What is it Hybrid DSm Rule for Combination of Belief Functions?*

Milan Daniel
Institute of Computer Science
Academy of Sciences of the Czech Republic
milan.daniel@cs.cas.cz

Abstract

This contribution is dealing with DSm theory of belief functions. Two different versions of hybrid DSm combination rule are presented. Inconsistencies of the newer version are observed and its corrections are suggested. New DSmH rules, which produce more specified results are suggested, and a schema, which covers a large family of combination rules derived from the generalized conjunctive combination rule, is presented.

1 Introduction

Belief functions are one of the widely used formalisms for uncertainty representation and processing. Belief functions enable representation of incomplete and uncertain knowledge, belief updating and combination of evidence. Originally belief functions were introduced as a principal notion of Dempster-Shafer Theory (DST) or the Mathematical Theory of Evidence [14].

For combination of beliefs Dempster's rule of combinations is used in DST. Under strict probabilistic assumptions its results are correct and probabilistically interpretable for any couple of belief functions. Nevertheless these assumptions are rarely fulfilled in real applications. There are not rare examples where the assumptions are not fulfilled and where results of Dempster's rule are counter intuitive, e.g. see [2, 15], thus a rule with more intuitive results is required in such situations.

Hence series of modifications of Dempster's rule were suggested and alternative approaches were created. The classical ones are Dubois-Prade's rule [11] and Yager's belief combination rule [20]. Among the others a wide class of weighted operators [13], Smets' Transferable Belief Model (TBM) using so called non-normalized Dempster's rule [18], disjunctive (or dual Dempster's)

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rule of combination [10], combination 'per elements' with its special case — minC combination, see [3, 5], and other combination rules. It is also necessary to mention the method for application of Dempster's rule in the case of partially reliable input beliefs [12].

A new approach provides the *Dezert-Smarandache (or Dempster-Shafer modified) theory* (DSmT) [8, 15] which allows overlapping of elements of a frame of discernment. Similarly to DST exhaustive frames of discernment are assumed.

In the special case of possible mutual overlapping of all the elements of the frame of discernment we refer to the *free DSm model* $\mathcal{M}^f$. As there are no conflicts among beliefs when using the free DSm model $\mathcal{M}^f$, we can use the conjunctive rule of combination, called the *classic DSm rule of combination* (DSmC) in DSmT, without any complications there.

In a general case some elements of a frame of discernment can overlap, whereas some other can be exclusive or recognized to be quite impossible, i.e., we admit some *exclusivity* or *non-existential constraints*. We speak about general *hybrid DSm model* $\mathcal{M}$ in such a case with (a) constraint(s). Thus conflicting belief masses can appear when combining belief on a general hybrid DSm model with (a) constraint(s). We use the *hybrid DSm combination rule* (DSmH) on general hybrid DSm models[1].

As another special example we can consider the classic case where all elements of a frame of discernment are allowed and they are all mutually exclusive, we refer to *Shafer's DSm model* $\mathcal{M}^0$ in this special case of a hybrid DSm model.

A new version of expression for DSmH rule has appeared in [9, 16] in 2005. Moreover we can notice that there is not only a different formula for belief combination, but there are also different results of combination. Thus there are two different hybrid DSm rules of combination in fact. We refer to DSmH_{02} the original version from 2002 [8, 15] and to DSmH_{05} the new version [9, 16] from 2005 to distinguish them.

A reason of a difference between both the DSmH rules is identified and discussed in the present contribution. When combining criteria for conflicting belief masses reallocation from both DSmH_{02} and DSmH_{05} rules we can define several other "refined" hybrid DSm rules. The criteria for conflicting belief masses reallocation are discussed on examples and some important open problems concerning the criteria are formulated in this contribution.

A question of interpretation of conflict redistribution criteria from DSmH_{02} and DSmH_{05} has been also opened. The original DSmH_{02} has a rational interpretation, whereas an reasonable interpretation of DSmH_{05} remains as an open problem for Dezert & Smarandache. Moreover the original DSmH_{02} fully corresponds to the generalized Dubois-Prade rule, whenever the generalization is possible, and to extended generalized Dubois-Prade rule in a fully general case of a dynamic fusion, see [4], for full text see [6]. DSmH_{05} seems to be an ad-hoc combination rule.

[1] New Proportional Conflict Redistribution (PCR) rules for combination of belief functions on hybrid DSm models have appeared recently, see [16, 17].

As belief representation and combination is performed on a mathematical structure of distributive lattice in DSmT, we have to mention also another related approaches, e.g. the approach by Besnard and his collaborators [1] and the minC combination [3] which was also generalized to hybrid DSm models, see [5, 7].

2 Preliminaries — Basic definitions

An exhaustive finite *frame of discernment* $\Theta = \{\theta_1, ..., \theta_n\}$, whose elements are mutually exclusive, is assumed in the classic Dempster-Shafer theory.

A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Theta) \longrightarrow [0, 1]$, such that $\sum_{A\subseteq\Theta} m(A) = 1$, the values of bba are called *basic belief masses (bbm)*. The value $m(A)$ is called the *basic belief mass (bbm) of A*.[2] A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Theta) \longrightarrow [0, 1]$, $bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$, belief function Bel uniquely corresponds to bba m and vice-versa. $\mathcal{P}(\Theta)$ is often denoted also by 2^Θ. A *focal element* is a subset X of the frame of discernment Θ, such that $m(X) > 0$. If a focal element is a one-element subset of Θ, we are referring to a *singleton*.

3 Introduction to the DSm theory

Because DSmT is a new theory which is in permanent dynamic evolution, we have to note that this text is related to its state described by formulas and text presented in the basic publication on DSmT — in the DSmT book Vol. 1 [15]. Rapid development of the theory is demonstrated by forthcoming second volume of the book. For new advances of DSmT see the second volume [17].

3.1 Dedekind lattice, basic DSm notions

DSm theory (Dempster-Shafer modified theory[3] or Dezert-Smarandache theory) by Dezert and Smarandache [8, 15] allows mutually overlapping elements of a frame of discernment. Thus, a frame of discernment is a finite exhaustive set of elements $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$, but not necessarily exclusive in DSmT. As an example, we can introduce a three-element set of colours $\{Red, Green, Blue\}$ from the DSmT homepage[4]. DSmT allows that an object can have 2 or 3 colours at the same time: e.g. it can be both red and blue, or red and green and blue in the same time, it corresponds to a composition of the colours from the 3 basic ones.

[2] $m(\emptyset) = 0$ is often assumed in accordance with Shafer's definition [14]. A classical counter example is Smets' Transferable Belief Model (TBM), see e.g. [18], which admits positive $m(\emptyset)$ as it assumes $m(\emptyset) \geq 0$.

[3] A motivation for DSmT is an endeavour to develop a more general theory than Dempster-Shafer theory is. A theory which contains a lot of different tools and techniques for various tasks of belief functions processing.

[4] www.gallup.unm.edu/∼smarandache/DSmT.htm

DSmT uses basic belief assignments and belief functions defined analogically to the classic Dempster-Shafer theory (DST), but they are defined on a so-called hyper-power set or Dedekind lattice instead of the classic power set of the frame of discernment. To be distinguished from the classic definitions, they are called generalized basic belief assignments and generalized basic belief functions.

The *Dedekind lattice*, more frequently called *hyper-power set* D^Θ in DSmT, is defined as the set of all composite propositions built from elements of Θ with union and intersection operators $\cup$ and $\cap$ such that $\emptyset, \theta_1, \theta_2, ..., \theta_n \in D^\Theta$, and if $A, B \in D^\Theta$ then also $A \cup B \in D^\Theta$ and $A \cap B \in D^\Theta$, no other elements belong to D^Θ ($\theta_i \cap \theta_j \neq \emptyset$ in general, $\theta_i \cap \theta_j = \emptyset$ iff $\theta_i = \emptyset$ or $\theta_j = \emptyset$).

Thus the hyper-power set D^Θ of Θ is closed to $\cup$ and $\cap$ and $\theta_i \cap \theta_j \neq \emptyset$ in general. Whereas the classic power set 2^Θ of Θ is closed to $\cup$, $\cap$ and complement, and $\theta_i \cap \theta_j = \emptyset$ for every $i \neq j$.

Examples of hyper-power sets. Let $\Theta = \{\theta_1, \theta_2\}$, we have $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, i.e. $|D^\Theta| = 5$. Let $\Theta = \{\theta_1, \theta_2, \theta_3\}$ now, we have $D^\Theta = \{\alpha_0, \alpha_1, ...\alpha_{18}\}$, where $\alpha_0 = \emptyset, \alpha_1 = \theta_1 \cap \theta_2 \cap \theta_3, \alpha_2 = \theta_1 \cap \theta_2, \alpha_3 = \theta_1 \cap \theta_3, ..., \alpha_{17} = \theta_2 \cup \theta_3, \alpha_{18} = \theta_1 \cup \theta_2 \cup \theta_3$, i.e., $|D^\Theta| = 19$ for $|\Theta| = 3$.

A *generalized basic belief assignment (gbba)* m is a mapping $m : D^\Theta \longrightarrow [0, 1]$, such that $\sum_{A \in D^\Theta} m(A) = 1$ and $m(\emptyset) = 0$. The quantity $m(A)$ is called the *generalized basic belief mass (gbbm) of A*. A *generalized belief function (gBF)* Bel is a mapping $Bel : D^\Theta \longrightarrow [0, 1]$, such that $Bel(A) = \sum_{X \subseteq A, X \in D^\Theta} m(X)$, generalized belief function Bel uniquely corresponds to gbba m and vice-versa.

3.2 DSm models

If we assume a Dedekind lattice (hyper-power set) according to the above definition without any other assumptions, i.e., all elements of an exhaustive frame of discernment can mutually overlap themselves, we refer to the *free DSm model* $\mathcal{M}^f(\Theta)$, i.e., about the DSm model free of constraints.

In general it is possible to add exclusivity or non-existential constraints into DSm models, we speak about *hybrid DSm models* in such cases.

An exclusivity constraint $\theta_1 \cap \theta_2 \overset{\mathcal{M}_1}{\equiv} \emptyset$ says that elements θ_1 and θ_2 are mutually exclusive in model $\mathcal{M}_1$, whereas both of them can overlap with θ_3. If we assume exclusivity constraints $\theta_1 \cap \theta_2 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_1 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, $\theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$, another exclusivity constraint directly follows them: $\theta_1 \cap \theta_2 \cap \theta_3 \overset{\mathcal{M}_2}{\equiv} \emptyset$. In this case all the elements of the 3-element frame of discernment $\Theta = \{\theta_1, \theta_2, \theta_3\}$ are mutually exclusive as in the classic Dempster-Shafer theory, and we call such hybrid DSm model as *Shafer's model* $\mathcal{M}^0(\Theta)$.

A non-existential constraint $\theta_3 \overset{\mathcal{M}_3}{\equiv} \emptyset$ brings additional information about a frame of discernment saying that θ_3 is impossible; it forces all the gbbm's of $X \subseteq \theta_3$ to be equal to zero for any gbba in model $\mathcal{M}_3$. It represents a sure meta-information with respect to generalized belief combination, which is used in a dynamic fusion.

In a degenerated case of the *degenerated DSm model* $\mathcal{M}_\emptyset$ (*vacuous DSm model* in [15]) we always have $m(\emptyset) = 1$, $m(X) = 0$ for $X \neq \emptyset$. It is the only case where $m(\emptyset) > 0$ is allowed in DSmT.

The total ignorance on Θ is the union $I_t = \theta_1 \cup \theta_2 \cup ... \cup \theta_n$. $\boldsymbol{\emptyset} = \{\boldsymbol{\emptyset}_\mathcal{M}, \emptyset\}$, where $\boldsymbol{\emptyset}_\mathcal{M}$ is the set of all elements of D^Θ which are forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical empty set [5].

For a given DSm model we can define (in addition to [15]) $\Theta_\mathcal{M} = \{\theta_i | \theta_i \in \Theta, \theta_i \not\in \boldsymbol{\emptyset}_\mathcal{M}\}$, $\Theta_\mathcal{M} \overset{\mathcal{M}}{\equiv} \Theta$, and $I_\mathcal{M} = \bigcup_{\theta_i \in \Theta_\mathcal{M}} \theta_i$, i.e. $I_\mathcal{M} \overset{\mathcal{M}}{\equiv} I_t$, $I_\mathcal{M} = I_t \cap \Theta_\mathcal{M}$, $I_{\mathcal{M}_\emptyset} = \emptyset$. $D^{\Theta_\mathcal{M}}$ is a hyper-power set on the DSm frame of discernment $\Theta_\mathcal{M}$, i.e., on Θ without elements which are excluded by the constraints of model $\mathcal{M}$. It holds $\Theta_\mathcal{M} = \Theta$, $D^{\Theta_\mathcal{M}} = D^\Theta$and $I_\mathcal{M} = I_t$ for any DSm model without non-existential constraint. Whereas *reduced (or constrained) hyper-power set* $D^\Theta_\mathcal{M}$ (or $D^\Theta(\mathcal{M})$ from Chapter 4 in [15] arises from D^Θ by identifying of all $\mathcal{M}$-equivalent elements. $D^\Theta_{\mathcal{M}^0}$ corresponds to classic power set 2^Θ.

3.3 The DSm rules of combination

The *classic DSm rule DSmC* is defined on the free DSm models as it follows[6]:

$$m_{\mathcal{M}^f(\Theta)}(A) = (m_1 \circledcirc m_2)(A) = \sum_{X,Y \in D^\Theta,\, X \cap Y = A} m_1(X) m_2(Y). \quad (1)$$

Since D^Θ is closed under operators $\cap$ and $\cup$ and all the $\cap$s are non-empty, the classic DSm rule guarantees that $(m_1 \circledcirc m_2)$ is a proper generalized basic belief assignment. The rule is commutative and associative. For n-ary version of the rule see [15].

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to a nature of the problem under consideration, which requires us to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such a case, the *hybrid DSm rule of combination DSmH* is used. Let us present its original version DSmH_{02} now. DSmH_{02}, based on the hybrid model $\mathcal{M}(\Theta)$, $\mathcal{M}^f(\Theta) \neq \mathcal{M}(\Theta) \neq \mathcal{M}_\emptyset(\Theta)$, for $k \geq 2$ independent sources of information, is defined as $m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledcirc m_2 \circledcirc ... \circledcirc m_k)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)]$, where $\phi(A)$ is a *characteristic non-emptiness function* of a set A. Specially, for combination of two independent sources we have:

$$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledcirc m_2)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A)], \quad (2)$$

where $\phi(A)$ is a *characteristic non-emptiness function* of a set A, i. e. $\phi(A) = 1$ if $A \notin \boldsymbol{\emptyset}$ and $\phi(A) = 0$ otherwise. $S_1 \equiv m_{\mathcal{M}^f(\Theta)}$, $S_2(A)$, and $S_3(A)$ are defined

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[5] $\boldsymbol{\emptyset}$ should be $\boldsymbol{\emptyset}_\mathcal{M}$ extended with the classical empty set $\emptyset$, thus more correct should be the expression $\boldsymbol{\emptyset} = \boldsymbol{\emptyset}_\mathcal{M} \cup \{\emptyset\}$.

[6] To distinguish the DSm rule from Dempster's rule, we use $\circledcirc$ instead of $\oplus$ for the DSm rule in this text.

for two sources (for n-ary versions see [15]) as it follows:

$$S_1(A) = \sum_{X,Y\in D^\Theta,\, X\cap Y=A} m_1(X)m_2(Y), \qquad (02-1)$$

$$S_2(A) = \sum_{X,Y\in\, \boldsymbol{\emptyset},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\, \boldsymbol{\emptyset})\wedge(A=I_t)]} m_1(X)m_2(Y), \qquad (02-2)$$

$$S_3(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ X\cap Y\in\, \boldsymbol{\emptyset}} m_1(X)m_2(Y), \qquad (02-3)$$

with $\mathcal{U} = u(X)\cup u(Y)$, where $u(X)$ is the union of all singletons θ_i that compose X and Y; all the sets A, X, Y are supposed to be in some canonical form, e.g. CNF. Unfortunately no mention about the canonical form is included in [15]. $S_1(A)$ corresponds to the classic DSm rule on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets in both the input gbba's, which arises due to non-existential constraints and which is transferred to the total or relative ignorance; and $S_3(A)$ transfers the sum of masses of relatively and absolutely empty sets, which arise as conflicts of the input gbba's, to the non-empty union of input sets [7].

On the degenerated DSm model $\mathcal{M}_\emptyset$ it must be $m_{\mathcal{M}_\emptyset}(\emptyset) = 1$ and $m_{\mathcal{M}_\emptyset}(A) = 0$ for $A \neq \emptyset$.

The hybrid DSm rule generalizes the classic DSm rule to be applicable to any DSm model. The hybrid DSm rule is commutative but not associative. It is the reason the n-ary version of the rule should be used in practical applications. For the n-ary version of $S_i(A)$, see [15]. For easier comparison with generalizations of the classic rules of combination [6] we suppose all formulas in CNF, thus we can include the compression step into formulas $S_i(A)$ as it follows[8]:

$S_1(A) = \sum_{X\equiv A,\ X\in D^\Theta} m_{\mathcal{M}^f(\Theta)}(X) = \sum_{X\cap Y\equiv A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,
$S_2(A) = \sum_{X,Y\in\, \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}\equiv A]\vee[(\mathcal{U}\in\, \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_{\mathcal{M}}$,

[7] As a given DSm model $\mathcal{M}$ is used, a final compression step must be applied, see Chapter 4 in [15], which is part of Step 2 of the hybrid DSm combination mechanism and which "consists in gathering (summing) all masses corresponding to same proposition because of the constraints of the model". I.e., gbba's of $\mathcal{M}$-equivalent elements of D^Θ are summed. Hence the final gbba m is computed as $m(A) = \sum_{X\equiv A} m_{\mathcal{M}(\Theta)}(X)$; it is defined on the reduced hyper-power set $D^\Theta_{\mathcal{M}}$.

[8] We can further simplify the formulas for DSmH rule by using a special canonical form related to the used hybrid DSm model, e.g. $CNF_{\mathcal{M}}(X) = CNF(X) \cap I_{\mathcal{M}}$. (Unfortunately, we have to note that it's form presented in [6], which is based on $\overset{\mathcal{M}}{\equiv}$, is not correct). Thus all subexpressions '$\equiv A$' can be replaced with '$= A$' in the definitions of $S_i(A)$ and '$S_i(A) = 0$ for $A \notin D^\Theta_{\mathcal{M}}$' can be removed from the definition. Hence we obtain a similar form to that published in DSmT book Vol. 1:
$S_1(A) = \sum_{X\cap Y=A,\ X,Y\in D^\Theta} m_1(X)m_2(Y)$,
$S_2(A) = \sum_{X,Y\in\, \boldsymbol{\emptyset}_{\mathcal{M}},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\, \boldsymbol{\emptyset}_{\mathcal{M}})\wedge(A=I_{\mathcal{M}})]} m_1(X)m_2(Y)$,
$S_3(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ X\cap Y\in\, \boldsymbol{\emptyset}_{\mathcal{M}}} m_1(X)m_2(Y)$.
Hence all the necessary assumptions of the definitions of $S_i(A)$ have been formalized.

$S_3(A) = \sum_{X,Y \in D^\Theta,\ X \cup Y \equiv A,\ X \cap Y \in \boldsymbol{\emptyset}_\mathcal{M}} m_1(X)m_2(Y)$ for $\emptyset \neq A \in D^\Theta_\mathcal{M}$,
$S_i(A) = 0$ for $A = \emptyset$ and for $A \notin D^\Theta_\mathcal{M}$
(where $\mathcal{U}$ is as it is above).

DSmH$_{05}$:
The 2005 version of DSmH rule [9, 16] is defined again with the formula (2). A difference(s) appear(s) in definitions of the combination steps as it follows:

$$S_1(A) = \sum_{X \cap Y = A,\ X,Y \in D^\Theta} m_1(X)m_2(Y), \qquad (05-1)$$

$$S_2(A) = \sum_{X,Y \in \boldsymbol{\emptyset},\ [\mathcal{U}=A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_t)]} m_1(X)m_2(Y), \qquad (05-2)$$

$$S_3(A) = \sum_{X,Y \in D^\Theta,\ u(c(X \cap Y)) = A,\ X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y), \qquad (05-3)$$

$c(X \cap Y)$ is explicitly distinguished from $X \cap Y$ in expression for S_3 thus it should be also in that one for S_1.

When including the compression step into formulas, we obtain the following:
$S_1(A) = \sum_{c(X \cap Y) \equiv A,\ X,Y \in D^\Theta} m_1(X)m_2(Y)$,
$S_2(A) = \sum_{X,Y \in \boldsymbol{\emptyset},\ [c(\mathcal{U}) \equiv A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = I_\mathcal{M})]} m_1(X)m_2(Y)$,
$S_3(A) = \sum_{X,Y \in D^\Theta,\ u(c(X \cap Y)) \equiv A,\ X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$.
Note that $\equiv$ already includes c, hence we can remove $c(\_)$ from the equations for S_1 and S_2.

Let us turn our attention back to formulas without the compression step. A contribution of $S_i(A)$ should be 0 for $A \in \boldsymbol{\emptyset}$ and for $A \notin D^\Theta_\mathcal{M}$. Positive $S_i(A)$ is eliminated by $\phi(A)$ for $A = \emptyset$, but not for $\emptyset \not\equiv A \notin D^\Theta_\mathcal{M}$. To manage it, we can either use $\phi_\mathcal{M}(A)$ related to DSm model $\mathcal{M}$ in question, such that $\phi_\mathcal{M}(A) = 0$ for $A \equiv \emptyset$ and for $A \notin D^\Theta_\mathcal{M}$ and $\phi_\mathcal{M}(A) = 1$ if $A \in D^\Theta_\mathcal{M} \& A \notin \boldsymbol{\emptyset}$, or we can add the conditions $A \in D^\Theta_\mathcal{M}$ directly to formulas as it follows:
$S_1(A) = \sum_{c_\mathcal{M}(X \cap Y) = A,\ X,Y \in D^\Theta} m_1(X)m_2(Y)$ for $A \in D^\Theta_\mathcal{M}$,
$S_2(A) = \sum_{X,Y \in \boldsymbol{\emptyset}_\mathcal{M},\ [c_\mathcal{M}(\mathcal{U}) = A] \vee [(\mathcal{U} \in \boldsymbol{\emptyset}) \wedge (A = c_\mathcal{M}(I_t))]} m_1(X)m_2(Y)$ for $A \in D^\Theta_\mathcal{M}$,
$S_3(A) = \sum_{X,Y \in D^\Theta,\ u(c_\mathcal{M}(X \cap Y)) = A,\ X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$ for $A \in D^\Theta_\mathcal{M}$,
$S_i(A) = 0$ for $A \notin D^\Theta_\mathcal{M}$.

If we explicitly assume all formulas in $c_\mathcal{M} = CNF(X) \cap I_\mathcal{M}$ (CNF related to DSm model $\mathcal{M}$) we can remove $c_\mathcal{M}$ from all formulas, thus DSmH$_{05}$ is fully defined by the set of equations $(05-1)$, $(05-2)$ and $(05-3')$,

$$S_3(A) = \sum_{X,Y \in D^\Theta,\ u(X \cap Y) = A,\ X \cap Y \in \boldsymbol{\emptyset}} m_1(X)m_2(Y). \qquad (05-3')$$

In this case positive values $S_i(A)$ are automatically generated only for $A \in D^\Theta_\mathcal{M}$, and $S_i(A) > 0$ are eliminated with classic $\phi(A)$.

We have presented two different definitions DSmH$_{02}$ and DSmH$_{05}$ of DSmH rule in this section. As it has already been mentioned in the introduction, these two definitions produce a different results, for examples see Section 5, thus there are two different DSmH rules.

Moreover there is another even more important problem with DSmH_{05}, that is its inconsistency: DSmH_{05} rule enables counting some belief masses twice in some cases, and on the other side, some belief masses are skipped in some cases. Hence the rule does not produce correct gbba's in general. We will show it on examples and suggest several possible corrections of the definition in the following section.

4 Inconsistencies and suggestion of corrections of DSmH_{05} rule

4.1 Doubling of generalized basic belief masses by DSmH_{05}

Let us suppose a very simple example for brief displaying of the problem. Let $m_1(A \cap B) = 1$ and $m_2(A \cap C) = 1$ and $A \cap B \equiv A \cap C \equiv \emptyset$ in DSm model $\mathcal{M}$ in question. Thus we obtain
$(A \cap B) \cap (A \cap C) = A \cap B \cap C \equiv \emptyset$,
$u(c((A \cap B) \cap (A \cap C))) = A \cup B \cup C \not\equiv \emptyset$,
$(A \cap B) \cup (A \cap C) = A \cap (B \cup C) \equiv \emptyset$,
$\mathcal{U} = \mathcal{U}(A \cap B, A \cap C) = u(A \cap B) \cup u(A \cap C) = A \cup B \cup C \not\equiv \emptyset$.
Thus under assumed constraints DSmH_{05} produces the following (on any frame of discernment which contains elements A, B, C):
$S_1(A \cap B \cap C) = 1$, $S_2(A \cup B \cup C) = 1$, $S_3(A \cup B \cup C) = 1$, $S_i(X) = 0$ otherwise, as $\phi(A \cap B \cap C) = 0$ and $\phi(A \cup B \cup C) = 1$ we obtain the following result:
$m_{12}(A \cap B \cap C) = 0 \cdot (1 + 0 + 0) = 0$
$m_{12}(A \cup B \cup C) = 1 \cdot (0 + 1 + 1) = 2$
$m_{12}(X) = 0$ otherwise.
Hence we have an incorect gbba m_{12}, where $m_{12}(A \cup B \cup C) = 2$, $m_{12}(X) = 0$ otherwise.

We have to note that this feature of DSmH_{05} rule can appear also in more general examples: let us suppose $m_1(A \cap B) = x$, $m_2(A \cap C) = y$, and $m_i(W)$ be arbitrary for another $W \in D^\Theta$, where $A, B, C \in \Theta$. $m_1(A \cap B)m_2(A \cap C) = xy$ is doubled in such examples and $\sum_{W \in D^\Theta_\mathcal{M}} m(W) \geq 1 + xy$ (unless some gbbm is ignored as in Subsection 4.2).

This problem does not appear in DSmH_{02} rule:
$S_1(A \cap B \cap C) = 1$, $S_2(A \cup B \cup C) = 1$, $S_3(A \cap (B \cup C)) = 1$, $S_i(X) = 0$ otherwise, as $\phi(A \cap (B \cup C) = 0$ we obtain the following result:
$m_{12}(A \cap B \cap C) = 0 \cdot (1 + 0 + 0) = 0$
$m_{12}(A \cap (B \cup C)) = 0 \cdot (0 + 0 + 1) = 0$
$m_{12}(A \cup B \cup C) = 1 \cdot (0 + 1 + 0) = 1$
$m_{12}(X) = 0$ otherwise.
Hence we have a correct gbba m_{12}, where $m_{12}(A \cup B \cup C) = 1$, $m_{12}(X) = 0$ otherwise.

We can simply correct the definition of DSmH_{05} rule as it follows:

$$S_2(A) = \sum_{X,Y\in \boldsymbol{\emptyset},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset})\wedge(A=I_t)],\ u(c(X\cap Y))\in \boldsymbol{\emptyset}} m_1(X)m_2(Y), \quad (05-2m)$$

or

$$S_2(A) = \sum_{X,Y\in \boldsymbol{\emptyset},\ [\mathcal{U}=A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset})\wedge(A=I_t)],\ u(X\cap Y)\in \boldsymbol{\emptyset}} m_1(X)m_2(Y), \quad (05-2m')$$

respectively, when assuming formulas in $CNF_{\mathcal{M}}$.

Using the suggested modification we obtain $S_2(A\cup B\cup C) = 0$, and $m_{12}(A\cup B\cup C) = 1\cdot(0+0+1) = 1$. Hence we have a correct resulting gbba m_{12}, where $m_{12}(A\cup B\cup C) = 1$, $m_{12}(X) = 0$ otherwise.

4.2 Ignoring of generalized basic belief masses by DSmH_{05}

There is another important problem with DSmH_{05}:
$\sum_{X,Y\in D^\Theta,\ u(c(X\cap Y))\in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$, is not included in the final $m_{\mathcal{M}(\Theta)}$, e.g. $X = A$, $Y = A\cup B$, when $A\equiv\emptyset$ we obtain $X\cap Y = A\cap(A\cup B) = A\equiv\emptyset$, $u(c(X\cap Y)) = u(c(A)) = A\equiv\emptyset$, $Y = A\cup B = B\notin\emptyset$, thus $m_1(A)m_2(A\cup B)$ and $m_1(A\cup B)m_2(A)$ are not added to final $m(\_)$ (they are canceled by $\phi(\_)$), hence $\sum_{A\in D^\Theta} m(A) < 1$ whenever $m_1(A)m_2(A\cup B) > 0$ or $m_1(A\cup B)m_2(A) > 0$. In an extreme example where $m_1(A) = 1$ and $m_2(A\cup B) = 1$ we obtain the resulting gbba $m_{12}(X)\equiv 0$.

In more general cases, where $m_1(A) = x$, $m_2(A\cup B) = y$, and $m_i(W)$ are arbitrary for other $W\in D^\Theta$, where $A, B\in\Theta$, $m_1(A)m_2(A\cup B) = xy$ is ignored and we obtain $\sum_{W\in D^\Theta_{\mathcal{M}}} m(W) \leq 1 - xy$ (unless some gbbm is doubled as in Subsection 4.1).

Neither this problem appears in original DSmH_{02} rule.

This weakness of DSmH_{05} rule must be necessarily corrected in DSmT. There are a lot of possibilities how to do it, e.g. with addition of an additional formula which assigns these belief masses in the same way as it DSmH_{02} does, i.e. assigning of the masses to union:

$$S_4(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y\equiv A,\ u(c(X\cap Y))\in \boldsymbol{\emptyset}} m_1(X)m_2(Y), \qquad (05-4)$$

or we can simply add the belief masses in question to $\mathcal{U} = \mathcal{U}(X,Y) = u(X)\cup u(Y)$ by modification of formula $(05-2m)$ for computing of $S_2(A)$ as it follows[9],

$$S_2(A) = \sum_{X,Y\in D^\Theta,\ [\mathcal{U}=A]\vee[(\mathcal{U}\in \boldsymbol{\emptyset})\wedge(A=I_{\mathcal{M}})],\ u(c(X\cap Y))\in \boldsymbol{\emptyset}} m_1(X)m_2(Y). \quad (05-2n)$$

[9]The simplest solution is, of course, a return back just to the idea from 2000-4, i.e., to DSmH_{02} rule, which is correct, but it would not be interesting from the point of view of a general discussion what it is the DSmH rule. As another simple correction of DSmH_{05} we can consider a simple assigning of whole forgotten belief mass $\sum_{X,Y\in D^\Theta,\ u(c(X\cap Y))\in \boldsymbol{\emptyset}} m_1(X)m_2(Y)$ to $I_{\mathcal{M}}$, i.e., modification of formula (05-2) for computing of $S_2(A)$.

It is possible to show that both the suggested modifications produce correct gbba's.

If we suppose CNF $c_{\mathcal{M}}$ related to the DSm hybrid model $\mathcal{M}$ in question, we can reformulate the corrected formulas as it follows:

$$S_4(A) = \sum_{X,Y\in D^\Theta,\ X\cup Y=A,\ u(X\cap Y)\in\ \boldsymbol{\emptyset},} m_1(X)m_2(Y), \qquad (05-4')$$

or with a modification of the formula $(05-2)$ for computing of $S_2(A)$,

$$S_2(A) = \sum_{X,Y\in D^\Theta,\ [\mathcal{U}=A]\vee[(\mathcal{U}\in\ \boldsymbol{\emptyset})\wedge(A=I_t)],\ u(X\cap Y)\in\ \boldsymbol{\emptyset},} m_1(X)m_2(Y), \quad (05-2n')$$

Using one of these formulas we obtain one of full definitions of the corrected DSmH_{05} rule: DSmH_{05m} given with the equations (3), $(05-1)$, $(05-2m')$, $(05-3')$ and $(05-4')$ or DSmH_{05n} given with the equations (2), $(05-1)$, $(05-2n')$ and $(05-3')$;

$$m_{\mathcal{M}(\Theta)}(A) = (m_1 \circledcirc m_2)(A) = \phi(A)[S_1(A) + S_2(A) + S_3(A) + S_4(A)]. \qquad (3)$$

Both of the above suggested versions of the DSmH rule correctly defines gbba's on hybrid DSm model, and when assuming formulas in $c_{\mathcal{M}}$ normal form, both of these above mentioned definitions already include also the compression step. Thus we have three different correct versions of the DSmH rule, the original one DSmH_{02} and two corrections of DSmH_{05}. A nature of these differences will be discussed in the next section.

5 Differences between DSmH_{02} and DSmH_{05} rules

Let us present differences between the rules DSmH_{02} and DSmH_{05} on examples. For simplicity we present very simple examples again.

Let suppose a frame of discernment which contains elements A, B, and C again and DSm model containing constraint $A \cap B \equiv \emptyset$. Let 4 gBF's be given by gbbma's m_1, m_2, m_3, and m_4 such that $m_1(A\cap B) = 1$, $m_2(A \cup B \cup C) = 1$, $m_3((A\cap B)\cup C) = 1$, $m_4(A) = 1$, $m_i(X) = 0$ otherwise. Let us compute DSmH combinations of m_1 with all other m_i's for both the DSmH rules.

DSmH_{02}: $m_{12}(A \cup B \cup C) = 1$,
DSmH_{05}: $m_{12}(A \cup B) = 1$,
DSmH_{02}: $m_{13}((A \cap B) \cup C) = 1$,
DSmH_{05}: $m_{13}(A \cup B) = 1$,
DSmH_{02}: $m_{14}((A) = 1$,
DSmH_{05}: $m_{14}(A \cup B) = 1$,
DSmH_{02}: $m_{1i}(X) = 0$ otherwise,
DSmH_{05}: $m_{1i}(X) = 0$ otherwise.

There are different results for all combinations of all three pairs of gbbm's. Hence DSmH$_{02}$ and DSmH$_{05}$ are really different rules[10].

Unfortunately we do not know either motivation or intention of creation of DSmH$_{05}$ different from the original version of the rule DSmH$_{02}$. For the first view to their formulas it seems, that the new version assigns multiples of bbm's to less or equal focal elements than the original version does. It would be a nice criteria for assigning of conflicting bbm's: if $X \cap Y \equiv \emptyset$ then to assign $m_1(X)m_2(X)$ to smallest as possible reasonable focal element. It holds in the first case of the example when combining m_1 and m_2, $A \cup B \subset A \cup B \cup C$, thus DSmH$_{05}$ assigns belief mass to less subset of the frame of discernment, i.e. belief mass is more specified than it is when DSmH$_{02}$ is used. But this does not hold in general, see the other cases of the example. Sets $(A \cap B) \cup C$ and $A \cup B$ are non-comparable from the point of view of inclusion, thus we cannot say which result is more specified in the case when m_1 and m_3 are combined. In the last case, the original version of the rule gives more specified result than the new version does, as $A \subset A \cup B$.

When comparing corrections DSmH$_{05m}$ and DSmH$_{05n}$ of the DSmH$_{05}$ rule, we can say that the first one produces more specified results than the first one. There is the only difference between them, DSmH$_{05m}$ assigns 'forgotten' multiples of bbm's to $X \cup Y$ whereas DSmH$_{05n}$ assigns them to $\mathcal{U}(X, Y)$, and it is possible to prove that $X \cup Y \subseteq \mathcal{U}(X, Y)$ in general.

Hence DSmH$_{05m}$ has more complicated formula including component S_4 but is produdes more specified results than DSmH$_{05n}$ does.

Similarly to the case of the uncorrected version of DSmH$_{05}$, it is not possible to say whether the corrected rules produce more or less specified results in comparison with DSmH$_{02}$ rule. We can look at the above example again, because both the suggested corrections produce for the given bbma's the same results as it original DSmH$_{05}$ version does.

We can present a schema of DSmH$_{02}$ and DSmH$_{05}$ rules and specificity of their results in Figure 1. To be more instructive we use $\mathcal{U}(X, Y) = \mathcal{U} = u(X) \cup u(Y)$, in general, it would be $\mathcal{U} = \mathcal{U}(X_1, ..., X_n) = u(X_1) \cup ... \cup u(X_n)$.

And similarly for all DSmH$_{02}$, DSmH$_{05}$, DSmH$_{05m}$ and DSmH$_{05n}$ rules in Figure 2.

We can use $(X \cup Y) \cup u(X \cap Y)$ instead of $X \cup Y$, in Figure 2, because it is used in DSmH$_{05m}$ rule only if $u(X \cap Y) \in \emptyset$, hence $(X \cup Y) \cup u(X \cap Y) \equiv X \cup Y$ in such a case.

[10]Both the suggested corrections DSmH$_{05m}$ and DSmH$_{05n}$ of DSmH$_{05}$ rule produce the same results here, hence we need not distinguish them in the presented examples. For distinguishing of rules DSmH$_{05m}$ and DSmH$_{05n}$, we have to use e.g. some modification of the example from Subsection 4.2, such that, $u(X \cap Y) \equiv \emptyset$ and $X \cup Y \neq \mathcal{U}(X, Y) = u(X) \cup u(Y)$, e.g. $X = A$ and $Y = A \cup (B \cap C)$, where $A \equiv \emptyset$: $u(X \cap Y) = u(A \cap (A \cup (B \cap C)) = u(A) = A \equiv \emptyset$, $X \cup Y = A \cup (A \cup (B \cap C) = A \cup (B \cap C)$ and $\mathcal{U}(X, Y) = u(A) \cup u(A \cup (B \cap C)) = A \cup B \cup C$, what is different from $A \cup (B \cap C)$ in general.

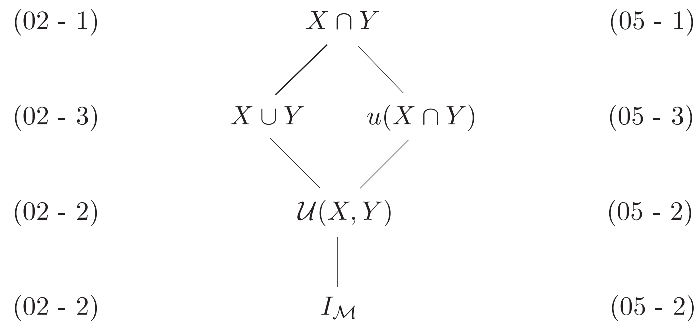

Figure 1: *A* schema of DSmH_{02} and DSmH_{05} combination rules.

6 A suggestion of DSmH rules with more specified results

Both the corrections DSmH_{05m} and DSmH_{05n} of DSmH_{05} rules produce correct bba's. We have shown that their results are non-comparable with those by DSmH_{02} from the point of view of their specificity. It corresponds with the fact that multiples of gbbm's $m_1(X)m_2(Y)$ are assigned along different paths from intersection $X \cap Y$ (the most specific element) to relative ignorance $I_{\mathcal{M}}$ (the less specific element) on Figures 1 and 2. It is based on the fact that the following holds: $X \cap Y \subseteq X \cup Y \subseteq \mathcal{U}(X, Y) \subseteq I_{\mathcal{M}}$, $X \cap Y \subseteq u(X \cap Y) \subseteq (X \cup Y) \cup u(X \cap Y) \subseteq \mathcal{U}(X, Y) \subseteq I_{\mathcal{M}}$, and $X \cup Y$ is incomparable with respect to inclusion either with $u(X \cap Y)$ or $(X \cup Y) \cup u(X \cap Y)$ in general. From this point of view we can briefly suggest an idea of another rules (another modifications of DSmH) which produce more specified results than DSmH_{02}, DSmH_{05m} and DSmH_{05n} do.

It holds both that $(X \cup Y) \cap u(X \cap Y) \subseteq X \cup Y$ and $(X \cup Y) \cap u(X \cap Y) \subseteq u(X \cup Y) \cap u(X \cap Y)$ in general, thus a rule which assigns multiples $m_1(X)m_2(Y)$ to $(X \cup Y) \cap u(X \cap Y)$ (if non-empty) instead of assigning it to $X \cup Y$ or $u(X \cup Y) \cap u(X \cap Y)$ produces more specified results than all DSmH_{02}, DSmH_{05}, DSmH_{05m} and DSmH_{05n}. We can present possible targets of assigning multiples $m_1(X)m_2(Y)$ in schema in Figure 3.

The schema is based on the following statements and hypothesis:

Statement 1 *The following holds:*

(i) $X \cap Y \subseteq (X \cup Y) \cap u(X \cap Y)$,

(ii) $(X \cup Y) \cap u(X \cap Y) \subseteq X \cup Y$,

(iii) $X \cup Y \subseteq u(X \cup Y)$,

(iv) $u(X \cup Y) \subseteq u(X \cup Y) \cup u(X \cap Y)$,

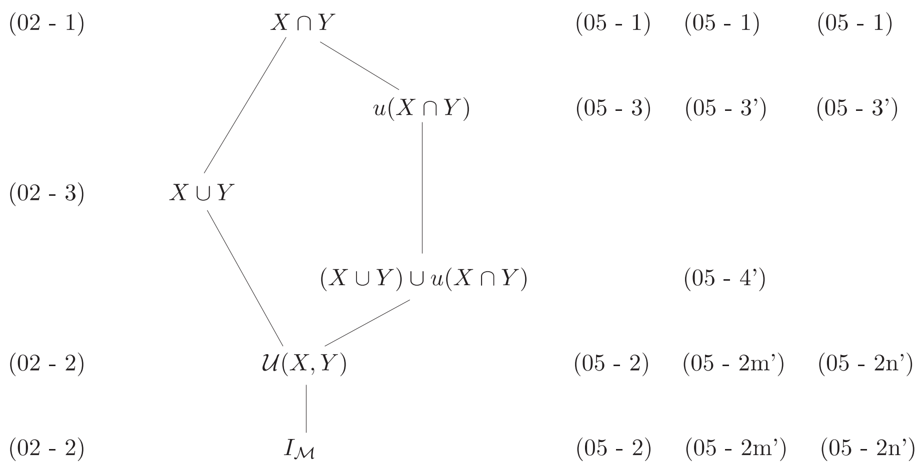

Figure 2: *A* schema of DSmH_{02}, DSmH_{05}, DSmH_{05m} and DSmH_{05n} combination rules.

(v) $u(X \cup Y) \cup u(X \cap Y) \subseteq \mathcal{U}(X, Y)$,

(vi) $\mathcal{U}(X, Y) \subseteq I_{\mathcal{M}}$,

(vii) $(X \cup Y) \cap u(X \cap Y) \subseteq u(X \cup Y) \cap u(X \cap Y)$,

(viii) $u(X \cup Y) \cap u(X \cap Y) \subseteq u(X \cap Y)$,

(ix) $u(X \cap Y) \subseteq (X \cup Y) \cup u(X \cap Y)$,

(x) $(X \cup Y) \cup u(X \cap Y) \subseteq u(X \cup Y) \cup u(X \cap Y)$,

(xi) $(X \cup Y) \subseteq (X \cup Y) \cup u(X \cap Y)$,

(xii) $u(X \cup Y) \cap u(X \cap Y) \subseteq u(X \cup Y)$.

Statement 2 *The following holds:*

(i) $X \cup Y$ *is incomparable with respect to inclusion either with* $u(X \cup Y) \cap u(X \cap Y)$ *or with* $u(X \cap Y)$ *in general.*

(ii) $u(X \cup Y)$ *is incomparable with respect to inclusion either with* $u(X \cap Y)$ *or with* $(X \cup Y) \cup u(X \cap Y)$ *in general.*

Hypothesis 3 *It holds that* $\mathcal{U}(X, Y) \subseteq u(X \cup Y) \cup u(X \cap Y)$*, i.e.,* $u(X \cup Y) \cup u(X \cap Y) = \mathcal{U}(X, Y)$.

The rules follow the paths from $X \cap Y$ to $I_{\mathcal{M}}$ possibly skipping some nodes. There are 4 rules producing the most specified results, i.e., rules which follow some of four different pathes not skipping any of their nodes. E.g. the right

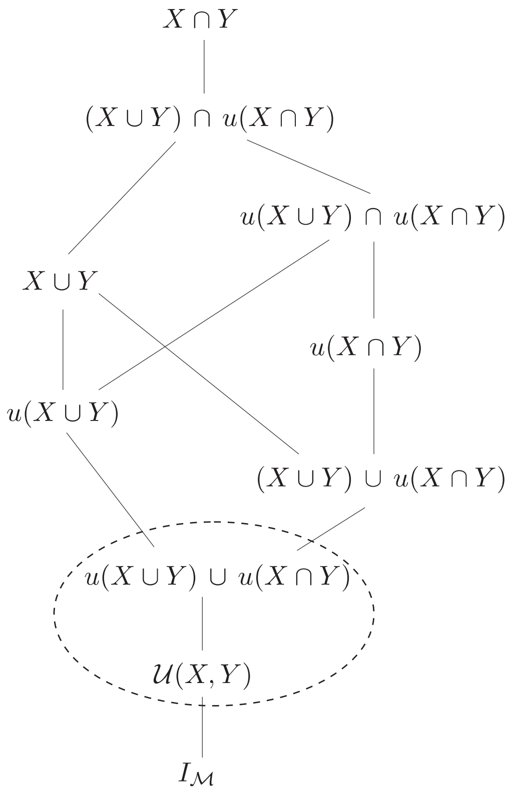

Figure 3: *A* general schema of the family of DSmH combination rules.

most path represents a DSmH rule which assignes $m_1(X)m_2(Y)$ to $X \cap Y$ if non-empty, else to $(X \cup Y) \cap u(X \cap Y)$ if non-empty, else to $u(X \cup Y) \cap u(X \cap Y)$ if non-empty, else to $u(X \cap Y)$ if non-empty, else to $(X \cup Y) \cup u(X \cap Y)$ if non-empty, else to $u(X \cup Y) \cup u(X \cap Y)$ if non-empty, (else to $\mathcal{U}(X, Y)$ if non-empty and different from $u(X \cup Y) \cup u(X \cap Y)$,) otherwise to $I_{\mathcal{M}}$.

The schema covers also all the above mentioned rules DSmH_{02}, DSmH_{05}, DSmH_{05m} and DSmH_{05n}, when skipping all the nodes which are not in Figure 2. Hence Figure 3 represents ideas of whole family of DSmH rules or alternatives to DSmH rule.

We have to note, that the schema on Figure 3 covers also Yager's rule generalized to DSm hyper-power sets (taking any path and skipping all the nodes with the exception of $X \cap Y$ and $I_{\mathcal{M}}$ and extended generalized Dubois-Prade rule (taking nodes $X \cap Y$, $X \cup Y$ and $I_{\mathcal{M}}$ on left most path), for both the generalized rules see [6].

7 Discussion

Two types of inconsistency of $DSmH_{05}$ rule was presented in this contribution.

It seems that problems of DSmH rules are caused because the quickly developing DSmT tries to cover very large area of gBFs processing, and/or because it starts from the real applications, which do not include all the theoretically possible cases of input gBFs.

Two of possible corrections of $DSmH_{05}$ rule have been suggested. It is an open question for authors of DSmT, whether they accept one of these suggested corrections or whether they suggest another correction or they return back to the original version $DSmH_{02}$ of DSmH rule.

Similarly it is an open question for the authors of DSmT and/or for authors of applications of DSmT, see e.g. second parts of both volumes of DSmT book [15, 17], to decide whether some new rules from the suggested family of DSmH rules are reasonable for enrichment of the DSm theory or for its applications.

In positive case a large area for further formalization and research of particular instance of suggested DSmH rules would be open.

8 Conclusion

A difference between two versions of the DSmH rule has been pointed out in this contribution. Moreover, it has been observed that the newer version, called $DSmH_{05}$ here, is not consistent for some classes of generalized belief functions combination tasks. Some possible corrections or a return back to the original version, called $DSmH_{02}$ here, has been suggested.

When attempting to interpret a reason for change of $DSmH_{02}$ to $DSmH_{05}$, new DSmH rules, which produce more specified results are suggested, and a general schema, which covers a large family of combination rules derived from the generalized conjunctive combination rule, is presented.

A series of open questions for authors of DSm theory is included.

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