How Many Extreme Points Does the Set of Probabilities Dominated by a Possibility Measure Have?

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Abstract

The paper contributes to the characterization of the convex set of all probabilities dominated by a possibility measure on a finite set. In particular, a lower and an upper bound for the number of extreme points of this convex set are derived by exploiting the geometrical nature of the problem alone and it is shown that in some cases the upper bound leads to a better estimate than the exponential bound of Miranda et al. [7].

1 Basic Concepts

Basic definitions and concepts of possibility theory [4] will be recalled at first. Let $X = \{x_1, \dots, x_n\}$ be a non-empty set. Throughout the paper we assume that $n \geq 2$. A *possibility measure* on 2^X is a mapping $\Pi : 2^X \to [0,1]$ such that $\Pi(\emptyset) = 0$ and for every $A, B \subseteq X$, we have $\Pi(A \cup B) = \max(\Pi(A), \Pi(B))$. In this paper only so-called *normal* possibility measures satisfying $\Pi(X) = 1$ are considered. A *possibility distribution* on X is a mapping $\pi : X \to [0,1]$ defined by $\pi(x) = \Pi(\{x\})$, for every $x \in X$. Without loss of generality we may assume that $\pi(x_1) \leq \cdots \leq \pi(x_n) = 1$ and denote $\pi_i = \pi(x_i)$, for every $i = 1, \dots, n$. Any possibility measure Π on 2^X is thus uniquely determined by a point $(\pi_1, \dots, \pi_{n-1}, 1) \in \mathbb{R}^n$.

We denote by $\mathscr{P}$ the set of all finitely additive probability measures on 2^X dominated by Π, that is, for each $P \in \mathscr{P}$ and every $A \subseteq X$, we have $P(A) \leq \Pi(A)$. Let $p_i = P(\{x_i\})$, $i = 1, \dots, n$. Every probability P on 2^X then uniquely corresponds to a point $(p_1, \dots, p_n) \in \mathbb{R}^n$. It was proven in [5] that

$$P \in \mathscr{P} \quad \text{if and only if} \quad \sum_{j=1}^{i} p_j \leq \pi_i, \quad i = 1, \dots, n-1.$$

Hence every probability from $\mathscr{P}$ is in one-to-one correspondence with a point $(p_1, \dots, p_n) \in \mathbb{R}^n$ satisfying the following conditions:

$$\begin{aligned} p_i &\geq 0, \quad i = 1, \dots, n, \\ \sum_{i=1}^{n} p_i &= 1, \\ \sum_{j=1}^{i} p_j &\leq \pi_i, \quad i = 1, \dots, n-1. \end{aligned}$$

Since p_n is uniquely determined by the equation $p_n = 1 - \sum_{i=1}^{n-1} p_i$, we can write equivalently

$$\begin{aligned} p_i &\geq 0, \quad i = 1, \dots, n-1, \\ \sum_{j=1}^{i} p_j &\leq \pi_i, \quad i = 1, \dots, n-1. \end{aligned} \tag{1}$$

The set defined by the system of inequalities (1) is clearly a convex polytope in $\mathbb{R}^{n-1}$. Notice that its dimension can be far less than $n-1$ due to the presence of zeros in $(\pi_1, \dots, \pi_{i-1})$. Also observe that its geometrical structure is not in general transparent as some of the inequalities in (1) can be redundant in the sense that their omission doesn't change the set of solutions of (1).

2 The Result

In the next paragraph we are going to find a more convenient representation of the set defined by the inequalities (1).

Let $i_0 = \min\{i \in \{1, \dots, n-1\} \mid \pi_i > 0\}$. We may assume that such i_0 exists since otherwise the possibility measure Π dominates only the probability measure given by $p_n = 1$ and $\mathscr{P}$ is a singleton. Put

$$S = \begin{cases} \{i \in \{i_0 \dots, n-2\} \mid \pi_{i+1} > \pi_i\} \cup \{n-1\}, & \text{if } n-2 \geq i_0, \\ \{n-1\}, & \text{otherwise.} \end{cases}$$

Observe that $\pi_k > 0$ for each $k \in S$, and if $k < l$ with $k, l \in S$, then $\pi_k < \pi_l$.

Lemma 1. *The system of $n - i_0 + |S|$ inequalities*

$$\begin{aligned} p_i &\geq 0, \quad i \in \{i_0, \dots, n-1\}, \\ \sum_{j=i_0}^{k} p_j &\leq \pi_k, \quad k \in S, \end{aligned} \tag{2}$$

is irreducible.

Proof. For each $m \in \{i_0, \dots, n-1\}$ consider the system of inequalities

$$\begin{aligned} p_i \geq 0, \quad & i \in \{i_0, \dots, n-1\} \setminus \{m\}, \\ \sum_{j=i_0}^{k} p_j \leq \pi_k, \quad & k \in S. \end{aligned} \tag{3}$$

It is clear that any $(p_{i_0}, \dots, p_{n-1}) \in \mathbb{R}^{n-i_0}$ such that $p_i < 0$, whenever $i = m$, and $p_i = 0$, otherwise, is a solution of (3) that is not a solution of (2). Analogously, for each $m \in S$ consider the system of inequalities

$$\begin{aligned} p_i \geq 0, \quad & i = i_0, \dots, n-1, \\ \sum_{j=i_0}^{k} p_j \leq \pi_k, \quad & k \in S \setminus \{m\}. \end{aligned} \tag{4}$$

Note that (4) has a solution with the property $p_i > \pi_m$, whenever $i = m$, and $p_i = 0$, otherwise, which is not a solution of (2). □

Lemma 2. *Let $I \subseteq \{i_0, \dots, n-1\}$ and $K \subseteq S$ with $|I| + |K| = n - i_0$. If the system of linear equations with $|I| + |K|$ variables*

$$p_i = 0, \quad i \in I, \tag{5a}$$

$$\sum_{j=i_0}^{k} p_j = \pi_k, \quad k \in K, \tag{5b}$$

has the unique solution $(p_{i_0}, \dots, p_{n-1})$, then

(i) $p_m = 0$ if and only if $m \in I$;

(ii) if $k, l \in K$ are such that $k + 1 < l$ and $K \cap \{k+1, \dots, l-1\} = \emptyset$, then there is exactly one variable taking non-zero value among $p_{k+1}, \dots, p_l$.

Proof. Since the system of linear equations (5a)-(5b) is uniquely solvable, each linear equation from (5b) determines — after substituting all zero variables p_i, $i \in I$ into (5b) — a value of some variable p_m for $m \in \{i_0, \dots, n-1\} \setminus I$. To prove (i) it is enough to note that the right-hand sides of linear equations from (5b) are positive and if $k < l$ for $k, l \in K$, then $\pi_k < \pi_l$. Hence $p_m > 0$ whenever $m \in \{i_0, \dots, n-1\} \setminus I$. The assertion of (ii) follows analogously. □

Theorem 1. *The set $\boldsymbol{P}$ of solutions of the system of inequalities* (2) *is a simple $(n - i_0)$-dimensional convex polytope in $\mathbb{R}^{n-i_0}$, which has precisely $n - i_0 + |S|$ facets and each of them is given by $\boldsymbol{P} \cap \boldsymbol{K}$, where either $\boldsymbol{K} = \{\boldsymbol{p} \in \mathbb{R}^{n-i_0} \mid p_i = 0\}$ for some $i \in \{i_0, \dots, n-1\}$ or $\boldsymbol{K} = \{\boldsymbol{p} \in \mathbb{R}^{n-i_0} \mid \sum_{j=i_0}^{k} p_j = \pi_k\}$ for some $k \in S$.*

Proof. Since $\boldsymbol{P}$ is defined by the system of inequalities (2), it is clear that $\boldsymbol{P}$ is a convex polytope in $\mathbb{R}^{n-i_0}$. The set $\boldsymbol{P}$ has a dimension $n-i_0$ since there exists an affine basis $\{\boldsymbol{b}^1,\dots,\boldsymbol{b}^{n-i_0+1}\}$ of $\mathbb{R}^{n-i_0}$ formed by elements of $\boldsymbol{P}$: for every $\boldsymbol{b}^i=(b^i_{i_0},\dots,b^i_{n-1})$, where $i=1,\dots,n-i_0$, put

$$b^i_j=\begin{cases}\pi_{i_0}, & \text{if } j=i+i_0-1,\\ 0, & \text{otherwise,}\end{cases}\qquad j=i_0,\dots,n-1,$$

and $\boldsymbol{b}^{n-i_0+1}=(0,\dots,0)$. It is clear from the definition of points $\boldsymbol{b}^i$ that the set $\{\boldsymbol{b}^1,\dots,\boldsymbol{b}^{n-i_0+1}\}$ is an affinely independent subset of $\boldsymbol{P}$ and hence it is an affine basis of $\mathbb{R}^{n-i_0}$. The number of facets and their complete characterization is a classical result for convex polytopes given by irreducible system of inequalities: see, for example, Theorem 8.2 in [3].

To show that $\boldsymbol{P}$ is simple we have to verify that every vertex of $\boldsymbol{P}$ is contained in precisely $n-i_0$ facets. Let $\boldsymbol{v}$ be a vertex of $\boldsymbol{P}$. Then $\boldsymbol{v}$ is the unique solution of some system (5) of $n-i_0$ linear equations as defined in Lemma 2, which means that $\boldsymbol{v}$ is the only point lying in the intersection of $n-i_0$ facets of $\boldsymbol{P}$. Since every facet of $\boldsymbol{P}$ corresponds to some linear equation, it suffices to show that if a linear equation is added to (5), then the resulting system has no solution.

If a linear equation $p_{i'}=0$, $i'\in\{i_0,\dots,n-1\}\setminus I$, is added to (5), then the resulting system has no solution as Lemma 2(i) gives $p_{i'}\neq 0$ for (5).

For every $k'\in\{i_0,\dots,n-1\}\setminus K$, let a linear equation $\sum_{j=i_0}^{k'}p_j=\pi_{k'}$ be added to (5). If $K=\emptyset$, then the extended system is not solvable since the unique solution of (5) is $(0,\dots,0)$. Let $K\neq\emptyset$.

Case 1. If $\min K<k'<\max K$, then put $k_1=\max\{k\in K\mid k<k'\}$ and $k_2=\min\{k\in K\mid k>k'\}$ and consider the following subsystem of (5b) extended with $\sum_{j=i_0}^{k'}p_j=\pi_{k'}$:

$$\begin{array}{ccccccccccccc}
p_{i_0} & + & \cdots & + & p_{k_1} & & & & & & & & =\pi_{k_1}\\
p_{i_0} & + & \cdots & + & p_{k_1} & + & \cdots & + & p_{k'} & & & & =\pi_{k'}\\
p_{i_0} & + & \cdots & + & p_{k_1} & + & \cdots & + & p_{k'} & + & \cdots & +\ p_{k_2} & =\pi_{k_2}
\end{array}$$

According to Lemma 2(ii) there is precisely one variable taking non-zero value among the variables $p_{k_1+1},\dots,p_{k_2}$. If it is contained in $p_{k_1+1},\dots,p_{k'}$, then, considering together the second and the third equation, we have $\pi_{k'}=\pi_{k_2}$, which is a contradiction. On the other hand, if the non-zero variable is among $p_{k'+1},\dots,p_{k_2}$, then the second and the first equation gives again a contradiction $\pi_{k_1}=\pi_{k'}$.

Case 2. If $k'<\min K$, then, analogously to the assertion of Lemma 2(ii) and the previous part of the proof, there must be precisely one variable taking non-zero value among $p_{i_0},\dots,p_{\min K}$, and its value is $\pi_{\min K}$. If it is among $p_{i_0},\dots,p_{k'}$, then we get the contradiction $\pi_{k'}=\sum_{j=i_0}^{k'}p_j=\sum_{j=i_0}^{\min K}p_j=\pi_{\min K}$. On the other hand, it the non-zero variable is among $p_{k'+1},\dots,p_{\min K}$, then $\pi_{k'}=\sum_{j=i_0}^{k'}p_j=0$, which is again a contradiction.

Case 3. If $k'>\max K$, then $p_j=0$ for each $j\in\{\max K+1,\dots,k'\}$ because

(5) is uniquely solvable and thus $\sum_{j=i_0}^{\max K} p_j = \pi_{\max K} < \pi_{k'} = \sum_{j=i_0}^{k'} p_j = \sum_{j=i_0}^{\max K} p_j$, which is contradictory. $\square$

Theorem 2. *Let* $\operatorname{ext}\mathscr{P}$ *be the set of all extreme points of the convex set* $\mathscr{P}$ *of all finitely additive probabilities dominated by a possibility measure. Then*

$$\begin{gathered}(n-i_0-1)(n-i_0+|S|)-(n-i_0+1)(n-i_0-2)\\ \le |\operatorname{ext}\mathscr{P}| \le\\ \binom{n-i_0+|S|-r_1-1}{r_2}+\binom{n-i_0+|S|-r_2-1}{r_1},\end{gathered}$$

where r_1 *is the greatest integer such that* $r_1 \le \frac{n-i_0-1}{2}$*, and* r_2 *is the greatest integer such that* $r_2 \le \frac{n-i_0}{2}$*.*

Proof. Clearly, the set $\mathscr{P}$ can be viewed as the set of points in $\mathbb{R}^{n-1}$ defined by (1) since the two convex sets are affinely isomorphic; the latter is also affinely isomorphic with $\boldsymbol{P}$ under the mapping

$$(p_{i_0},\dots,p_{n-1}) \in \boldsymbol{P} \mapsto (\underbrace{0,\dots,0}_{i_0-1},p_{i_0},\dots,p_{n-1}) \in \mathbb{R}^{n-1}.$$

Hence the convex structure of $\mathscr{P}$ is the same as that of $\boldsymbol{P}$; in particular, the two convex sets have the same number of extreme points. The lower and the upper bound for $|\operatorname{ext}\mathscr{P}|$ are fundamental results in combinatorial theory of convex polytopes, which are known under the name Lower Bound Theorem (see [1, 2]) and Upper Bound Theorem (see [6]), respectively. The two inequalities are thus in this case direct consequences of the characterization of the set $\boldsymbol{P}$ by Theorem 1. $\square$

Miranda et al. [7] derived the exponential upper bound 2^{n-1} bound for $|\operatorname{ext}\mathscr{P}|$. It turns out that the upper bound from Theorem 2 can be a better estimate for the actual number of extreme points: Table 1 documents that this is the case when the possibility distribution contains many zeros or when the range of possibility distribution is "poor".

Example 1. *Let* π *be a possibility distribution on* $X=\{x_1,\dots,x_5\}$ *given by* $\pi_1=0$, $\pi_2=\pi_3=\frac{1}{2}$, $\pi_4=\frac{3}{4}$*, and* $\pi_5=1$*. We have* $i_0=2$ *and* $S=\{3,4\}$*. Hence we obtain the irreducible system of inequalities*

$$\begin{aligned}&p_2\ge 0,\ p_3\ge 0,\ p_4\ge 0,\\ &p_2+p_3\le\frac{1}{2},\\ &p_2+p_3+p_4\le\frac{3}{4},\end{aligned}\tag{6}$$

which defines the 3-dimensional convex polytope

$$\boldsymbol{P}=\{\boldsymbol{p}=(p_2,p_3,p_4)\in\mathbb{R}^3 \mid \boldsymbol{p} \textit{ is a solution of } (6)\}.$$

n	i_0	$\|S\|$	**Lower Bound**	**Upper Bound**	2^{n-1}
4	1	3	8	8	8
4	1	2	6	6	8
4	1	1	4	4	8
4	2	2	4	4	8
4	2	1	3	3	8
5	1	4	14	20	16
5	1	3	11	14	16
5	1	2	8	9	16
5	2	3	8	8	16
10	1	9	74	1430	512
10	1	7	58	660	512
10	1	6	50	420	512

Table 1: Comparison of the upper bounds

Note that the inequality $p_2 \leq \frac{1}{2}$ *was redundant in the description of* $\boldsymbol{P}$*. The extreme points of* $\boldsymbol{P}$ *are the following:* $\boldsymbol{p}^1 = (0, 0, 0)$, $\boldsymbol{p}^2 = (0, 0, \frac{3}{4})$, $\boldsymbol{p}^3 = (0, \frac{1}{2}, 0)$, $\boldsymbol{p}^4 = (0, \frac{1}{2}, \frac{1}{4})$, $\boldsymbol{p}^5 = (\frac{1}{2}, 0, 0)$, $\boldsymbol{p}^6 = (\frac{1}{2}, 0, \frac{1}{4})$. *While the upper bound for the number of extreme points of Miranda et al. [7] equals* $2^{n-1} = 16$*, the lower and the upper bound in Theorem 2 coincides in this case so that the exact number of* 6 *extreme points is recovered.*

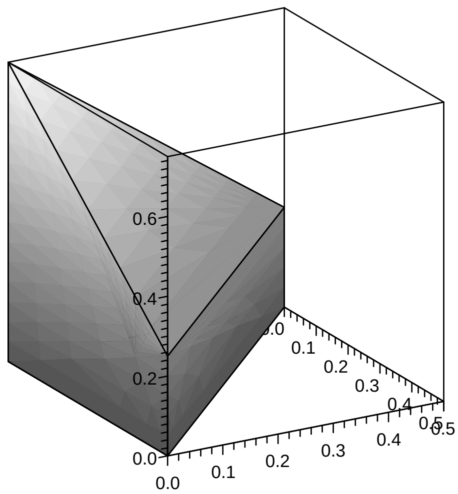

Figure 1: Convex polytope $\boldsymbol{P}$ from Example 1

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