Sensitivity analysis of extreme inaccuracies in Gaussian Bayesian Networks

Miguel A. Gómez-Villegas and Paloma Maín
ma_gv@mat.ucm.es, pmain@mat.ucm.es
Departamento de Estadstica e Investigacin Operativa
Universidad Complutense de Madrid
Plaza de Ciencias 3,
28040 Madrid, Spain

Rosario Susi
rsusi@estad.ucm.es
Departamento de Estadstica e Investigacin Operativa III
Universidad Complutense de Madrid
Avda. Puerta de Hierro s/n,
28040 Madrid, Spain

Abstract

We present the behavior of a sensitivity measure defined to evaluate the impact of model inaccuracies over the posterior marginal density of the variable of interest, after the evidence propagation is executed, for extreme perturbations of parameters in Gaussian Bayesian networks. This sensitivity measure is based on the Kullback-Leibler divergence and yields different expressions depending on the type of parameter (mean, variance or covariance) to be perturbed. This analysis is useful to know the extreme effect of uncertainty about some of the initial parameters of the model in a Gaussian Bayesian network. These concepts and methods are illustrated with some examples.

Keywords: Gaussian Bayesian Network, Sensitivity analysis, Kullback-Leibler divergence.

1 Introduction

Bayesian network is a graphical probabilistic model that provides a graphical framework for complex domains with lots of inter-related variables. Among other authors, Bayesian networks have been studied, by Pearl (1988), Lauritzen (1996), Heckerman (1998) and Jensen (2001).

A sensitivity analysis in a Bayesian network is necessary to study how sensitive is the network's output to inaccuracies or imprecisions in the parameters of the initial network, and therefore to evaluate the network robustness.

In recent years, some sensitivity analysis techniques for Bayesian networks have been developed. In Discrete Bayesian networks Laskey (1995) presents a sensitivity analysis based on computing the partial derivative of a posterior marginal probability with respect to a given parameter, Coupé, et al. (2002) develop an efficient sensitivity analysis based on inference algorithms and Chan, et al. (2005) introduce a sensitivity analysis based on a distance measure. In Gaussian Bayesian networks Castillo, et al. (2003) present a sensitivity analysis based on symbolic propagation and Gómez-Villegas, et al. (2006) develop a sensitivity measure, based on the Kullback-Leibler divergence, to perform the sensitivity analysis.

In this paper, we study the behavior of the sensitivity measure presented by Gómez-Villegas, et al. (2006) for extreme inaccuracies (perturbations) of parameters that describe the Gaussian Bayesian network. To prove that this is a well-

defined measure we are interested in studying the sensitivity measure when one of the parameters is different from the original value. Moreover, we want to proof that if the value of one parameter is similar to the real value then the sensitivity measure is close to zero.
The paper is organized as follows. In Section 2 we briefly introduce definitions of Bayesian networks and Gaussian Bayesian networks, review how propagation in Gaussian Bayesian networks can be performed, and present our working example. In Section 3, we present the sensitivity measure and develop the sensitivity analysis proposed. In Section 4, we obtain the limits of the sensitivity measure for extreme perturbations of the parameters giving the behavior of the measure in the limit so as their interpretation. In Section 5, we perform the sensitivity analysis with the working example for some extreme imprecisions. Finally, the paper ends with some conclusions.

2 Gaussian Bayesian Networks and Evidence propagation

Definition 1 (Bayesian network)**.** A Bayesian network is a pair (G,P) where G is a directed acyclic graph (DAG), which nodes corresponding to random variables $\mathbf{X}=\{X_1,...,X_n\}$ and edges representing probabilistic dependencies, and $P=\{p(x_1|pa(x_1)),...,p(x_n|pa(x_n))\}$ is a set of conditional probability densities (one for each variable) where $pa(x_i)$ is the set of parents of node X_i in G. The set P defines the associated joint probability density as

$$p(\mathbf{x}) = \prod_{i=1}^{n} p(x_i|pa(x_i)). \qquad (1)$$

As usual we work with a variable of interest, so the network's output is the information about this variable of interest after the evidence propagation.

Definition 2 (Gaussian Bayesian network)**.** A Gaussian Bayesian network is a Bayesian network over $\mathbf{X}=\{X_1,...,X_n\}$ with a multivariate normal distribution $N(\mu,\Sigma)$, then the joint density is given by $f(\mathbf{x}) =$

$$= (2\pi)^{-n/2}|\Sigma|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)\right\}$$

where μ is the n-dimensional mean vector, Σ $n\times n$ the covariance matrix and $|\Sigma|$ the determinant of Σ.

The conditional density associated with X_i for $i = 1,...,n$ in equation (1), is the univariate normal distribution, with density

$$f(x_i|pa(x_i)) \sim N\left(\mu_i + \sum_{j=1}^{i-1}\beta_{ij}(x_j-\mu_j), \nu_i\right)$$

where β_{ij} is the regression coefficient of X_j in the regression of X_i on the parents of X_i, and $\nu_i = \Sigma_{ii} - \Sigma_{iPa(x_i)}\Sigma^{-1}_{Pa(xi)}\Sigma^{'}_{iPa(x_i)}$ is the conditional variance of X_i given its parents.

Different algorithms have been proposed to propagate the evidence of some nodes in Gaussian Bayesian networks. We present an incremental method, updating one evidential variable at a time (see Castillo, et al. 1997) based on computing the conditional probability density of a multivariate normal distribution given the evidential variable X_e.
For the partition $\mathbf{X} = (\mathbf{Y}, E)$, with $\mathbf{Y}$ the set of non-evidential variables, where $X_i \in \mathbf{Y}$ is the variable of interest, and E is the evidence variable, then, the conditional probability distribution of $\mathbf{Y}$, given the evidence $E = \{X_e = e\}$, is multivariate normal with parameters

$$\mu^{\mathbf{Y}|E=e} = \mu_{\mathbf{Y}} + \Sigma_{\mathbf{Y}E}\Sigma^{-1}_{EE}(e-\mu_E)$$

$$\Sigma^{\mathbf{Y}|E=e} = \Sigma_{\mathbf{YY}} - \Sigma_{\mathbf{Y}E}\Sigma^{-1}_{EE}\Sigma_{E\mathbf{Y}}$$

Therefore, the variable of interest $X_i \in \mathbf{Y}$ after the evidence propagation is

$$X_i|E=e \sim N(\mu_i^{\mathbf{Y}|E=e}, \sigma_{ii}^{\mathbf{Y}|E=e}) \equiv$$

$$\equiv N\left(\mu_i + \frac{\sigma_{ie}}{\sigma_{ee}}(e-\mu_e), \sigma_{ii} - \frac{\sigma^2_{ie}}{\sigma_{ee}}\right) \qquad (2)$$

where μ_i and μ_e are the means of X_i and X_e respectively before the propagation, σ_{ii} and σ_{ee} the variances of X_i and X_e respectively before propagating the evidence, and σ_{ie} the covariance between X_i and X_e before the evidence propagation.

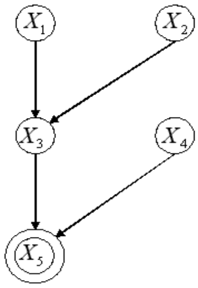

Figure 1: DAG of the Gaussian Bayesian network

We illustrate the concept of a Gaussian Bayesian network and the evidence propagation method with next example.

Example 1. Assume that we are interested in how a machine works. This machine is made up of 5 elements, the variables of the problem, connected as the network in Figure 1. Let us consider the time each element is working has a normal distribution and we are interested in the last element of the machine (X_5).

Being $\mathbf{X} = \{X_1, X_2, X_3, X_4, X_5\}$ normally distributed, $\mathbf{X} \sim N(\mu, \Sigma)$, with parameters

$$\mu = \begin{pmatrix} 2 \\ 3 \\ 3 \\ 4 \\ 5 \end{pmatrix}; \Sigma = \begin{pmatrix} 3 & 0 & 6 & 0 & 6 \\ 0 & 2 & 2 & 0 & 2 \\ 6 & 2 & 15 & 0 & 15 \\ 0 & 0 & 0 & 2 & 4 \\ 6 & 2 & 15 & 4 & 26 \end{pmatrix}$$

Considering the evidence $E = \{X_2 = 4\}$, after evidence propagation we obtain that $X_5|X_2 = 4 \sim N(6, 24)$ and the joint distribution is normal with parameters

$$\mu^{\mathbf{Y}|X_2=4} = \begin{pmatrix} 2 \\ 4 \\ 4 \\ 6 \end{pmatrix};$$

$$\Sigma^{\mathbf{Y}|X_2=4} = \begin{pmatrix} 3 & 6 & 0 & 6 \\ 6 & 13 & 0 & 13 \\ 0 & 0 & 2 & 4 \\ 6 & 13 & 4 & 24 \end{pmatrix}$$

3 Sensitivity Analysis and non-influential parameters

The Sensitivity Analysis proposed is as follows: Let us suppose the model before propagating the evidence is $N(\mu, \Sigma)$ with one evidential variable X_e, whose value is known. After propagating this evidence we obtain the marginal density of interest $f(x_i|e)$. Next, we add a *perturbation* δ to one of the parameters in the model before propagating the evidence (this parameter is supposed to be inaccurate thus δ reflects this inaccuracy) and perform the evidence propagation, to get $f(x_i|e, \delta)$. In some cases, the perturbation δ has some restrictions to get admissible parameters.
The effect of adding the perturbation is computed by comparing those density functions by means of the sensitivity measure. That measure is based on the Kullback-Leibler divergence between the target marginal density obtained after evidence propagation, considering the model with and without the perturbation

Definition 3 (Sensitivity measure)**.** Let (G,P) be a Gaussian Bayesian network $N(\mu, \Sigma)$. Let $f(x_i|e)$ be the marginal density of interest after evidence propagation and $f(x_i|e, \delta)$ the same density when the perturbation δ is added to one parameter of the initial model. Then, the sensitivity measure is defined by

$$S^{p_j}(f(x_i|e), f(x_i|e, \delta)) =$$
$$= \int_{-\infty}^{\infty} f(x_i|e) \ln \frac{f(x_i|e)}{f(x_i|e, \delta)} dx_i \qquad (3)$$

where the subscript p_j is the inaccurate parameter and δ the proposed perturbation, being the new value of the parameter $p_j^\delta = p_j + \delta$.

For small values of the sensitivity measure we can conclude our Bayesian network is robust for that perturbation.

3.1 Mean vector inaccuracy

Three different situations are possible depending on the element of μ to be perturbed, i.e., the perturbation can affect the mean of the variable of interest $X_i \in \mathbf{Y}$, the mean of the evidence variable $X_e \in E$, or the mean of any other variable $X_j \in \mathbf{Y}$ with $j \neq i$. Developing the sensitivity measure (3), we obtain next propositions,

Proposition 1. *For the perturbation $\delta \in \Re$ in the mean vector μ, the sensitivity measure is as follows*

• *When the perturbation is added to the mean of the variable of interest, $\mu_i^\delta = \mu_i + \delta$, and $X_i|E = e, \delta \sim N(\mu_i^{\mathbf{Y}|E=e} + \delta, \sigma_{ii}^{\mathbf{Y}|E=e})$,*

$$S^{\mu_i}(f(x_i|e), f(x_i|e,\delta)) = \frac{\delta^2}{2\sigma_{ii}^{\mathbf{Y}|E=e}}.$$

• *If the perturbation is added to the mean of the evidential variable, $\mu_e^\delta = \mu_e + \delta$, the posterior density of the variable of interest, with the perturbation, is $X_i|E = e, \delta \sim N(\mu_i^{\mathbf{Y}|E=e} - \frac{\sigma_{ie}}{\sigma_{ee}}\delta, \sigma_{ii}^{\mathbf{Y}|E=e})$, then*

$$S^{\mu_e}(f(x_i|e), f(x_i|e,\delta)) = \frac{\delta^2}{2\sigma_{ii}^{\mathbf{Y}|E=e}}\left(\frac{\sigma_{ie}}{\sigma_{ee}}\right)^2.$$

• *The perturbation δ added to the mean of any other non-evidential variable, different from the variable of interest, has no influence over X_i, then $f(x_i|e,\delta) = f(x_i|e)$, and the sensitivity measure is zero.*

3.2 Covariance matrix inaccuracy

There are six possible different situations, depending on the parameter of the covariance matrix Σ to be changed; three if the perturbation is added to the variances (elements in the diagonal of Σ) and other three if the perturbation is added to the covariances of Σ. When the covariance matrix is perturbed, the structure of the network can change. Those changes are presented in the precision matrix of the perturbed network, that is, the inverse of the covariance matrix with perturbation δ. To guarantee the normality of the network it is necessary $\Sigma^{\mathbf{Y}|E=e,\delta}$ to be a positive definite matrix in all cases presented in next proposition

Proposition 2. *Adding the perturbation $\delta \in \Re$ to the covariance matrix Σ, the sensitivity measure obtained is*

• *If the perturbation is added to the variance of the variable of interest, being $\sigma_{ii}^\delta = \sigma_{ii} + \delta$ with $\delta > -\sigma_{ii} + \frac{\sigma_{ie}^2}{\sigma_{ee}}$, then $X_i|E = e, \delta \sim N(\mu_i^{\mathbf{Y}|E=e}, \sigma_{ii}^{\mathbf{Y}|E=e,\delta})$ where $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} = \sigma_{ii} + \delta - \frac{\sigma_{ie}^2}{\sigma_{ee}}$ and $S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) =$*

$$= \frac{1}{2}\left[\ln\left(1 + \frac{\delta}{\sigma_{ii}^{\mathbf{Y}|E=e}}\right) - \frac{\delta}{\sigma_{ii}^{\mathbf{Y}|E=e,\delta}}\right].$$

• *When the perturbation is added to the variance of the evidential variable, being $\sigma_{ee}^\delta = \sigma_{ee} + \delta$ and $\delta > -\sigma_{ee}(1 - \max_{X_j \in \mathbf{Y}} \rho_{je}^2)$ with ρ_{je} the corresponding correlation coefficient, the posterior density of interest is $X_i|E = e, \delta \sim N(\mu_i^{\mathbf{Y}|E=e,\delta}, \sigma_{ii}^{\mathbf{Y}|E=e,\delta})$ with $\mu_i^{\mathbf{Y}|E=e,\delta} = \mu_i + \frac{\sigma_{ie}}{\sigma_{ee} + \delta}(e - \mu_e)$ and $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} = \sigma_{ii} - \frac{\sigma_{ie}^2}{\sigma_{ee} + \delta}$ therefore*

$$S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) = \frac{1}{2}\left[\ln\left(\frac{\sigma_{ii}^{\mathbf{Y}|E=e,\delta}}{\sigma_{ii}^{\mathbf{Y}|E=e}}\right) + \right.$$
$$\left. + \frac{\frac{\sigma_{ie}^2}{\sigma_{ee}}\left(\frac{-\delta}{\sigma_{ee}+\delta}\right)\left(1 + (e-\mu_e)^2\left(\frac{-\delta}{(\sigma_{ee}+\delta)\sigma_{ee}}\right)\right)}{\sigma_{ii}^{\mathbf{Y}|E=e,\delta}}\right].$$

• *The perturbation δ added to the variance of any other non-evidential variable $X_j \in \mathbf{Y}$ with $j \neq i$, $\sigma_{jj}^\delta = \sigma_{jj} + \delta$, has no influence over X_i, therefore $f(x_i|e,\delta) = f(x_i|e)$ and the sensitivity measure is zero.*

• *When the covariance between the variable of interest X_i and the evidential variable X_e is perturbed, $\sigma_{ie}^\delta = \sigma_{ie} + \delta = \sigma_{ei}^\delta$ and $-\sigma_{ie} - \sqrt{\sigma_{ii}\sigma_{ee}} < \delta < -\sigma_{ie} + \sqrt{\sigma_{ii}\sigma_{ee}}$, then $X_i|E = e, \delta \sim N(\mu_i^{\mathbf{Y}|E=e,\delta}, \sigma_{ii}^{\mathbf{Y}|E=e,\delta})$ with $\mu_i^{\mathbf{Y}|E=e,\delta} = \mu_i + \frac{(\sigma_{ie}+\delta)}{\sigma_{ee}}(e - \mu_e)$ and $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} = \sigma_{ii} - \frac{(\sigma_{ie}+\delta)^2}{\sigma_{ee}}$ the sensitivity measure obtained is*

$S^{\sigma_{ie}}(f(x_i|e), f(x_i|e,\delta)) =$

$$= \frac{1}{2}\left[\ln\left(1-\frac{\delta^2+2\sigma_{ie}\delta}{\sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e}}\right)+\right.$$

$$\left.+\frac{\sigma_{ii}^{\mathbf{Y}|E=e}+\left(\frac{\delta}{\sigma_{ee}}(e-\mu_e)\right)^2}{\sigma_{ii}^{\mathbf{Y}|E=e,\delta}}-1\right].$$

• *If we add the perturbation to any other covariance, i.e., between the variable of interest X_i and any other non-evidential variable X_j or between the evidence variable X_e and $X_j \in \mathbf{Y}$ with $j \neq i$, the posterior probability density of the variable of interest X_i is the same as without perturbation and therefore the sensitivity measure is zero.*

4 Extreme behavior of the Sensitivity Measure

When using the sensitivity measure, that describes how sensitive is the variable of interest when a perturbation is added to a inaccurate parameter, it would be interesting to know how is the sensitivity measure when the perturbation $\delta \in \Re$ is extreme. Then, we study the behavior of that measure for extreme perturbations so as the limit of the sensitivity measure.

Next propositions present the results about the limits of the sensitivity measures in all cases given in Propositions 1 and 2,

Proposition 3. *When the perturbation added to the mean vector is extreme, the sensitivity measure is as follows,*

1. • $\lim_{\delta \longrightarrow \pm\infty} S^{\mu_i}(f(x_i|e), f(x_i|e,\delta)) = \infty$
   • $\lim_{\delta \longrightarrow 0} S^{\mu_i}(f(x_i|e), f(x_i|e,\delta)) = 0$

2. • $\lim_{\delta \longrightarrow \pm\infty} S^{\mu_e}(f(x_i|e), f(x_i|e,\delta)) = \infty$
   • $\lim_{\delta \longrightarrow 0} S^{\mu_e}(f(x_i|e), f(x_i|e,\delta)) = 0.$

Proof. It follows directly. □

Proposition 4. *When the extreme perturbation is added to the elements of the covariance matrix and the correlation coefficient of X_i and X_e is $0 < \rho_{ie}^2 < 1$, the results are,*

1. • $\lim_{\delta \longrightarrow \infty} S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) = \infty$ *but* $S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) = o(\delta)$

   • $\lim_{\delta \longrightarrow M_{ii}} S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) = \infty$ *with* $M_{ii} = -\sigma_{ii} + \frac{\sigma_{ie}^2}{\sigma_{ee}} = -\sigma_{ii}(1-\rho_{ie}^2)$

   • $\lim_{\delta \longrightarrow 0} S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) = 0$

2. • $\lim_{\delta \longrightarrow \infty} S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) =$
   $= \frac{1}{2}\left[-ln(1-\rho_{ie}^2) - \rho_{ie}^2\left(1-\frac{(e-\mu_e)^2}{\sigma_{ee}}\right)\right]$

   • $\lim_{\delta \longrightarrow M_{ee}} S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) =$
   $= \frac{1}{2}\left[ln\left(\frac{M_{ee}^* - \rho_{ie}^2}{M_{ee}^*(1-\rho_{ie}^2)}\right) + \frac{\rho_{ie}^2(1-M_{ee}^*)}{M_{ee}^*-\rho_{ie}^2}\left(1+\frac{(e-\mu_e)^2}{\sigma_{ee}}\left(\frac{1-M_{ee}^*}{M_{ee}^*}\right)\right)\right]$
   where $M_{ee} = -\sigma_{ee}(1-M_{ee}^*)$ *being* $M_{ee}^* = \max_{X_j}\rho_{je}^2$

   • $\lim_{\delta \longrightarrow 0} S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) = 0$

3. • $\lim_{\delta \longrightarrow M_{ie}^1} S^{\sigma_{ie}}(f(x_i|e), f(x_i|e,\delta)) = \infty$

   • $\lim_{\delta \longrightarrow M_{ie}^2} S^{\sigma_{ie}}(f(x_i|e), f(x_i|e,\delta)) = \infty$
   with $M_{ie}^1 = -\sigma_{ie} - \sqrt{\sigma_{ii}\sigma_{ee}}$ *and* $M_{ie}^2 = -\sigma_{ie} + \sqrt{\sigma_{ii}\sigma_{ee}}$

   • $\lim_{\delta \longrightarrow 0} S^{\sigma_{ie}}(f(x_i|e), f(x_i|e,\delta)) = 0.$

Proof. 1. • It follows directly.

• When $\sigma_{ii}^\delta = \sigma_{ii} + \delta$ the new variance of X_i is $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} = \sigma_{ii}^{\mathbf{Y}|E=e} + \delta$.
Being $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} > 0$ then $\delta > -\sigma_{ii}^{\mathbf{Y}|E=e}$.
Naming $M_{ii} = -\sigma_{ii}^{\mathbf{Y}|E=e}$ and considering $x = \sigma_{ii}^{\mathbf{Y}|E=e} + \delta$ we have

$\lim_{\delta \longrightarrow M_{ii}} S^{\sigma_{ii}}(f(x_i|e), f(x_i|e,\delta)) =$
$= \lim_{x \longrightarrow 0} \frac{1}{2x}\left[x\ln(x) - x\ln(\sigma_{ii}^{\mathbf{Y}|E=e}) - x + \right.$
$\left. + \sigma_{ii}^{\mathbf{Y}|E=e}\right] = \infty.$

- It follows directly.

2. - $\lim_{\delta \longrightarrow \infty} S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) =$
$= \frac{1}{2}\left[\ln\left(\frac{\sigma_{ii}}{\sigma_{ii}^{\mathbf{Y}|E=e}}\right) + \frac{\frac{-\sigma_{ie}^2}{\sigma_{ee}}\left(1 - \frac{(e-\mu_e)^2}{\sigma_{ee}}\right)}{\sigma_{ii}}\right]$
with $\sigma_{ii}^{\mathbf{Y}|E=e} = \sigma_{ii}(1-\rho_{ie}^2)$ and $\rho_{ie}^2 = \frac{\sigma_{ie}^2}{\sigma_{ii}\sigma_{ee}}$ the limit is
$= \frac{1}{2}\left[-ln(1-\rho_{ie}^2) - \rho_{ie}^2\left(1 - \frac{(e-\mu_e)^2}{\sigma_{ee}}\right)\right].$

   - When $\sigma_{ee}^{\delta} = \sigma_{ee} + \delta$, the new conditional variance for all non evidential variables is $\sigma_{jj}^{\mathbf{Y}|E=e,\delta} = \sigma_{jj} - \frac{\sigma_{je}^2}{\sigma_{ee}+\delta}$ for all $X_j \in \mathbf{Y}$.
If we impose $\sigma_{jj}^{\mathbf{Y}|E=e,\delta} > 0$ for all $X_j \in \mathbf{Y}$ then δ must satisfy next condition $\delta > -\sigma_{ee}(1 - \max_{X_j \in \mathbf{Y}} \rho_{je}^2)$.
Naming $M_{ee}^* = \max_{X_j} \rho_{je}^2$ and $M_{ee} = -\sigma_{ee}(1-M_{ee}^*)$ then we have
$\lim_{\delta \longrightarrow M_{ee}} S^{\sigma_{ee}}(f(x_i|e), f(x_i|e,\delta)) =$
$= \lim_{\delta \longrightarrow M_{ee}} \frac{1}{2}\left[\ln\left(\frac{\sigma_{ii} - \frac{\sigma_{ie}^2}{\sigma_{ee}+\delta}}{\sigma_{ii} - \frac{\sigma_{ie}^2}{\sigma_{ee}}}\right) + \right.$
$\left. \frac{\frac{\sigma_{ie}^2}{\sigma_{ee}}\left(\frac{-\delta}{\sigma_{ee}+\delta}\right)\left(1 + (e-\mu_e)^2\left(\frac{-\delta}{(\sigma_{ee}+\delta)\sigma_{ee}}\right)\right)}{\sigma_{ii} - \frac{\sigma_{ie}^2}{\sigma_{ee}+\delta}}\right]$
with $\rho_{ie}^2 = \frac{\sigma_{ie}^2}{\sigma_{ii}\sigma_{ee}}$
$= \frac{1}{2}\left[\ln\left(\frac{\sigma_{ii}\sigma_{ee}M_{ee}^* - \sigma_{ie}^2}{M_{ee}^*(\sigma_{ii}\sigma_{ee} - \sigma_{ie}^2)}\right) + \right.$
$\left. + \frac{\frac{\sigma_{ie}^2}{\sigma_{ee}}\left(\frac{1-M_{ee}^*}{M_{ee}^*}\right)\left(1 + \frac{(e-\mu_e)^2}{\sigma_{ee}}\left(\frac{1-M_{ee}^*}{M_{ee}^*}\right)\right)}{M_{ee}^* - \rho_{ie}^2}\right] =$
$= \frac{1}{2}\left[ln\left(\frac{M_{ee}^* - \rho_{ie}^2}{M_{ee}^*(1-\rho_{ie}^2)}\right) + \frac{\rho_{ie}^2(1-M_{ee}^*)}{M_{ee}^* - \rho_{ie}^2}\right.$
$\left.\left(1 + \frac{(e-\mu_e)^2}{\sigma_{ee}}\left(\frac{1-M_{ee}^*}{M_{ee}^*}\right)\right)\right].$

   - It follows directly.

3. - If we make $\sigma_{ie}^{\delta} = \sigma_{ie} + \delta$, the new conditional variance is $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} = \sigma_{ii}^{\mathbf{Y}|E=e} - \frac{\delta^2 + 2\delta\sigma_{ie}}{\sigma_{ee}}$.
Then, if we impose $\sigma_{ii}^{\mathbf{Y}|E=e,\delta} > 0$, δ must satisfy the next condition $-\sigma_{ie} - \sqrt{\sigma_{ii}\sigma_{ee}} < \delta < -\sigma_{ie} + \sqrt{\sigma_{ii}\sigma_{ee}}$.
First, naming $M_{ie}^2 = -\sigma_{ie} + \sqrt{\sigma_{ii}\sigma_{ee}}$, we calculate $\lim_{\delta \longrightarrow M_{ie}^2} S^{\sigma_{ie}}(f(x_i \mid e), f(x_i \mid e,\delta))$.
But $\delta \rightarrow M_{ie}^2$ is equivalent to $(\delta^2 + 2\delta\sigma_{ie}) \rightarrow \sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e}$ and given that
$S^{\sigma_{ie}}(f(x_i \mid e), f(x_i \mid e,\delta)) =$
$= \frac{1}{2}\left[\ln\left(\frac{\sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e} - (\delta^2 + 2\delta\sigma_{ie})}{\sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e}}\right) + \right.$
$\left. + \frac{\sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e} + \left(\frac{\delta}{\sigma_{ee}}(e-\mu_e)\right)^2}{\sigma_{ee}\sigma_{ii}^{\mathbf{Y}|E=e} - (\delta^2 + 2\delta\sigma_{ie})} - 1\right]$
and as $\lim_{x \longrightarrow 0}\left[\ln x + \frac{k}{x}\right] = \infty$ for every k, then we get $\lim_{\delta \longrightarrow M_{ie}^2} S^{\sigma_{ie}}(f(x_i \mid e), f(x_i \mid e,\delta)) = \infty$.

   - Analogously, the other limit is also $\lim_{\delta \longrightarrow M_{ie}^1} S^{\sigma_{ie}}(f(x_i \mid e), f(x_i \mid e,\delta)) = \infty$.

   - It follows directly.

□

The behavior of the sensitivity measure is the expected one, except when the extreme perturbation is added to the evidential variance, because with known evidence, the posterior density of interest with the perturbation in the model $f(x_i|e,\delta)$ is not so different of the posterior density of interest without the perturbation $f(x_i|e)$, therefore although an extreme perturbation added to the evidential variance can exist, the sensitivity measure tends to a finite value.

5 Experimental results

Example 2. Consider the Gaussian Bayesian network given in Example 1. Experts disagree with definition of the variable of interest X_5, then the mean could be $\mu_5^{\delta_1} = 2 = \mu_5 + \delta_1$ (with $\delta_1 = -3$), the variance could be $\sigma_{55}^{\delta_2} = 24$ with $\delta_2 = -2$ and the covariances between X_5 and evidential variable X_2 could be $\sigma_{52}^{\delta_3} = 3$ with $\delta_3 = 1$ (the same to σ_{25}); with the variance of the evidential variable being $\sigma_{22}^{\delta_4} = 4$ with $\delta_4 = 2$ and between X_5 and other non-evidential variable X_3 that could be $\sigma_{53}^{\delta_5} = 13$ with $\delta_5 = -2$ (the same to σ_{35}); moreover, there are different opinions about X_3, because they suppose that μ_3 could be $\mu_3^{\delta_6} = 7$ with $\delta_6 = 4$, that σ_{33} could be $\sigma_{33}^{\delta_7} = 17$ with $\delta_7 = 2$, and that σ_{32} could be $\sigma_{32}^{\delta_8} = 1$ with $\delta_8 = -1$ (the same to σ_{23}).

The sensitivity measure for those inaccuracy parameters yields
$S^{\mu_5}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_1)) = 0.1875$
$S^{\sigma_{55}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_2)) = 0.00195$
$S^{\sigma_{52}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_3)) = 0.00895$
$S^{\sigma_{22}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_4)) = 0.00541$
$S^{\sigma_{53}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_5)) = 0$
$S^{\mu_3}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_6)) = 0$
$S^{\sigma_{33}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_7)) = 0$
$S^{\sigma_{32}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_8)) = 0$
As these values of the sensitivity measures are small we can conclude that the perturbations presented do not affect the variable of interest and therefore the network can be considered robust. Also, the inaccuracies about the non-evidential variable X_3 do not disturb the posterior marginal density of interest, being the sensitivity measure zero in all cases. If experts think that the sensitivity measure obtained for the mean of the variable of interest is large enough then they should review the information about this variable.
Moreover, we have implemented an algorithm (see Appendix 1) to compute all the sensitivity measures that can influence over the variable of interest X_i; this algorithm compare those sensitivity measures computed with a threshold s fixed by experts. Then, if the sensitivity measure is larger than the threshold, the parameter should be reviewed.

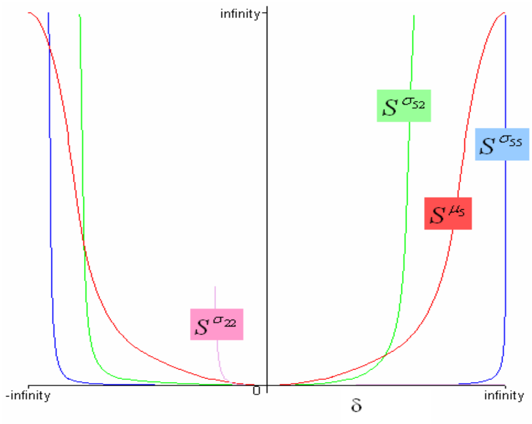

Figure 2: Sensitivity measures obtained in the example for any perturbation value

The extreme behavior of the sensitivity measure for some particular cases, is given as follows
When $\delta_1 = 20$, the sensitivity measure is $S^{\mu_5}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_1)) = 8.33$ and with the perturbation $\delta_1 = -25$, $S^{\mu_5}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_1)) = 13.02$.
If the perturbation $\delta_2 = 1000$, $S^{\sigma_{55}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_2)) = 1.39$ therefore as the perturbation increases to infinity the sensitivity measure grows very slowly, in fact $S^{\sigma_{55}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_2)) = o(\delta)$ as stated before. However if $\delta_2 = -23$, the sensitivity measure is $S^{\sigma_{55}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_2)) = 9.91$.
We do not present the sensitivity measure when δ_3 is extreme because δ_3 must be in $(-\sqrt{2}, \sqrt{2})$ to keep the covariance matrix of the network with the perturbation δ_3 positive definite.
Finally, when $\delta_4 = 100$, the sensitivity measure is $S^{\sigma_{22}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_4)) = 0.02$ (where the limit of $S^{\sigma_{22}}$ when δ tends to infinity is 0.0208) and with $\delta_4 = -1.73$, $S^{\sigma_{22}}(f(x_5|X_2 = 4), f(x_5|X_2 = 4, \delta_4)) = 2.026$ (being the limit when δ tends to M_{ee} 2.1212).
In Figure 2, we observed the sensitivity measures, considered as a function of δ; the graph shows the behavior of the measure when $\delta \in \Re$.

6 Conclusions

In this paper we study the behavior of the sensitivity measure, that compares the marginal density of interest when the model of the Gaussian Bayesian network is described with and without a perturbation $\delta \in \Re$, when the perturbation is extreme. Considering a large perturbation the sensitivity measure is large too except when the extreme perturbation is added to the evidence variance. Therefore, although the evidence variance were large and different from the variance in the original model, the sensitivity measure would be limited by a finite value, that is because the evidence about this variable explains the behavior of the variable of interest regardless its inaccurate variance.

Moreover, in all possible cases of the sensitivity measure, if the perturbation added to a parameter tends to zero, the sensitivity measure is zero too.

The study of the behavior of the sensitivity measure is useful to prove that this is a well-defined measure to develop a sensitivity analysis in Gaussian Bayesian networks even if the proposed perturbation is extreme.

The posterior research is focused on perturbing more than one parameter simultaneously so as with more than one variable of interest.

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Appendix 1

The algorithm that computes the sensitivity measures and determines the parameters in the network is available at the URL:
http://www.ucm.es/info/eue/pagina/
/APOYO/RosarioSusiGarcia/S_algorithm.pdf

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