the material presented in Section 4 to infinite universes.

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