Decision-Making Under Uncertainty Based on Rank Reducible Possibilistic Measures*

Ivan Kramosil
Institute of Computer Science
Academy of Science of the Czech Republic
Pod Vodárenskou věží 2
182 07 Prague, Czech Republic
Fax: (+420) 268 585 789, E-mail: kramosil@cs.cas.cz

Abstract

Ranking functions are qualitative degrees of uncertainty ascribed to events charged by uncertainty and taking as their values non-negative integers in the sense of ordinal numbers. Introduced are ranking functions induced by real-valued possibilistic measures and it is shown that different possibilistic measures with identical ranking functions yield the same results when applied in decision procedures based on qualitative comparation of the magnitudes of the possibilistic measures in question ascribed to the uncertain events.

Keywords: Possibilistic distribution, rank degree, ranking distribution, rank equivalent possibilistic measures, qualitative decision making under uncertainty.

1 Introduction and Motivation

This paper proposes and analyses in more detail some tools for rational and more or less sophisticated decision making under uncertainty. Uncertainty will be understood in the sense of randomness, i.e., as the lack of knowledge of values of some hidden parameters generating the data being at a subject's disposal when taking a decision, not in the sense of fuzziness or vagueness charging these data. As a matter of fact, the tools proposed and investigated below will not be based on the standard theory of probability and statistical decision making but rather

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on their alternatives replacing additivity or σ-additivity of probability measures by the principle of maxitivity, in the more common terms, by possibilistic or possibility measures. However, we will take profit of the fact that the problem to be presented can be described also in the terms of probability theory, perhaps more intuitive to be understood by the reader.

Consider the case when we have to choose among a finite list $H_1, H_2, \dots, H_n$ of mutually disjoint and exhaustive hypotheses (i.e., just one among then is the true or valid one) on the ground of some empirical data being at our disposal (experiments, observations, tests) and charged by the uncertainty of the kind of randomness (e.g., statistical results). Let $P(H_1), P(H_2), \dots, P(H_n), \sum_{i=1}^{n} P(H_i) = 1$, be the a posteriori probabilities of particular hypotheses obtained by an adequate actualization of their apriori probabilities when using the empirical data in question (the details are not relevant for our purposes and may be omitted). Let us still simplify the situation by accepting the trivial $0-1$ loss function according to which the suffered loss is 1, if the decision is wrong, i.e., if the hypothesis chosen as the valid hypothesis is not identical with the only true one, and the suffered loss is 0, if the decision is correct. As a matter of fact, such an approach is obviously impossible in many real decision problems, e.g., when testing new medicaments (when the two kinds of losses must be qualitatively distinguished), but let us limit to this simple loss function in what follows.

Hence, under these simplifying assumptions, the maximum likelihood decision procedure choosing the hypothesis H_{i_0} such that $P(H_{i_0}) \geq P(H_i)$ is the case for each $i = 1, 2, \dots, n$, is reasonable in the sense that it minimizes the expected loss $1 - P(H_i)$ connected with the choice of the hypothesis H_i (leaving aside the solution in the case when the maximum value of $P(H_i)$ is taken for two or more values of i). However, what is completely lost, when applying the maximum likelihood principle, is the difference or the ratio between the maximum value $P(H_{i_0})$ and the second largest value $P(H_{i_1})$. E.g., if $n = 2$, and $P(H_1) = 0.99$, $P(H_2) = 0.01$, the maximum likelihood decision is the same as in the case when $P(H_1) = 0.51$ and $P(H_2) = 0.49$.

So, what is also lost is the great part of the effort and work expended in order to specify the values $P(H_i), i = 1, 2, \dots, n$, or at least to obtain their good and reliable enough approximations and estimations. From an alternative (and in a sense dual) point of view, when the values $P(H_i)$ have been obtained due to some additional conditions imposed on the probabilitstic structures under consideration (statistical independence of certain random events and/or variables, Laplace principle applied to unknown apriori probabilities, etc.), these additional conditions may apper to be unnecessarily strong when only the ordering of the values $P(H_i)$ according to their sizes matters.

To summarize, what we need is a characteristic of the probability values $P(H_1), \dots, P(H_n)$ (probability distribution of $\{H_1, H_2, \dots, H_n\}$, in other terms) perhaps too rough to specify this distribution in all detail, but enabling to distinguish from each other two such probability distributions $P_1(H_i)$ and $P_2(H_i), i = 1, 2, \dots, n$, in the cases when the orderings of the hypotheses $H_1, \dots, H_n$ according to the sizes of the values $P_1(H_i)$ and $P_2(H_i)$ are different. As already introduced above, even when illustrating our motivation by the case of probability measures and statistical decision functions, we will apply the same approach, below, to the case of real-valued possibilistic measures taken as the set functions quantifying the degree of uncertainty under consideration. The characteristics of possibilistic distributions fitted for separation of these distributions from each other supposing that the orderings of their values according to their sizes are different will be called ranking distributions and functions.

2 Rank Reducible Real-Valued Possibilistic Distributions

Definition 2.1 (Real-valued) possibilistic distribution *over a nonempty space* Ω *is a mapping* $\pi : \Omega \to [0,1]$ *such that* $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$; *here and below,* $\bigvee, \vee (\bigwedge, \wedge,$ *resp.) denotes the supremum (infimum, resp.) operation in the unit interval of real numbers induced by its standard linear ordering* $\leq$. *Given a real* $\varepsilon, 0 \leq \varepsilon < 1$, *a possibilistic distribution* π *over* Ω *is called* ε-rank reducible, *if there exists a finite or infinite sequence* $\alpha_0 > \alpha_1 > \alpha_2 > \ldots > \varepsilon$ *of real numbers such that, for each* $\omega \in \Omega$, *either* $\pi(\omega) \leq \varepsilon$ *or* $\pi(\omega) = \alpha_k$ *for some* $k \geq 0$ *and, for each* α_k, *there exists* $\omega \in \Omega$ *such that* $\pi(\omega) = \alpha_k$. *The (real-valued) possibilistic measure induced by the possibilistic distribution* π *is the mapping* $\Pi : \mathcal{P}(\Omega) \to [0,1]$ *defined by* $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$, *if* $\emptyset \neq A \subset \Omega$, *and by* $\Pi(\emptyset) = 0$ *for the empty subset of* Ω. *If the possibilistic distribution* π *is* ε*-rank reducible, also the induced possibilistic measure* Π *on* $\mathcal{P}(\Omega)$ *is called* ε*-rank reducible. If* $\varepsilon = 0$, *we use the term "rank reducible" instead of "0-rank reducible".*

The following almost immediate consequences of this definition are perhaps worth being introduced explicitly. If π is ε-rank reducible, then the length of the sequence $\langle \alpha_0, \alpha_1, \ldots \rangle$ and every α_i are uniquely defined, moreover, $\alpha_0 = 1$ in any case. Indeed, $\bigvee_{\omega \in \Omega} \pi(\omega) = 1$, but $\bigvee \{\pi(\omega) : \pi(\omega) \neq 1\} = \alpha_1 < 1$, so that at least one $\omega \in \Omega$ such that $\pi(\omega) = 1$ exists, consequently, $\alpha_0 = 1$ (let us recall that $\varepsilon < 1$ holds). As a matter of fact, far not every possibilistic distribution on Ω is ε-rank reducible, e.g., take π such that for infinitely many values $\alpha_0 > \alpha_1 > \ldots > \alpha_* > \varepsilon$ the sets $\{\omega \in \Omega : \pi(\omega) = \alpha_i\}, i = 0, 1, \ldots, *$ are nonempty. Roughly speaking, a possibilistic distribution π is ε-rank reducible, if for each value taken by π and greater than ε we can uniquely define that it is the k-th largest (in the order of the magnitude) value of π for some nonnegative integer. If Ω is finite then each possibilistic distribution π over Ω is obviously ε-rank reducible for each $0 \leq \varepsilon < 1$.

Definition 2.2 *Let* π *be an* ε*-rank reducible possibilistic distribution on* Ω, *let* $\kappa(\varepsilon, \omega) = k$, *if* $\pi(\omega) = \alpha_k$, *let* $\kappa(\varepsilon, \omega) = \infty$, *if* $\pi(\omega) \leq \varepsilon$. *The value* $\kappa(\varepsilon, \omega)$ *is called the* ε*-rank of the element* $\omega \in \Omega$ *defined by the possibilistic distribution* π *and the mapping* $\kappa(\varepsilon, \cdot) : \Omega \to \mathcal{N}^* = \{1, 2, \ldots\} \cup \{\infty\}$ *is called the* ε*-ranking distribution defined by* π *on* Ω. *Again, if* $\varepsilon = 0$, *the index* 0 *– is omitted.*

Definition 2.3 *Let* π *be an* ε*-rank reducible posibilistic disribution on* Ω. *The* *induced ε-ranking function $K(\varepsilon)$ on $\mathcal{P}(\Omega)$ is defined by $K(\varepsilon, A) = k$, if $\Pi(A) = \alpha_k > \varepsilon$, and $K(\varepsilon, A) = \infty$, if $\Pi(A) \leq \varepsilon$.*

As $\Pi(A) = \bigvee_{\omega \in A} \pi(\omega)$, obviously $\Pi(A)$ must be a value from the subset $\{\alpha_0(=1), \alpha_1, \ldots\} \cup [\varepsilon, 0]$ of $[0,1]$, so that $K(\varepsilon, A)$ is defined for every $A \subset \Omega$. $\Pi(\emptyset) = 0$ by definition, so that $K(\varepsilon, \emptyset) = \infty$ for every ε. For every ε and every $\omega \in \Omega$, $\Pi(\{\omega\}) = \pi(\omega)$ and $K(\varepsilon, \{\omega\}) = \kappa(\varepsilon, \omega)$ so that, in this sense, the ε-ranking function $K(\varepsilon)$ on $\mathcal{P}(\Omega)$ can be taken as a conservative extension of $\kappa(\varepsilon)$ from Ω to $\mathcal{P}(\Omega)$.

Lemma 2.1 *Let π be an ε-rank reducible possibilistic distribution on a nonempty set Ω. Then, for each $\varepsilon_1, \varepsilon \leq \varepsilon_1 < 1$, π is also ε_1-rank reducible and the sequence $\alpha_1(\varepsilon_1) > \alpha_2(\varepsilon_1) > \ldots > \varepsilon_1$, corresponding to ε_1, is an initial segment of, or is identical with, the sequence $\alpha_1(\varepsilon) > \alpha_2(\varepsilon) > \ldots > \varepsilon$ corresponding to ε.*

Proof: Obvious. □

Lemma 2.2 *Let π be as in Lemma 2.1. Then, for each $A \subset \Omega$, the relation*

$$K(\varepsilon, A) = \bigwedge\{\kappa(\varepsilon, \omega) : \omega \in A\} \tag{2.1}$$

holds.

Proof: Let $\Pi(A) \leq \varepsilon$ hold, so that $K(\varepsilon, A) = \infty$. As $\Pi(A) = \bigvee_{\omega \in \Omega} \pi(\omega)$ by definition, then $\pi(\omega) \leq \varepsilon$ and $\kappa(\varepsilon, \omega) = \infty$ for every $\omega \in A$, so that $\bigwedge\{\kappa(\varepsilon, \omega) : \omega \in A\} = \infty$ and (2.1) holds. Let $\Pi(A) > \varepsilon$ be the case, so that $\Pi(A) = \alpha_k$ and $K(\varepsilon, A) = k$ for some nonnegative integer k. Consequently, $\pi(\omega) \leq \alpha_k$ for every $\omega \in A$ and $\pi(\omega) = \alpha_k$ for at least one $\omega \in A$, as the inequality $\alpha_{k+1} < \alpha_k$ holds, so that $\kappa(\varepsilon, \omega) \geq k$ for every $\omega \in A$ and $\kappa(\varepsilon, \omega) = k$ for at least one $\omega \in A$ follows. Hence, $\bigwedge\{\kappa(\varepsilon, \omega) : \omega \in A\} = k = K(\varepsilon, A)$ and the assertion is proved. □

Definition 2.4 *ε-rank reducible possibilistic distributions π_1, π_2 on the same nonempty space Ω ore called* ε-rank equivalent *($\pi_1 \approx_\varepsilon \pi_2$, in symbols), if their ε-ranking distributions are identical, i.e., if $\kappa_1(\varepsilon, \omega) = \kappa_{\pi_1}(\varepsilon, \omega) = \kappa_{\pi_2}(\varepsilon, \omega) = \kappa_2(\varepsilon, \omega)$ for each $\omega \in \Omega$.*

As can be easily seen, for each $\varepsilon < 1, \approx_\varepsilon$ defines an equivalence relation in the space of ε-rank reducible possibilistic distributions over Ω. If π_1 and π_2 are ε-rank equivalent, also the corresponding ε-ranking functions are identical, i.e., $K_1(\varepsilon, A) = K_2(\varepsilon, A)$ for each $A \subset \Omega$.

Theorem 2.1 *ε-rank reducible possibilistic distributions π_1, π_2 over a nonempty space Ω are ε-rank equivalent, if and only if for each $A, B \subset \Omega$ such that $\Pi_j(A) \geq \varepsilon, \Pi_j(B) \geq \varepsilon$ holds for both $j = 1, 2$, the relations $\Pi_1(A) \leq \Pi_1(B)$ and $\Pi_2(A) \leq \Pi_2(B)$ hold simultaneously, i.e., $\Pi_1(A) \leq \Pi_1(B)$ is valid iff $\Pi_2(A) \leq \Pi_2(B)$ holds.*

Proof: For both $j = 1,2$, let $\alpha_0^j > \alpha_1^j > \ldots > \varepsilon$ be the sequence corresponding to π_j due to the assumption that both π_1, π_2 are ε-rank reducible. Let $\alpha_\infty^j = \varepsilon < \alpha_i^j$ for both $i = 1,2$ and for each $i = 1,2,\ldots$, supposing that there exists $\omega \in \Omega$ such that $\pi_j(\omega) = \varepsilon$. Let A, B be such that $\varepsilon \le \Pi_1(A) \le \Pi_1(B)$ holds. As $\Pi_1(A) = \alpha^1_{k_1(A)}$ and $\Pi_1(B) = \alpha^1_{k_1(B)}$ for some $k_1(A), k_1(B) \in \{0,1,\ldots\} \cup \{\infty\}$, the inequality $k_1(A) \ge k_1(B)$ follows. But $k_1(A) = K_1(\varepsilon, A)$ and $k_1(B) = K_1(\varepsilon, B)$ by definition and the identities $K_1(\varepsilon, A) = K_2(\varepsilon, A)$ and $K_1(\varepsilon, B) = K_2(\varepsilon, B)$ are valid, as π_1 and π_2 are supposed to be ε-rank equivalent. Hence, $K_2(\varepsilon, A) \ge K_2(\varepsilon, B)$ follows, so that $\Pi_2(A) = \alpha^2_{K_2(\varepsilon,A)} \le \Pi_2(B) = \alpha^2_{k_2(\varepsilon,B)}$ holds. As the roles of π_1 and π_2 are completely exchangeable, we obtain that for π_1 and π_2 ε-rank equivalent and for each $A, B \subset \Omega$ such that $\varepsilon \le \Pi_j(A)$, $\varepsilon \le \Pi_j(B), j = 1,2$ holds the inequality $\Pi_1(A) \le \Pi_1(B)$ is valid iff $\Pi_2(A) \le \Pi_2(B)$ is the case.

Let for each $A, B \subset \Omega$ such that $\varepsilon \le \Pi_j(A), \varepsilon \le \Pi_j(B)$, $j = 1,2$, holds, the equivalence $\Pi_1(A) \le \Pi_1(B)$ iff $\Pi_2(A) \le \Pi_2(B)$ be valid and suppose, in order to arrive at a contradiction, that there exists $\omega_0 \in \Omega$ such that $\kappa_1(\varepsilon, \omega_0) > 0$, $\kappa_2(\varepsilon, \omega_0) = 0$ holds. Then $K_1(\varepsilon, \Omega - \{\omega_0\}) = 0$, as $K_1(\varepsilon, \Omega) = 0 = K_1(\varepsilon, \Omega - \{\omega_0\}) \wedge K_1(\varepsilon, \{\omega_0\})$, so that $\Pi_1(\Omega - \{\omega_0\}) = 1 > \Pi_1(\{\omega_0\})$ follows, as $\Pi_1(\{\omega_0\}) = \alpha^1_{\kappa_1(\varepsilon,\omega_0)} < 1$ holds. However, $\Pi_2(\Omega - \{\omega_0\}) \le 1 = \Pi_2(\{\omega_0\}) = \alpha_0^2 = 1$, so that the pair $\{\omega_0\}$, $\Omega - \{\omega_0\}$ of subsets of Ω violates the conditions imposed on Π_1, Π_2. Hence, $\{\omega \in \Omega : \kappa_1(\varepsilon, \omega) = 0\} = \{\omega \in \Omega : \kappa_2(\varepsilon, \omega) = 0\}$.

Applying the principle of induction, let the identity

$$\Omega_i^1 = \{\omega \in \Omega : \kappa_1(\varepsilon, \omega) = i\} = \{\omega \in \Omega : \kappa_2(\varepsilon, \omega) = i\} = \Omega_i^2 \tag{2.2}$$

hold for each $i = 1,2,\ldots,k$. Let $\omega_0 \in \Omega_0 = \Omega - (\bigcup_{i=0}^k \Omega_i^1)(= \Omega - (\bigcup_{i=0}^k \Omega_i^2))$ be such that $\kappa_1(\varepsilon, \omega_0) > k+1$, $\kappa_2(\varepsilon, \omega_0) = k+1$. Then $K_1(\varepsilon, \Omega_0 - \{\omega_0\}) = k+1$, as $K_1(\varepsilon, \Omega_0) = K_1(\varepsilon, \Omega_0 - \{\omega_0\}) \wedge K_1(\varepsilon, \{\omega_0\}) = k+1$, so that $\Pi_1(\Omega_0 - \{\omega_0\}) = \alpha_{k+1} > \Pi_1(\{\omega_0\})$ holds. However, $K_2(\varepsilon, \Omega - \{\omega_0\}) \ge k+1 = K_2(\varepsilon, \{\omega_0\})$ holds, so that $\Pi_2(\Omega_0 - \{\omega_0\}) \le \Pi_2(\{\omega_0\})$ follows. So, the pair $\{\omega_0\}$, $\Omega_0 - \{\omega_0\}$ of subsets of Ω violates the conditions imposed on Π_1 and Π_2. Consequently, the identity $\kappa_1(\varepsilon, \omega) = i$ iff $\kappa_2(\varepsilon, \omega) = i$ holds for each $i = 1,2,\ldots$, such that $\kappa_j(\varepsilon, \omega) = \alpha_k^j$ for some $k = 0,1,2,\ldots$, i.e., for each $\omega \in \Omega$ such that $\pi_j(\omega) > \varepsilon$ holds. Consequently, π_1 and π_2 are ε-rank reducible and the assertion is proved. □

3 Cartesian Products of ε-Rank Reducible Possibilistic Distributions

According to what we proved in the foregoing chapter, ε-rank reducible real-valued possibilistic measures with identical ranking distributions yield identical results when applying them to decision-making procedures based on qualitative (i.e., "greater than", "smaller than") comparisons of the values ascribed by such measures to the sets in question, supposing that these values reach or exceed the treshold value ε. In this chapter our aim will be to generalize this result to a particular case of lattice-valued possibilistic measures which may be taken, in the sense to be specified below, as Cartesian products of ε-rank reducible real-valued possibilistic measures.

Let us recall, for the reader's convenience, the following, more or less elementary notions and relations. As above, let $\mathcal{T} = \langle [0,1], \leq \rangle$ denote the unit interval of real numbers with their standard linear ordering, let $[0,1]^n = \{\langle x_1, \ldots, x_n \rangle : x_i \in [0,1], i = 1, 2, \ldots, n\}$. Given $\langle x_1, \ldots, x_n \rangle, \langle y_1, \ldots, y_n \rangle \in [0,1]^n$, set

$$\langle x_1, \ldots, x_n \rangle \leq_* \langle y_1, \ldots, y_n \rangle \Leftrightarrow_{df} x_i \leq y_i \text{ for } i = 1, 2, \ldots, n \tag{3.1}$$

As can be easily seen, $\leq_*$ defines a partial ordering on $[0,1]^n$. Denoting by $\vee, \bigvee$ ($\wedge, \bigwedge$, resp.) the standard supremum (infimum, resp.) induced by $\leq$ on $[0,1]$, and by $\vee_*, \bigvee_*$ ($\wedge_*, \bigwedge_*$, resp.) the supremum (infimum, resp.) induced by $\leq_*$ on $[0,1]^n$, we obtain that

$$\langle x_1, \ldots, x_n \rangle \vee_* \langle y_1, \ldots, y_n \rangle = \langle x_1 \vee y_1, \ldots, x_n \vee y_n \rangle, \tag{3.2}$$
$$\langle x_1, \ldots, x_n \rangle \wedge_* \langle y_1, \ldots, y_n \rangle = \langle x_1 \wedge y_1, \ldots, x_n \wedge y_n \rangle, \tag{3.3}$$

In general, for each $\emptyset \neq A \subset [0,1]^n$,

$$\begin{aligned} \bigvee\nolimits_* A &= \bigvee\nolimits_{*\langle x_1,\ldots,x_n\rangle \in A} \langle x_1, \ldots, x_n \rangle = \\ &= \Big\langle \bigvee\nolimits_{\langle x_1,\ldots,x_n\rangle \in A} x_1, \ldots, \bigvee\nolimits_{\langle x_1,\ldots,x_n\rangle \in A} x_n \Big\rangle, \end{aligned} \tag{3.4}$$
$$\begin{aligned} \bigwedge\nolimits_* A &= \bigwedge\nolimits_{*\langle x_1,\ldots,x_n\rangle \in A} \langle x_1, \ldots, x_n \rangle = \\ &= \Big\langle \bigwedge\nolimits_{\langle x_1,\ldots,x_n\rangle \in A} x_1, \ldots, \bigwedge\nolimits_{\langle x_1,\ldots,x_n\rangle \in A} x_n \Big\rangle, \end{aligned} \tag{3.5}$$

Hence, for each $A \subset [0,1]^n$, $\bigvee_* A$ and $\bigwedge_* A$ are defined (for $A = \emptyset$ the standard conventions apply), so that the partially ordered set $\mathcal{T}_n = \langle [0,1]^n, \leq_* \rangle$ defines a complete lattice over $[0,1]^n$, which can be called the *n-th Cartesian power* of the complete lattice $\mathcal{T}(= \mathcal{T}_1) = \langle [0,1], \leq \rangle$.

Let Ω be a nonempty set, let $\pi_1, \pi_2, \ldots, \pi_n$ be real-valued possibilistic distributions over Ω, i.e., $\pi_i : \Omega \to [0,1]$ and $\bigvee_{\omega \in \Omega} \pi_i(\omega) = 1$ holds for each $i = 1, 2, \ldots, n$. Define the mapping $\pi : \Omega \to [0,1]$ setting, for each $\omega \in \Omega$, $\pi(\omega) = \langle \pi_1(\omega), \pi_2(\omega), \ldots, \pi_n(\omega) \rangle$. Obviously,

$$\bigvee_{*\omega \in \Omega} \pi(\omega) = \Big\langle \bigvee_{\omega \in \Omega} \pi_1(\omega), \ldots, \bigvee_{\omega \in \Omega} \pi_n(\omega) \Big\rangle = \langle 1, \ldots, 1 \rangle = \mathbf{1}_{\mathcal{T}_n} \tag{3.6}$$

(the unit element of the complete lattice $\mathcal{T}_n$), so that $\pi : \Omega \to [0,1]^n$ defines a $\mathcal{T}_n$-valued possibilistic distribution on Ω. Introducing the real-valued possibilistic measure $\Pi_i : \mathcal{P}(\Omega) \to [0,1]$ by $\Pi_i(A) = \bigvee_{\omega \in A} \pi_i(\omega)$ for each $A \subset \Omega$ and each $i = 1, 2, \ldots, n$, and the $\mathcal{T}_n$-valued possibilistic measure $\Pi : \mathcal{P}(\Omega) \to [0,1]^n$ by $\Pi(A) = \bigvee_{*\omega \in A} \pi(\omega)$, $A \subset \Omega$ we obtain easily that $\Pi(A) = \langle \Pi_1(A), \ldots, \Pi_n(A) \rangle \in [0,1]^n$ holds for each $A \subset \Omega$. The mapping π (Π, resp.) will be called the

Cartesian product of the possibilistic distributions $\pi_1, \dots, \pi_n$ (of the possibilistic measures $\Pi_1, \dots, \Pi_n$, resp.). If each possibilistic distribution π_i (each possibilistic measure Π_i, resp.) is ε_i-rank reducible, $0 \le \varepsilon_i < 1$, $i = 1, 2, \dots, n$, their Cartesian product π (Π, resp.) is called ε^*-*rank reducible*, where $\varepsilon^* = \langle \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n \rangle \in [0,1]^n$.

Also the ε^*-rank distribution induced by ε^*-rank reducible $\mathcal{T}_n$-valued possibilistic distribution can be defined by the vector of the values of ε_i-rank distributions induced by the particular ε_i-rank reducible possibilistic distributions $\Pi_i, i = 1, 2, \dots, n$. So, given $i = 1, 2, \dots, n$, let $\alpha^i_j, j = 0, 1, \dots, \infty$ be the sequence of real numbers from $[0,1]$ with respect to which π_i is ε_i-rank reducible. Hence, $\alpha^i_0 = 1$, $\alpha^i_j > \alpha^i_{j+1}$ holds for each j such that $\alpha^i_j > \varepsilon_i$ is the case. Moreover, for each $\omega \in \Omega$ such that $\pi_i(\omega) \ge \varepsilon_i$ holds there exists $j = 0, 1, \dots, \infty$ such that $\pi_i(\omega) = \alpha^i_j$ and for each j with $\alpha^i_j > \varepsilon_i$ there exists $\omega \in \Omega$ such that $\pi_i(\omega) = \alpha^i_j$. Set, for each $\omega \in \Omega, \kappa^i(\varepsilon^*, \omega) = j$, if $\pi_i(\omega) = \alpha^i_j > \varepsilon_i$, set $\kappa^i(\varepsilon^*, \omega) = \infty$, if $\pi_i(\omega) \le \varepsilon_i$ holds, let us recall that $\varepsilon^* = \langle \varepsilon_1, \dots, \varepsilon_n \rangle$. So we obtain, for each $\omega \in \Omega$ and each $i = 1, \dots, n$, the uniquely defined value $\kappa^i(\varepsilon^*, \omega) \in \mathcal{N}^* = \{0, 1, \dots, \infty\}$, i.e., the value of ε_i-rank distribution for $\omega \in \Omega$ induced by the possibilistic distribution π_i on Ω. The value $\kappa(\varepsilon^*, \omega)$ of the ε^*-rank distribution induced by the $\mathcal{T}_n$-valued possibilistic distribution $\pi = \langle \pi_1, \dots, \pi_n \rangle$ and ascribed to $\omega \in \Omega$ is defined by $\kappa(\varepsilon^*, \omega) = \langle \kappa^1(\varepsilon^*, \omega), \dots, \kappa^n(\varepsilon^*, \omega) \rangle (= \langle \kappa^1(\varepsilon_1, \omega), \dots, \kappa^n(\varepsilon_n, \omega) \rangle$, as a matter of fact), and this value belongs to $(\mathcal{N}^*)^n$. So, the value of ε^*-rank distribution induced by $\mathcal{T}_n$-valued possibilistic distribution π on Ω is an n-tuple consisting of non-negative integers or of the value ∞.

As π_i is ε_i-rank reducible, for each $A \subset \Omega$ the value $\pi_i(A)$, if $\Pi_i(A) \ge \varepsilon_i$ holds, if identical with $\alpha^i_{j(A)}$ for just one $j(A) = 0, 1, 2 \dots$ (if $\Pi_i(A) > \varepsilon_i$ holds), or with $\alpha^i_\infty = \varepsilon_i$ (if $\Pi_i(A) = \varepsilon_i$ is the case), so that $K^i(\varepsilon^*, A)$ can be defined by $\kappa^i(\varepsilon^*, \omega)$ for each $\omega \in \Omega$ such that $\pi_i(\omega) = \alpha^i_{j(A)}$ (if $\alpha^i_{j(A)} > \varepsilon_i$ is the case, such an ω always exists. As we proved above (Lemma 2.1), the identity $K^i(\varepsilon^*, A) = K^i(\varepsilon_i, A) = \bigwedge_{\omega \in A} K^i(\varepsilon^*, \omega)$ holds for each $A \subset \Omega$, where $\bigwedge$ is the minimum operation on $\mathcal{N}^*$ induced by the standard linear ordering $\le$ on $\mathcal{N} = \{0, 1, 2, \dots\}$ extended to $\mathcal{N}^*$ when setting $j < \infty$ for each $j \in \mathcal{N}$. Let us consider the partial ordering $\le_*$ on $(\mathcal{N}^*)^n$ defined by $\langle j_1, \dots, j_n \rangle \le_* \langle k_1, \dots, k_n \rangle$, if $j_i \le k_i$ holds for each $i = 1, 2, \dots, n$ let $\bigwedge_*$ denote the infimum operation on $\langle (\mathcal{N}^*)^n, \le_* \rangle$, and set $K(\varepsilon^*, A) = \langle K^1(\varepsilon^*, A), \dots, K^n(\varepsilon^*, A) \rangle$ for every $A \subset \Omega$. Applying the same way of reasoning as when considering the partial ordering $\le_*$ on $[0,1]^n$, we obtain easily that

$$\bigwedge\nolimits_* \{\kappa(\varepsilon^*, \omega) : \omega \in A\} = \bigwedge\nolimits_* \{\langle \kappa^1(\varepsilon^*, \omega), \dots, \kappa^n(\varepsilon^*, \omega) \rangle : \omega \in A\} =$$

$$= \left\langle \bigwedge_{\omega \in A} \kappa^1(\varepsilon^*, \omega), \dots, \bigwedge_{\omega \in A} \kappa^n(\varepsilon^*, \omega) \right\rangle = \langle K^1(\varepsilon^*, A), \dots, K^n(\varepsilon^*, A) \rangle \qquad (3.7)$$

holds for each $A \subset \Omega$. So, not only the $\mathcal{T}_n$-valued possibilistic measure Π on $\mathcal{P}(\Omega)$, but also the corresponding ε^*-rank function $K(\varepsilon^*)$ on $\mathcal{P}(\Omega)$, supposing that π is ε^*-rank reducible, can be expressed and processed as the n-tuple composed from the ε_i-rank functions $K^i(\varepsilon_i)$ induced on $\mathcal{P}(\Omega)$ by the particular ε_i-rank reducible possibilistic distributions $\pi_i, i = 1, 2, \ldots, n$. The mapping $K(\varepsilon^*) : \mathcal{P}(\Omega) \to (\mathcal{N}^*)^n$ will be called the *(n-dimensional)* ε^**-rank function* induced on $\mathcal{P}(\Omega)$ by the $\mathcal{T}_n$-valued possibilistic distribution π on Ω.

ε^*-rank reducible $\mathcal{T}_n$-valued possibilistic distributions π^1 and π^2, $\pi^1 = \langle \pi_1^1, \ldots, \pi_n^1 \rangle, \pi^2 = \langle \pi_1^2, \ldots, \pi_n^2 \rangle$, over a nonempty space Ω are called ε^**-rank equivalent,* $\pi^1 \approx (\varepsilon^*)\pi^2$, in symbols, if $\kappa^1(\varepsilon^*, \omega) = \kappa^2(\varepsilon^*, \omega)$ for each $\omega \in \Omega$. As $\kappa^i(\varepsilon^*, \omega) = \langle \kappa_n^i(\varepsilon^*, \omega), \ldots, \kappa_n^i(\varepsilon^*, \omega) \rangle$ for both $i = 1, 2, \pi^1 \approx (\varepsilon^*)\pi^2$ holds iff $\kappa_j^1(\varepsilon^*, \omega) = \kappa_j^2(\varepsilon^*, \omega)$ for every $\omega \in \Omega$ and every $j = 1, 2, \ldots, n$.

Theorem 3.1 *Let* $\pi^1 = \langle \pi_1^1, \ldots, \pi_n^1 \rangle, \pi_2 = \langle \pi_1^2, \ldots, \pi_n^2 \rangle$ *be* ε^**-rank reducible* $\mathcal{T}_n$*-valued possibilistic distributions over a nonempty space* Ω*. These distributions are* ε^**-rank equivalent if and only if, for each* $A, B \subset \Omega$ *such that* $\Pi^i(A) \geq_* \varepsilon^*$, $\Pi^i(B) \geq_* \varepsilon^*$ *holds for both* $i = 1, 2$*, the inequality* $\Pi^1(A) \leq_* \Pi^1(B)$ *is valid if and only if* $\Pi^2(A) \leq_* \Pi^2(B)$ *is the case. Written in symbols, the equivalence*

$$\begin{aligned} \pi^1 \approx (\varepsilon^*)\pi^2 \quad \Leftrightarrow \quad & (\forall A, B \subset \Omega, \Pi^1(A) \geq_* \varepsilon^*, \Pi^1(B) \geq_* \varepsilon^*, \Pi^2(A) \geq_* \varepsilon^*, \\ & \Pi^2(B) \geq_* \varepsilon^*)[(\Pi^1(A) \leq_* \Pi^1(B)) \Leftrightarrow \\ \Leftrightarrow \quad & (\Pi^2(A) \leq_* \Pi^2(B))] \end{aligned} \tag{3.8}$$

holds.

Proof: Let $\pi^1 \approx (\varepsilon^*)\pi^2$ be the case, where $\varepsilon^* = \langle \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \rangle$, so that $K_i^1(\varepsilon^*, \omega) = K_i^1(\varepsilon_i, \omega) = K_i^2(\varepsilon^*, \omega) = K_i^2(\varepsilon_i, \omega)$ holds for each $\omega \in \Omega$ and each $i = 1, 2, \ldots, n$. Hence, the possibilistic distributions π_i^1 and π_i^2 are ε_i-rank reducible and ε_i-rank equivalent for each $i = 1, 2, \ldots, n$ so that, for each $A, B \subset \Omega$ such that $\Pi_i^j(A) \geq \varepsilon_i$, $\Pi_i^j(B) \geq \varepsilon_i$ holds for both $j = 1, 2$, the inequalities $\Pi_i^1(A) \leq \Pi_i^1(B)$ and $\Pi_i^2(A) \leq \Pi_i^2(B)$ hold simultaneously. So, if $\pi^1 \approx (\varepsilon^*)\pi^2$, $\Pi^j(A) = \langle \Pi_1^j(A), \ldots, \Pi_n^j(A) \rangle \geq_* \varepsilon^* = \langle \varepsilon_1, \ldots, \varepsilon_n \rangle$, $\Pi^j(B) = \langle \Pi_1^j(B), \ldots, \Pi_n^j(B) \rangle \geq \varepsilon^*$ for both $j = 1, 2$, and $\Pi^1(A) = \langle \Pi_1^1(A), \ldots, \Pi_n^1(A) \rangle \leq_* \Pi^1(B) = \langle \Pi_1^1(B), \ldots, \Pi_n^1(B) \rangle$ is the case, then $\varepsilon_i \leq \Pi_i^1(A) \leq \Pi_i^1(B)$ holds for each $i = 1, 2, \ldots, n$, consequently, also $\varepsilon_i \leq \Pi_i^2(A) \leq \Pi_i^2(B)$ holds for each $i = 1, 2, \ldots, n$, and $\Pi^2(A) \leq_* \Pi^2(B)$ follows. Replacing mutually the roles of Π^1 and Π^2 we obtain immediately, that $\pi^1 \approx (\varepsilon^*)\pi^2$ and $\varepsilon^* \leq_* \Pi^2(A) \leq_* \Pi^2(B)$ yields that $\Pi^1(A) \leq_* \Pi^2(A)$ holds, hence, the implication $\pi^1 \approx (\varepsilon^*)\pi^2 \Rightarrow \ldots$ in (3.8) is proved.

The inverse implication will be proved by contradiction. Let $n = 2$, let there exist $A, B \subset \Omega$ such that $\varepsilon^* = \langle \varepsilon_1, \varepsilon_2 \rangle \leq_* \Pi^1(A) \leq_* \Pi^1(B)$, $\varepsilon^* \leq_* \Pi^2(A)$, $\varepsilon^* \leq_* \Pi^2(B)$ holds, but not $\Pi^2(A) \leq_* \Pi^2(B)$ (the case with $\Pi^2(A) \leq_* \Pi^2(B)$ but not $\Pi^1(A) \leq_* \Pi^1(B)$ is the same just with the roles of Π^1 and Π^2 mutually interchanged). Hence, as $\Pi^1(A) = \langle \Pi_1^i(A), \Pi_2^i(A) \rangle$ and $\Pi^i(B) = \langle \Pi_1^i(B), \Pi_2^i(B) \rangle$ for both $i = 1, 2$, both the inequalities $\varepsilon_1 \leq \Pi_1^1(A) \leq \Pi_1^1(B), \varepsilon_2 \leq \Pi_2^1(A) \leq \Pi_2^1(B)$ hold, $\varepsilon_i \leq \Pi_i^2(A)$ and $\varepsilon_i \leq \Pi_i^2(B)$ holds for both $i = 1, 2$, but either $\Pi_1^2(A) \leq \Pi_1^2(B)$ or $\Pi_2^2(A) \leq \Pi_2^2(B)$ does not hold. Let us suppose, without any loss of generality, that $\Pi_1^2(A) \leq \Pi_1^2(B)$ does not hold, hence, that $\Pi_1^2(A) > \Pi_1^2(B)$ is the case.

Both the real-valued possibilistic distributions π_1^1 and π_1^2 over Ω are supposed to be ε_1-rank reducible. Hence, for each $A \subset \Omega$ such that $\Pi_1^1(A) > \varepsilon_1$ and $\Pi_1^2(A) > \varepsilon_1$ holds there exist $\omega_A^1 \in A$ and $\omega_A^2 \in A$ such that $\Pi_1^1(A) = \pi_1^1(\omega_A^1)$ and $\Pi_1^2(A) = \pi_1^2(\omega_A^2)$, and analogously for $B \subset \Omega$, $\Pi_1^1(B) = \pi_1^1(\omega_B^1)$, $\Pi_1^2(B) = \pi_1^2(\omega_B^2)$. The remaining cases to cover the inequalities $\Pi_1^i(C) \geq \varepsilon_1$ for $i = 1, 2$ and $C = A$ or B will be considered below, here $\varepsilon^* = \langle \varepsilon_1, \varepsilon_2 \rangle$.

In order to arrive at a contradiction we suppose, as already introduced, that the inequalities $\Pi_1^1(A) \leq \Pi_1^1(B)$ and $\Pi_1^2(A) > \Pi_1^2(B)$ hold simultaneously. Hence, there exist $\omega_A \in A$ and $\omega_B \in B$ such that $\pi_1^1(\omega_1) \leq \pi_1^1(\omega_B)$ holds for each $\omega_1 \in A$ and $\pi_1^2(\omega_A) > \pi_1^2(\omega_2)$ holds for each $\omega_2 \in B$. As can be easily seen, for each ε-rank reducible real-valued possibilistic distribution π over Ω the equivalence $\kappa(\varepsilon, \omega^1) \geq \kappa(\varepsilon, \omega^2)$ iff $\pi(\omega^1) \leq \pi(\omega^2)$ holds for each ω^1, ω^2 such that $\pi(\omega^1) \geq \varepsilon$ and $\pi(\omega^2) \geq \varepsilon$ holds, and the stronger equivalence $\kappa(\varepsilon, \omega^1) > \kappa(\varepsilon, \omega^2)$ iff $\pi(\omega^1) < \pi(\omega^2)$ is valid holds for each ω^1, ω^2 such that $\pi(\omega^1) > \varepsilon$, $\pi(\omega^2) > \varepsilon$ is the case. Hence, our assumption that $\Pi_1^1(A) \leq \Pi_1^1(B)$ and $\Pi_1^2(A) > \Pi_1^2(B)$ hold simultaneously implies that there exist $\omega_A \in A$ and $\omega_B \in B$ such that $\kappa_1^1(\varepsilon_1, \omega_1) \geq \kappa_1^1(\varepsilon_1, \omega_B)$ holds for each $\omega_1 \in A$ and $\kappa_1^2(\varepsilon_1, \omega_A) < \kappa_1^2(\varepsilon_1, \omega_B)$ holds for each $\omega_2 \in B$. Applying the first inequality to $\omega_1 = \omega_A \in A$ and the other one to $\omega_2 = \omega_B \in B$, we obtain that the inequalities $\kappa_1^1(\varepsilon_1, \omega_A) \geq \kappa_1^1(\varepsilon_1, \omega_B)$ and $\kappa_1^2(\varepsilon_1, \omega_A) < \kappa_1^2(\varepsilon_1, \omega_B)$ should be valid simultaneously, but this contradicts the assumption that $\pi^1 \approx (\varepsilon^*)\pi^2$ holds, hence, in particular, $\pi_1^1 \approx (\varepsilon_1)\pi_1^2$ holds, according to what $\kappa_1^1(\varepsilon_1)$ and $\kappa_1^2(\varepsilon_1)$ should be identical on the whole space Ω.

Let us weaken our former assumption that $\Pi_1^i(C) > \varepsilon_1$ holds for $i = 1, 2$ and $C = A, B$, replacing $>$ by $\geq$. Due to the assumption $\Pi_1^2(A) > \Pi_1^2(B) \geq \varepsilon_1$ we obtain that $\Pi_1^2(A) > \varepsilon$ holds, hence, as proved above, there exists $\omega_A^2 \in A$ such that $\kappa_1^2(\varepsilon_1, \omega_A^2) < \infty$ is the case no matter whether $\Pi_1^2(B) > \varepsilon_1$ or $\Pi_1^2(B) = \varepsilon_1$ is the case. Having analyzed the case with $\Pi_1^1(A) > \varepsilon_1$ above and keeping in mind that, due to the assumption $\Pi_1^1(A) \leq \Pi_1^1(B)$, $\Pi_1^1(B) = \varepsilon_1$ implies that $\Pi_1^1(A) = \varepsilon_1$ holds, we have to consider just this case. However, $\Pi_1^1(A) = \varepsilon_1$ yields that $\pi_1^1(\omega) \leq \varepsilon_1$, hence, $\kappa_1^1(\varepsilon_1, \omega) = \infty$ holds for each $\omega \in A$. So, $\kappa_1^2(\varepsilon_1)$ and $\kappa_1^1(\varepsilon_1)$ are not identical at least for $\omega = \omega_A^2$ and a contradiction is reached again. So, the implication $\pi^1 \approx (\varepsilon^*)\pi^2 \Leftarrow \ldots$ in (3.8) and Theorem 3.1 as a whole are proved. $\square$

4 Axiomatic and Empirical Approaches to Ranking Distributions and Ranking Functions

Ranking distributions and ranking functions were introduced as a secondary notion and useful tool when investigating possibilistic measures and decision making under uncertainty supposing that this uncertainty is quantified and processed by possibilistic measures. However, like numerous mathematically formalized notions, also ranking distributions and ranking functions may be introduced axiomatically as primary notions. A definition may read as follows.

Definition 4.1 *Let Y be a nonempty set, let $\mathcal{N} = \{0, 1, 2, \ldots\}$, let $\mathcal{N}^* = \mathcal{N} \cup \{\infty\}$, let $n \in \mathcal{N}^*$. A mapping $\kappa : Y \to \mathcal{N}^*$ is called an n-ranking distribution on Y, if there exists $y \in Y$ such that $\kappa(y) = 0$ and if for each $k \in \mathcal{N}$, $0 \leq k < n$, the implication*

$$\{y \in Y : \kappa(y) = k\} \neq \emptyset \Rightarrow \{y \in Y : \kappa(y) = k - 1\} \neq \emptyset \tag{4.1}$$

is valid. If $n = \infty$, we write simply ranking distribution instead of ∞-ranking distribution. Each n-ranking distribution κ on Y can be extended to n-ranking function K on $\mathcal{P}(Y)$, setting $K(B) = \bigwedge_{y \in B} \kappa(y)$ for each $\emptyset \neq B \subset Y$ and setting $K(\emptyset) = \infty$, here $\bigwedge$ denotes the standard infimum on $\mathcal{N}^*$ supposing that $k < \infty$ holds for each $k \in \mathcal{N}$. An n-ranking distribution κ on Y is simple, if for each $k \in \mathcal{N}$, $k < n$, there exists at most one $y \in Y$ such that $\kappa(y) = k$.

As can be easily seen, for each $n \in \mathcal{N}^*$, each n-ranking distribution κ on Y and each nonempty system $\mathcal{B}$ of subsets of Y, the relation

$$K\left(\bigcup \mathcal{B}\right) = \bigwedge_{y \in \bigcup \mathcal{B}} \kappa(y) = \bigwedge_{B \in \mathcal{B}} \left(\bigwedge_{y \in \mathcal{B}} \kappa(y) \right) = \bigwedge_{B \in \mathcal{B}} K(B) \tag{4.2}$$

holds, where $\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B$. Each n-ranking distribution κ on Y is obviously also an n-ranking distribution on Y for each $1 \leq m \leq n$. Consequently, for each mapping $\kappa : Y \to \mathcal{N}^*$ such that $\{y \in Y : \kappa(y) = 0\} \neq \emptyset$ the value

$$n(K) = \sup\{m \in \mathcal{N}^* : \kappa \text{ is an } m\text{-ranking distribution on } Y\} \tag{4.3}$$

is uniquely defined and will be called the *ranking order* of the mapping κ. If κ is an n-ranking distribution on Y, its *standard modification* κ^* is the ranking distribution on Y such that $\kappa^*(y) = \kappa(y)$, if $\kappa(y) < n$, and $\kappa^*(y) = \infty$ otherwise. Hence, for each ranking distribution κ on Y the identity $\kappa \equiv \kappa^*$ holds.

Two n-ranking distributions κ_1, κ_2 defined on a nonempty set Y are called n-rank equivalent, if $\kappa_1(y) = \kappa_2(y)$ for each $y \in Y$ such that $\kappa_1(y) < n$ holds, in symbols, $\kappa_1 \approx_n \kappa_2$. As can be easily seen, $\approx_n$ defines an equivalence relation on the space of n-ranking distributions.

Theorem 4.1 *For each n-ranking distribution κ on Y and each $1 > \varepsilon \geq 0$ there exists an ε-rank reducible possibilistic distribution $\pi(\kappa)$ on Y with this property: if π_0 is an ε-rank reducible possibilistic distribution on Y which is ε-rank equivalent to $\pi(\kappa)$, then the n-ranking distribution $\kappa(\pi_0)$ defined by π_0 on Y and the n-ranking distribution κ are n-rank equivalent, i.e., $\kappa \approx_n \kappa(\pi_0)$ holds.*

Proof: For each $k \in \mathcal{N}^*$, let $Y_k = \{y \in Y : \kappa(y) = k\}$. Given $1 > \varepsilon \geq 0$, take real numbers $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ (if $n \in \mathcal{N}$), or $\alpha_0, \alpha_1, \alpha_2, \ldots$ (if $n = \infty$) in such a way that $\alpha_0 = 1 > \alpha_1 > \alpha_2 > \ldots > \alpha_{n-1} > \varepsilon$ ($\alpha_0 = 1 > \alpha_1 > \alpha_2 > \ldots > \varepsilon$, resp.) holds; as the inequality $1 > \varepsilon$ is valid, such a sequence of real numbers

α_i always exists. For each $k \in \mathcal{N}^*$, set $\pi_1(y) = \alpha_k$, if $y \in Y_k$ and $k < n$ holds, set $\pi_1(y) \in [\varepsilon, 0]$ arbitrarily, if $y \in Y_k$ and $k \geq n$ is the case. Then, for $k < n$, $\kappa(\pi_1)(y) = k$ iff $\pi_1(y) = \alpha_k$ iff $y \in Y_k$ iff $\kappa(y) = k$ holds, so that $\kappa(\pi_1)$ is n-rank equivalent to κ. If π_0 is ε-rank equivalent to π_1 defined above, then $\kappa(\pi_1)(y) = \kappa(\pi_0)(y)$ for each $y \in Y$ such that $\kappa(\pi_0)(y) < n$ holds (due to the definition of ε-rank equivalent possibilistic distributions on Y). Hence, $\kappa(\pi_0)$ is n-rank equivalent to $\kappa(\pi_1)$, so that $\kappa(\pi_0)$ is also n-rank equivalent to κ. The assertion is proved. □

Let us mention explicitly, that the n-ranking distribution $\kappa(\pi_1)$ is n-rank equivalent to κ, but need not be identical with κ. Indeed, for $y \in Y$ such that $\pi_1(y) \leq \varepsilon$ holds the definition of $\kappa(\pi_1)$ yields that $\kappa(\pi_1)(y) = \infty$, but this need not hold for $\kappa(y)$, where only the inequality $\kappa(y) \geq n$ follows.

Let Y be a nonempty set, let $\mathbf{y} = \langle y_1, y_2 \ldots \rangle$ be a finite or infinite sequence of elements of Y, let $\mathbf{y}^0 = \langle y_1^0, y_2^0, \ldots \rangle$ be the sequence obtained from $\mathbf{y}$ when preserving just the first occurrences of elements from Y in $\mathbf{y}$ and erasing repeated occurrences. For each $y \in Y$ occurring in $\mathbf{y}$ there exists just one $i_1 \in \mathcal{N}$ such that $y_{i_1}^0 = y$.

Given $\mathbf{y}, y \in Y$ and $n \in \mathcal{N}^*$, let $\kappa_n(\mathbf{y}) = i_1 - 1$, if $y_{i_1}^0 = y$ and $i_1 - 1 < n$ (i.e., $i_1 \leq n$) holds, let $\kappa_n(\mathbf{y}) = \infty$, if $y = y^0)i_1$ for $i_1 > n$ or if y does not occur in $\mathbf{y}$.

In general, different sequences $\mathbf{y}^1, \mathbf{y}^2$ can induce different n-ranking distributions $\kappa_n(\mathbf{y}^1)$ and $\kappa_n(\mathbf{y}^2)$ on Y. The sequences $\mathbf{y}^1, \mathbf{y}^2$ are called *n-rank equivalent*, if $\kappa(\mathbf{y}^1)(y) = \kappa(\mathbf{y}^2)(y)$ holds for each y such that $\kappa(\mathbf{y}^1)(y) < n$ holds. Consequently, $\mathbf{y}^1$ and $\mathbf{y}^2$ are n-rank equivalent, if the initial segments $\langle y_1^{1,0}, \ldots, y_n^{1,0} \rangle$ and $\langle y_1^{2,0}, \ldots, y_n^{2,0} \rangle$ are identical, where $\mathbf{y}^1 = \langle y_1^1, y_2^1, \ldots \rangle$ and $\mathbf{y}^2 = \langle y_1^2, y_2^2, \ldots \rangle$. A reasonable interpretation and intuition behind may read as follows. Let $\mathbf{y} = \langle X_1(\omega), X_2(\omega), \ldots \rangle$ be a sequence of random samples, i.e., a sequence of empirical data (observations, results of experiments, etc.) being at our disposal when we have to decide reasonably among two or more hypotheses or alternatives, but charged by the lack of knowledge as far as the precise value of the parameter is concerned (cf. the next chapter for a more detailed formalization). Taking into consideration only decision procedures based on qualitative comparison ("greater than", "not smaller than", etc.) of values ascribed to various cases by a possibilistic measure possessing $\kappa_n(\mathbf{y})$ as its n-ranking distribution, the case when $\mathbf{y}^1 = \langle X_1(\omega_1), X_2(\omega_1) \ldots \rangle$ and $\mathbf{y}^2 = \langle X_1(\omega_2, X_2(\omega_2), \ldots \rangle$ are n-rank equivalent implies that the decision procedure in question will give identical results no matter whether ω_1 or ω_2 may be the actual value of ω, but supposing that only the first n different results occurring in $\langle X_1(\omega, X_2(\omega, \ldots \rangle$ are used in the decision procedure. A natural weakening of the notion of n-rank equivalence, leading to almost-n-rank equivalence and δ-n-rank equivalence will be introduced below having introduced appropriate tools from probability theory and mathematical statistics.

5 Randomized Ranking Distributions and Functions

Let us develop, in more detail, the idea sketched in the end of the foregoing chapter, supposing that the mappings $X_1, X_2, \ldots$ are random variables defined on a probability space over a nonempty set Ω and taking their values in a measurable space over a nonempty set Y, with perhaps some more specific conditions to be imposed later. To formalize our considerations mathematically, let $\mathcal{A}$ be a nonempty σ-field of subsets of Ω, let $P : \mathcal{A} \to [0,1]$ be a probability measure on $\mathcal{A}$ (i.e., a σ-additive real-valued normalized set function), so that the triple $\langle \Omega \mathcal{A}, P \rangle$ defines a probability space, and let $\mathcal{Y}$ be a σ-field of subsets of Y. Each mapping $X_i : \Omega \to Y$, $i = 1, 2, \ldots$, is a *random variable,* i.e., a measurable mapping defined on $\langle \Omega, \mathcal{A}, P \rangle$ and taking its values in $\langle Y, \mathcal{Y} \rangle$, so that the inverse image of each $B \in \mathcal{Y}$ induced by each X_i is in $\mathcal{A}$. In symbols, the inclusion

$$\bigcup_{i=1}^{\infty} \{\{\omega \in \Omega : X_i(\omega) \in B\} : B \in \mathcal{Y}\} \subset \mathcal{A} \tag{5.1}$$

is valid. Consequently, for each $B \in \mathcal{Y}$ and each $i = 1, 2, \ldots$, the probability $P(\{\omega \in \Omega : X_i(\omega) \in B\})$ is defined. Random variables $X_1, X_2, \ldots$ are *identically distributed,* if for each $B \in \mathcal{Y}$ and each $i = 1, 2, \ldots$ the identity

$$P(\{\omega \in \Omega : X_i(\omega) \in B\}) = P(\{\omega \in \Omega : X_1(\omega) \in B\}) \tag{5.2}$$

is the case. The random variables $X_1, X_2, \ldots$ are *statistically (stochastically) independent*, if for each $n = 1, 2, \ldots$ and each $B_1, B_2, \ldots, B_n \in \mathcal{Y}$ the relation

$$P\left(\bigcap_{i=1}^{n} \{\omega \in \Omega : X_i(\omega) \in B_i\}\right) = \Pi_{i=1}^{n} P(\{\omega \in \Omega : X_i(\omega) \in B_i\}) \tag{5.3}$$

holds. If the random variables $X_1, X_2, \ldots$ are statistically independent and identically distributed, they are shortly denoted as i.i.d. – random variables.

For the sake of simplicity let us limit ourselves, in this chapter, to n-ranking distributions and functions for $n = \infty$, i.e., applying the convention introduced above, to ranking distributions and functions. Consequently, we will write simply $\kappa(\mathbf{y})$ instead of $\kappa_\infty(\mathbf{y})$.

As can be easily proved, if Y is countable and $\mathcal{Y} = \mathcal{P}(Y)$, then for each $k \in \mathcal{N}^*$ and each $y \in Y$ the set $\{\omega \in \Omega : \kappa(\mathbb{X}(\omega)) = k\}$ is in $\mathcal{A}$, where $\mathbb{X}(\omega) = \langle X_1(\omega), X_2(\omega), \ldots \rangle$. Indeed, if $k = \infty$, then the relation

$$\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y) = \infty\} = \bigcap_{i=1}^{\infty} \{\omega \in \Omega : X_i(\omega) \in Y - \{y\}\} \in \mathcal{A} \tag{5.4}$$

is obvious. Let $k < \infty$, let $n \geq k$, then there exist at most countable number of n-tuples $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle$ of elements of Y. Consequently,

$$\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y) = k\} =$$
$$= \bigcup_{n=0}^{\infty} \Big(\bigcup_{\langle \alpha_1,\dots,\alpha_n\rangle \in Y^n, card\{\alpha_1,\dots,\alpha_n\}=k} \{\omega \in \Omega : X_1(\omega) = \alpha_1, X_2(\omega) =$$
$$= \alpha_2, \dots, X_n(\omega) = \alpha_n, X_{n+1}(\omega) = y\} \Big) \tag{5.5}$$

As each finite cylinder $\{\omega \in \Omega : X_1(\omega) = \alpha_1, \dots, X_n(\omega) = \alpha_n, X_{n+1}(\omega) = y\}$ is in $\mathcal{A}$, the set $\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y) = k\}$ is in $\mathcal{A}$ as well.

Consequently, both the ranking distribution $\kappa(\mathbb{X}(\cdot))(\cdot) : \Omega \times Y \to \mathcal{N}^*$, defined in Chapter 4 when taking $\mathbf{y} = \mathbb{X}(\omega) = \langle X_1(\omega), X_2(\omega), \dots\rangle$, as well as the ranking function $K(\mathbb{X}(\cdot))(\cdot) : \Omega \times \mathcal{P}(Y) \to \mathcal{N}^*$ induced by this ranking distribution, are random variables defined on the probability space $\langle \Omega, \mathcal{A}, P\rangle$ and taking their values in $\mathcal{N}^*$, more precisely, in the measurable space $\langle \mathcal{N}^*, \mathcal{P}(\mathcal{N}^*)\rangle$.

The sequence $\mathbb{X} = \langle X_1, X_2, \dots\rangle$ of random variables, each of them taking $\langle \Omega, \mathcal{A}, P\rangle$ into $\langle Y, \mathcal{Y}\rangle$, is called *rank equivalent,* if for each $\omega_1, \omega_2 \in \Omega$ the sequences $X_1(\omega_1), X_2(\omega_1) \dots$ and $X_1(\omega_2), X_2(\omega_2), \dots$ are rank equivalent in the sense defined in Chapter 4, i.e., if $\kappa(\mathbb{X}(\omega_1))(y) = \kappa(\mathbb{X}(\omega_2))(y)$ for every $y \in Y$. This notion can be weakened in two consecutive steps. The sequence $\mathbb{X}$ of random variables is *almost rank equivalent*, if there exists a subset Ω_0 of Ω such that $\Omega_0 \in \mathcal{A}$, $P(\Omega_0) = 1$, and the sequences $\mathbb{X}(\omega_1)\mathbb{X}(\omega_2)$ are rank equivalent for every $\omega_1, \omega_2 \in \Omega_0$. Given a real number $\varepsilon \geq 0$, the sequence $\mathbb{X}$ is *ε-rank equivalent*, if there exists $\Omega_\varepsilon \subset \Omega, \Omega_\varepsilon \in \mathcal{A}$, such that $P(\Omega_\varepsilon) \geq 1 - \varepsilon$ and the sequences $\mathbb{X}(\omega_1), \mathbb{X}(\omega_2)$ are rank equivalent for every $\omega_1, \omega_2 \in \Omega_\varepsilon$ (hence, 0-rank equivalence coincides with almost rank equivalence).

Using the tools and results offered by probability theory and mathematical statistics, some simple results on ranking distributions and functions, taken as numerically – valued random variables, can be introduced and proved. In order to simplify our considerations we suppose, in the rest of this chapter, that the sample space Y is finite or countable with the power-set $\mathcal{P}(Y)$ in the role of the σ-field $\mathcal{Y}$. Random variables $X_1, X_2, \dots$ are supposed to be i.i.d. with $p(y)$ ($p(B)$, resp.) denoting the value $P(\{\omega \in \Omega : X_1(\omega) = y\})$ (the value $P(\{\omega \in \Omega : X_1(\omega) \in B\})$, resp.) for each $y \in Y$ (each $B \subset Y$, resp.). The relation $p(B) = \sum_{y \in B} p(y)$ is evident.

Theorem 5.1 *Under the conditions just stated and for each $y_1, y_2 \in Y$ such that $p(y_1) + p(y_2) > 0$ holds, the relation*

$$P(\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y_1) < \kappa(\mathbb{X}(\omega))(y_2)\}) = p(y_1)(p(y_1) + p(y_2))^{-1} \tag{5.6}$$

is valid.

Proof: The inequality $\kappa(\mathbb{X}(\omega))(y_1) < \kappa(\mathbb{X}(\omega))(y_2)$ is the case if and only if y_1 occurs in the sequence $X_1(\omega), X_2(\omega), \dots$ sooner than y_2 (including the case that

y_2 does not occur at all, as in this case $\kappa(\mathbb{X}(\omega))(y_2) = \infty$). This happens, if there exists $i \in \mathcal{N}$ such that $X_j(\omega) \in Y - \{y_1, y_2\}$ for each $j < i$ and $X_i(\omega) = y_1$, hence,

$$P(\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y_1) < \kappa(\mathbb{X}(\omega))(y_2)\}) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i), \quad (5.7)$$

where

$$A_i = \left(\bigcap_{j=1}^{i-1} \{\omega \in \Omega : X_j(\omega) \in Y - \{y_1, y_2\}\}\right) \cap \{\omega \in \Omega : X_i(\omega) = y_1\} \quad (5.8)$$

for $i > 1$, $A_1 = \{\omega \in \Omega : X_1(\omega) = y_1\}$ for $i = 1$. These random events are obviously mutually disjoint for different i's. Due to the supposed i.i.d. property of the random variables $X_1, X_2, \ldots$ we obtain easily that

$$P(A_i) = (1 - (p(y_1) + p(y_2)))^{i-1} p(y_1) \quad (5.9)$$

holds for every $i = 1, 2, \ldots$, so that

$$\begin{aligned} & P(\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y_1) < \kappa(\mathbb{X}(\omega))(y_2)\}) = \\ = & \sum_{i=1}^{\infty} (1 - (p(y_1) + p(y_2)))^{-1} p(y_1) = p(y_1)(p(y_1) + p(y_2))^{-1}. \quad (5.10) \end{aligned}$$

The assertion is proved. □

The relation (5.6) can be easily generalized by induction as follows.

Theorem 5.2 *Let the conditions of Theorem 5.1 hold, let $y_1, y_2, \ldots, y_n$ be different elements of Y such that $p(y_i) > 0$ is the case for every $i = 1, 2, \ldots, n$. Then the relation*

$$\begin{aligned} & P(\{\omega \in \Omega : \kappa(\mathbb{X}(\omega))(y_1) < \kappa(\mathbb{X}(\omega))(y_2) < \ldots < \kappa(\mathbb{X}(\omega))(y_n)\}) = \\ = & \Pi_{i=1}^{n-1} [p(y_i)(p(y_i) + p(y_{i+1} + \ldots + p(y_n))^{-1}] \quad (5.11) \end{aligned}$$

holds.

Proof: The proof is by induction on $n \geq 2$. For $n = 2$, (5.11) reduces to (5.6). Let (5.11) hold for n, let $y_1, y_2, \ldots, y_{n+1}$ be different elements of Y such that $p(y_i) > 0$, $i = 1, 2, \ldots, n+1$ holds. Without any loss of generality we may suppose that $Y = \{y_1, y_2, \ldots, y_{n+1}\}$, so that $p(y_1) + p(y_2) + \ldots + p(y_n) = 1$. Indeed, if there are other values than $y_1, y_2, \ldots, y_{n+1}$ in Y taken by some $X_i(\omega)$, erase all these $X_i(\omega)$ from the sequence $\mathbb{X}(\omega)$ and replace each $p(y)$

by $p(y)(p(y_1)+p(y_2)+\ldots+p(y_{n+1}))^{-1}$ for every $y \in \{y_1, y_2, \ldots, y_{n+1}\}$. Consequently, writing $\kappa(y_i)$ instead of $\kappa(\mathbb{X}(\omega))(y_i)$ to simplify our notation during this proof, we obtain that $\kappa(y_1) < \kappa(y_j)$ for each $j = 2, 3, \ldots, n+1$ may occur if and only if $X_1(\omega) = y_1$ holds, as in this case $\kappa(X_1(\omega)) = \kappa(y_1) = 0$ and $\kappa(y_i) \geq 1$ holds for each $i = 2, 3, \ldots, n+1$. Due to the supposed i.i.d. assumption imposed on the random variables and due to the induction assumption for n-tuples of elements of Y we obtain that

$$
\begin{aligned}
& P(\{\omega \in \Omega : \kappa(y_1) < \kappa(y_2) < \ldots < \kappa(y_n) < \kappa(y_{n+1})\}) = \\
= \; & P(\{\omega \in \Omega : X_1(\omega) = y_1\} \cap \{\omega \in \Omega : \kappa(y_2) < \kappa(y_3) < \ldots < \kappa(y_{n+1})\}) = \\
= \; & P(\{\omega \in \Omega : X_1(\omega) = y_1\}) P(\{\omega \in \Omega : \kappa(y_2) < \kappa(y_3) < \ldots < \kappa(y_{n+1})\}) = \\
= \; & p(y_1)\Pi_{i=2}^{n}[p(y_i)(p(y_i) + p(y_{i+1}) + \ldots + p(y_{n+1}))^{-1}] = \\
= \; & p(y_1)(p(y_1) + \ldots + p(y_{n+1}))\Pi_{i=2}^{n}[p(y_i)(p(y_i) + p(y_{i+1} + \\
& + \ldots + p(y_{n+1}))^{-1}] = \\
= \; & \Pi_{i=1}^{n}(p(y_i)(p(y_i) + p(y_{i+1}) + \ldots + p(y_{n+1}))^{-1}),
\end{aligned}
$$

as $p(y_1) + p(y_2) + \ldots + p(y_{n+1}) = 1$. The assertion is proved. □

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