An application of non-smooth mechanics in real analysis

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Abstract

Singular and non-smooth constrained evolution problems in mechanics often lead to evolution (quasi-)variational inequalities for regulated functions with values in a Hilbert space X. The goal of this contribution is to show that conversely, the technique of variational inequalities and hysteresis operators in the Kurzweil integral setting can be used for showing the rich topological structure of the space of regulated functions including two independent weak convergence concepts.

Introduction

The theory of regulated functions goes back to Aumann[1] and has been substantially developed by Hönig[2], Tvrdý[3] and Fraňková[4]. Recall that a function f defined in a compact interval $[a,b]$ with values in a Banach space X is said to be *regulated* if for every $t \in [a,b]$ there exist both one-sided limits $f(t+), f(t-) \in X$ with the convention $f(a-) = f(a)$, $f(b+) = f(b)$. We focus our attention here on two independent weak convergence concepts in the Banach space $G(a,b;X)$ of regulated functions endowed with the usual sup-norm. The classical weak convergence is based on representations of bounded linear functionals by functions of bounded variation in terms of the Kurzweil integral. The so-called wbo-convergence ("wbo" stands for "weak bounded oscillation") exists in parallel to the weak convergence and is useful e. g. in the investigation of the limit hysteresis behaviour as $\varepsilon \to 0+$ of solutions to the singularly perturbed ODE

$$\varepsilon \dot{x}(t) + \Psi(x(t)) = y(t)$$

with a non-monotone function Ψ, see Ref. [5].

The bounded oscillation property is equivalent to the concept of "uniformly bounded ε-variation" introduced by Fraňková[4] in connection with a generalized version of the Helly Selection Principle. The present text is a survey of results of Refs. [6]–[7] and its title is motivated by the fact that the equivalence proof is based on analytical properties of the classical dry friction model.

1 Regulated functions

Let $[a,b] \subset \mathbb{R}$ be a nondegenerate compact interval. By $\mathcal{D}_{a,b}$ we denote the set of all divisions of the form

$$d \;=\; \{t_0, \ldots, t_m\}\,, \quad a = t_0 < t_1 < \cdots < t_m = b\,. \tag{1.1}$$

In what follows, we restrict ourselves to functions $[a,b] \to X$, where X is a separarable Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|x| = \sqrt{\langle x, x\rangle}$ for $x \in X$.

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For a given function $g : [a,b] \to X$ and a given division $d \in \mathcal{D}_{a,b}$ of the form (1.1) we define the *variation* $\mathcal{V}_d(g)$ of g on d by the formula

$$\mathcal{V}_d(g) \;=\; \sum_{j=1}^{m} |g(t_j) - g(t_{j-1})|$$

and the *total variation* $\operatorname{Var}_{[a,b]} g$ of g by

$$\operatorname*{Var}_{[a,b]} g \;=\; \sup\{\mathcal{V}_d(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

In a standard way (cf. Ref. [8]) we denote the set of functions of bounded variation by

$$BV(a,b;X) \;=\; \{g : [a,b] \to X\,;\, \operatorname*{Var}_{[a,b]} g < \infty\}\,. \tag{1.2}$$

Let us further introduce the set $S(a,b;X)$ of all *step functions* of the form

$$w(t) \;=\; \sum_{k=0}^{m} \hat{c}_k \chi_{\{t_k\}}(t) \;+\; \sum_{k=1}^{m} c_k \chi_{]t_{k-1},t_k[}(t)\,, \quad t \in [a,b]\,, \tag{1.3}$$

where $d = \{t_0, \ldots, t_m\} \in \mathcal{D}_{a,b}$ is a given division, $\hat{c}_0, \ldots, \hat{c}_m, c_1, \ldots, c_m$ are given elements from X, and χ_A for $A \subset [a,b]$ is the characteristic function of A, that is, $\chi_A(t) = 0$ if $t \notin A$, $\chi_A(t) = 1$ if $t \in A$.

It is well-known (see e. g. the Appendix of Ref. [8]) that every function of bounded variation is regulated. According to Refs. [2]–[3], we denote by $G(a,b;X)$ the set of all regulated functions $f : [a,b] \to X$, and by $G_L(a,b;X)$ and $G_R(a,b;X)$ the space of left-continuous and right-continuous regulated functions on $[a,b]$, respectively. We further set $BV_L(a,b;X) = BV(a,b;X) \cap G_L(a,b;X)$, and $BV_R(a,b;X) = BV(a,b;X) \cap G_R(a,b;X)$. Let us introduce in $G(a,b;X)$ a system of seminorms

$$\|f\|_{[s,t]} \;=\; \sup\{|f(\tau)|\,;\, \tau \in [s,t]\} \tag{1.4}$$

for any subinterval $[s,t] \subset [a,b]$. Indeed, $\|\cdot\|_{[a,b]}$ is a norm which transforms $G(a,b;X)$ into a (non-reflexive and non-separable) Banach space. Proposition 1.1 (iii) below implies that $G(a,b;X)$ is in fact the closure of $S(a,b;X)$, hence also of $BV(a,b;X)$, with respect to this norm.

Let us denote by $\mathbb{R}_+$ the interval $[0,\infty[$ and by Φ the set of all increasing functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\varphi(0) = \varphi(0+) = 0$, $\varphi(+\infty) = +\infty$. For $\varphi \in \Phi$, $g : [a,b] \to X$ and a division $d \in \mathcal{D}_{a,b}$ of the form (1.1) we define the φ*-variation* $\mathcal{V}_d^{\varphi}(g)$ of g on d by the formula

$$\mathcal{V}_d^{\varphi}(g) \;=\; \sum_{j=1}^{m} \varphi\left(|g(t_j) - g(t_{j-1})|\right)$$

and the *total* φ*-variation* $\varphi\text{–}\operatorname{Var}_{[a,b]} g$ of g by

$$\varphi\text{–}\operatorname*{Var}_{[a,b]} g \;=\; \sup\{\mathcal{V}_d^{\varphi}(g)\,;\, d \in \mathcal{D}_{a,b}\}\,.$$

We recall a series of equivalent characterizations of regulated functions. Proofs can be found in Refs. [6]–[7].

Proposition 1.1. *Let $f : [a,b] \to X$ be a given function. Then the following four conditions are equivalent.*

(i) *f is regulated;*

(ii) *For every $\varepsilon > 0$ there exists a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that for every $j = 1, \dots, m$ we have*

$$t_{j-1} < \tau < t < t_j \quad \Longrightarrow \quad |f(t) - f(\tau)| < \varepsilon\,. \tag{1.5}$$

(iii) *For each $\varepsilon > 0$ there exists $w \in S(a,b;X)$ such that $\|f - w\|_{[a,b]} \le \varepsilon$, $\cup_{t\in[a,b]}\{w(t)\} \subset \cup_{t\in[a,b]}\{f(t)\}$, and $\operatorname{Var}_{[a,b]} w \le \operatorname{Var}_{[a,b]} f$.*

(iv) *There exists $\varphi \in \Phi$ such that φ–$\operatorname{Var}_{[a,b]} f \le 1$.*

2 The Kurzweil integral

We briefly outline here an integration concept introduced in Kurzweil[9] which is particularly suitable for the integration of regulated functions. With a division $a = t_0 < t_1 < \cdots < t_m = b$ of the interval $[a,b]$ we associate *partitions* D defined as

$$D = \{(\tau_j, [t_{j-1}, t_j])\,;\, j = 1, \dots, m\}\,; \quad \tau_j \in [t_{j-1}, t_j] \quad \forall j = 1, \dots, m\,. \tag{2.1}$$

The Kurzweil integration theory is based on the notion of a *δ-fine partition*. We define the set

$$\Gamma(a,b) \;=\; \{\delta : [a,b] \to \mathbb{R}\,;\, \delta(t) > 0 \quad \text{for every} \quad t \in [a,b]\}\,. \tag{2.2}$$

An element $\delta \in \Gamma(a,b)$ is called a *gauge*. For $t \in [a,b]$ and $\delta \in \Gamma(a,b)$ we denote

$$I_\delta(t) \;=\;]t - \delta(t), t + \delta(t)[\;. \tag{2.3}$$

Definition 2.1. *Let $\delta \in \Gamma(a,b)$ be a given gauge. A partition D of the form (2.1) is said to be δ-*fine *if for every $j = 1, \dots, m$ we have*

$$\tau_j \;\in\; [t_{j-1}, t_j] \;\subset\; I_\delta(\tau_j)\,. \tag{2.4}$$

The set of all δ-fine partitions is denoted by $\mathcal{F}_\delta(a,b)$.

An easy argument (often referred to as Cousin's Lemma) shows that $\mathcal{F}_\delta(a,b)$ is non-empty for every $\delta \in \Gamma(a,b)$.

For given functions $f, g : [a,b] \to X$ and a partition D of the form (2.1) we define the Kurzweil integral sum $K_D(f,g)$ by the formula

$$K_D(f,g) \;=\; \sum_{j=1}^{m} \langle f(\tau_j), g(t_j) - g(t_{j-1})\rangle\;. \tag{2.5}$$

Definition 2.2. *Let $f, g : [a,b] \to X$ be given. We say that $J \in \mathbb{R}$ is the K-*integral *over $[a,b]$ of f with respect to g and denote*

$$J \;=\; \int_a^b \langle f(t), dg(t)\rangle\;, \tag{2.6}$$

if for every $\varepsilon > 0$ there exists $\delta \in \Gamma(a,b)$ such that for every $D \in \mathcal{F}_\delta(a,b)$ we have

$$|J \;-\; K_D(f,g)| \;\le\; \varepsilon\,. \tag{2.7}$$

Using the fact that the implication

$$\delta \le \min\{\delta_1\,,\ \delta_2\} \quad \Rightarrow \quad \mathcal{F}_\delta(a,b)\ \subset\ \mathcal{F}_{\delta_1}(a,b)\cap\mathcal{F}_{\delta_2}(a,b) \tag{2.8}$$

holds for every $\delta\,,\ \delta_1\,,\ \delta_2\ \in\ \Gamma(a,b)$, we easily check that the value J in Definition 2.2 is uniquely determined. The K-integral has the usual additivity properties with respect to both integrands as well as with respect to the integration domain.

Theorem 2.3. *If $f\ \in\ G(a,b;X)$ and $g\ \in\ BV(a,b;X)$ or $g\ \in\ G(a,b;X)$ and $f\ \in BV(a,b;X)$, then $\int_a^b \langle f(t), dg(t)\rangle$ exists and satisfies the inequality*

$$\left|\int_a^b \langle f(t), dg(t)\rangle\right| \ \le\ \min\left\{\begin{array}{l} \|f\|_{[a,b]}\ \mathrm{Var}_{\,[a,b]}\, g \\ \big(|f(a)|+|f(b)|+\mathrm{Var}_{\,[a,b]}\, f\big)\ \|g\|_{[a,b]} \end{array}\right\}. \tag{2.9}$$

3 Weak convergences in G(a,b; X)

We first cite a representation theorem for bounded linear functionals on the space of regulated functions. For our purposes, it suffices to restrict ourselves to functionals on $G_R(a,b;X)$ or $G_L(a,b;X)$; the general case is treated in Ref. [6] as a slight generalization of results in Refs. [2]–[3]. The prime denotes the topological dual.

Theorem 3.1. *For every functionals $P_R \in G_R(a,b;X)'$, $P_L \in G_L(a,b;X)'$ there exist uniquely determined functions $f, \hat f \in BV(a,b;X)$ such that*

$$P_R(g) \ = \ \langle f(a), g(a)\rangle + \int_a^b \langle f(t), dg(t)\rangle \qquad \forall g \in G_R(a,b;X)\,, \tag{3.1}$$

$$P_L(g) \ = \ \left\langle \hat f(b), g(b)\right\rangle - \int_a^b \left\langle \hat f(t), dg(t)\right\rangle \qquad \forall g \in G_L(a,b;X)\,, \tag{3.2}$$

and we have

$$\|P_R\|\ =\ |f(b)| + \operatorname*{Var}_{[a,b]} f\,, \qquad \|P_L\|\ =\ |\hat f(a)| + \operatorname*{Var}_{[a,b]} \hat f\,. \tag{3.3}$$

In particular, both $G_R(a,b;X)'$, $G_L(a,b;X)'$ are isometrically isomorphic to $BV(a,b;X)$.

Since $G_R(a,b;X)$ or $G_L(a,b;X)$ are not separable, we cannot directly conclude from the Banach-Alaoglu Theorem (cf. Ref. [10]) that bounded sets in $BV(a,b;X)$ are sequentially weakly-star compact. This nevertheless holds true as a consequence of the well-known Helly Selection Principle which we cite in the following form, see Ref. [11].

Theorem 3.2. *Let $\{g_n; n\in\mathbb{N}\}$ be a bounded sequence in $BV(a,b;X)$ such that $\mathrm{Var}_{\,[a,b]}\, g_n \le C$ for every $n\in\mathbb{N}$. Then there exist $g\in BV(a,b;X)$ and a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\mathrm{Var}_{\,[a,b]}\, g \le C$ and the sequence $g_{n_k}(t)$ weakly converges in X to $g(t)$ for every $t\in[0,T]$.*

Fraňková[4] generalized this statement in a remarkable way to the space of regulated functions. The original result was restricted to the case $\dim X < \infty$, a straightforward extension to the Hilbert case has been done in Ref. [12]. Fraňková's idea was to replace the bounded variation condition by the condition **(F)** in Theorem 3.3 below which she called "uniformly bounded ε-variation".

Theorem 3.3. *Let $\{f_n\,;\, n \in \mathbb{N}\}$ be a bounded sequence in $G(a,b;X)$ satisfying the condition*

$$
\textbf{(F)} \qquad \left\{
\begin{array}{l}
\forall \varepsilon > 0 \quad \exists L(\varepsilon) > 0 \quad \forall n \in \mathbb{N} \quad \exists g_n \in BV(a,b;X) : \\[2ex]
\qquad\qquad \|f_n - g_n\|_{[a,b]} \le \varepsilon\,, \ \operatorname{Var}_{[a,b]} g_n \le L(\varepsilon)\,.
\end{array}
\right.
$$

Then there exist $f \in G(a,b;X)$ and a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $f_{n_k}(t)$ weakly converge in X to $f(t)$ for every $t \in [0,T]$.

Clearly, the limit of a merely pointwise convergent sequence of regulated functions is not necessarily regulated; the result that f belongs to $G(a,b;X)$ is therefore highly non-trivial. On the other hand, it is not easy to check in applications that condition **(F)** holds. We list now two criteria which are equivalent to **(F)**; for a detailed proof see Refs. [6]–[7].

Theorem 3.4. *For every sequence $\{f_n\}$ in $G(a,b;X)$, each of the following two conditions is equivalent to* **(F)***:*

(UBO) ("uniformly bounded oscillation") *There exists $R > 0$ such that $\|f_n(\cdot) - f_n(a)\|_{[a,b]} \le R$ for all $n \in \mathbb{N}$, and*

$$
\left\{
\begin{array}{l}
\forall r > 0 \quad \exists N(r) \in \mathbb{N} \quad \forall n \in \mathbb{N} \quad \forall\ ]a_k, b_k[\ \subset [a,b]\,, \quad k = 1, \dots m\,, \\[2ex]
\qquad\qquad]a_k, b_k[\ \cap\ ]a_j, b_j[\ = \emptyset \quad \textit{for} \quad j \ne k : \\[2ex]
\Big(|f_n(b_k) - f_n(a_k)| \ge r \quad \forall k = 1, \dots, m \Big) \quad \Rightarrow \quad m \le N(r)\,.
\end{array}
\right.
$$

(Φ) *There exists a function $\varphi \in \Phi$ such that φ–$\operatorname{Var}_{[a,b]} f_n \le 1$ for all $n \in \mathbb{N}$.*

We will call the convergence in Theorem 3.3 the *wbo-convergence*. Let us briefly comment on the proof of Theorem 3.4. The implication **(F)** $\Rightarrow$ **(UBO)** is obvious. The equivalence **(UBO)** $\Leftrightarrow$ **(Φ)** is easy as well; in fact, one finds an almost "explicit" relation between the functions $N(r)$ and $\varphi(x)$. The implication **(UBO)** $\Rightarrow$ **(F)** is indeed crucial, because it raises the question how one can find a function $g \in BV(a,b;X)$ in an ε-neighbourhood of $f \in G(a,b;X)$ such that the total variation of g is dominated by the value of $N(r)$ from **(UBO)** at some $r(\varepsilon)$. The case $\dim X = 1$ can be treated separately: for every regulated function f, every $\varepsilon > 0$ and every initial condition $g(a) \in [f(a) - \varepsilon, f(a) + \varepsilon]$, there exists a unique function g with the property that its total variation is minimal on each subinterval $[a,t]$, $t \in [a,b]$, among all functions in the ε-neighbourhood of f. Moreover, this function g is piecewise monotone and its total variation can be directly computed in terms of $N(2\varepsilon)$. This result goes back to A. Vladimirov and a proof for continuous functions can be found in Ref. [13]. An extension to $L^\infty(a,b)$ (that is, even beyond regulated functions) has been done in Ref. [14]. Examples show that this BV-minimization property cannot be expected in the vector-valued case. We propose instead a technique based on dry friction models similar to those in Ref. [15] to find an upper bound for the total variation of solutions to a variational inequality with a regulated input in the form of Problem **(P)** below.

4 Variational inequalities

Consider a family of convex sets $Z(v) \subset X$ parametrized by elements v of a closed subset V of a Banach space Y. We make the following hypothesis.

(H) (i) The mapping $\Delta : V \times V \to \mathbb{R}_+ : (v,w) \mapsto d_H(Z(v), Z(w))$, where d_H denotes the Hausdorff distance, is continuous.

(ii) For every $v \in V$ there exist $x(v) \in X$ and $\varrho(v) > 0$ such that the closed ball $B_{\varrho(v)}(x(v))$ of radius $\varrho(v)$ and centered in $x(v)$ is contained in $Z(v)$.

According to Section 3 of Ref. [7], we state Problem **(P)** in Kurzweil integral form as follows.

(P) Given $T > 0$, $f \in G_R(0,T\,;V)$, and $g^0 \in Z(f(0))$, find $g \in BV_R(0,T\,;X)$ such that

(i) $g(t) \in Z(f(t)) \qquad \forall t \in [0,T]$,

(ii) $g(0) = g^0$,

(iii) $\int_0^T \langle w(t) - g(t), dg(t)\rangle \ \geq\ 0 \qquad \forall w \in \mathcal{T}(f)$,

where $\mathcal{T}(f)$ is the set of admissible test functions

$$\mathcal{T}(f) \ = \ \{w \in G(0,T\,;X)\,;\, w(t) \in Z(f(t)) \ \ \forall t \in [0,T]\}\,. \tag{4.1}$$

We now summarize the main results from Sections 3 and 4 in Ref. [7].

Theorem 4.1. *Let Hypothesis* **(H)** *hold. Then for every* $f \in G_R(0,T\,;V)$ *and* $g^0 \in Z(f(0))$ *there exists a unique solution* $g \in BV_R(0,T\,;X)$ *to Problem* **(P)**. *Moreover, if* $\{f_n\}$ *is a uniformly convergent sequence in* $G_R(0,T\,;Y)$ *and the initial conditions* g_n^0 *converge strongly in* X, *then the corresponding solutions* g_n *converge uniformly in* $G(0,T\,;X)$, *have uniformly bounded variation, and their limit solves the limit problem.*

We actually need a strengthened form of the above statement, where a bound for the total variation of the solution sequence $\{g_n\}$ is obtained under the weaker assumption of uniformly bounded oscillation of f_n. In the whole generality of Hypothesis **(H)**, this would lead to technical difficulties. Instead, we restrict ourselves to a simpler special case.

(HH) We assume that $V = Y = X$, and that there exists a convex closed set $Z \subset X$ and $\varrho > 0$ such that

(i) $Z(v) = v - Z$ for all $v \in X$,

(ii) $B_\varrho(0) \subset Z$.

The solution operator $(f, g^0) \mapsto g$ is then called the *play operator* which is often used in the theory of constitutive models in plasticity, where the boundary of Z represents the yield surface. We are interested in its following property.

Theorem 4.2. *Let Hypothesis* **(HH)** *hold, let* $f \in G_R(0,T\,;X)$ *and* $g^0 \in f(0) - Z$ *be given, and let* $g \in BV_R(0,T\,;X)$ *be the solution to Problem* **(P)**. *Let* $N(r)$ *be the function from the* **(UBO)** *property in Theorem 3.4 relative to* f. *Then there exists a polynomial* p *with coeficients depending only on* ϱ *and* $|f(0) - g^0|$ *such that* $\mathrm{Var}_{[a,b]}\, g \leq p(N(\varrho/2))$.

This estimate enables us to finish the proof of the implication **(UBO)** $\Rightarrow$ **(F)**: for every $f \in G_R(a,b;X)$ and $\varepsilon > 0$ we use Theorem 4.2 with $Z = B_\varepsilon(0)$ and initial condition shifted to $t = a$ to obtain $L(\varepsilon) = p(N(\varepsilon/2))$. If f is not right-continuous, we refer to Proposition 1.1 (ii) which states that there are only finitely many discontinuities of amplitude larger than, say, $\varepsilon/2$. We find a function $\tilde{g}$ for the right-continuous representative $\tilde{f}$ of f according to the above recipe with ε replaced by $\varepsilon/2$ and modify it accordingly at points where $|f - \tilde{f}| > \varepsilon/2$.

5 Strong and weak convergences in G(a,b; X)

Compact sets in $G(a,b;X)$ obviously have uniformly bounded oscillation. In particular, the uniform convergence implies both the weak and the wbo-convergence. The following variant of the Arzelà-Ascoli compactness criterion extends the results of Section 2 of Ref. [4] to the infinite-dimensional case.

Theorem 5.1. *Let $U \subset G(a,b;X)$ be a given set. Then the following two conditions are equivalent.*

(i) *The set U is relatively compact in $G(a,b;X)$;*

(ii) *There exists a compact set $K \subset X$ such that $f(t) \in K$ for each $f \in U$ and $t \in [a,b]$, and for every $\varepsilon > 0$ there exists a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that for every $f \in U$ and every $j = 1, \dots, m$ we have*

$$t_{j-1} < \tau < t < t_j \quad \Longrightarrow \quad |f(t) - f(\tau)| < \varepsilon \,. \tag{5.1}$$

Condition (5.1) can be restated independently of the compactness concept as follows.

Proposition 5.2. *For any set $U \subset G(a,b;X)$, the following two conditions are equivalent.*

(i) *The set U is bounded, and for every $\varepsilon > 0$ there exists a division $d = \{t_0, \dots, t_m\} \in \mathcal{D}_{a,b}$ such that for every $f \in U$ and every $j = 1, \dots, m$ condition (5.1) holds;*

(ii) *The set $U_a := \{f(a)\,;\, f \in U\}$ is bounded, and there exist an increasing function $v : [a,b] \to \mathbb{R}_+$ and a bounded concave non-decreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $v(a) = 0$, $\psi(0) = 0$, and for all $f \in U$ and $a \le \tau < t \le b$ we have*

$$|f(t) - f(\tau)| \;<\; \psi(v(t) - v(\tau))\,. \tag{5.2}$$

We conclude the paper by illustrating the relationship between the two weak convergence concepts. Easy examples show that there exist weakly convergent sequences which do not wbo-converge as well as wbo-convergent sequences which do not converge weakly. Integral convergence characterizations also exhibit interesting differences (notice in particular the different roles of f and g in the integrals).

Proposition 5.3. *Let $f_n, f \in G_R(a,b;X)$ for $n \in \mathbb{N}$ be given. Then*

(i) *f_n converges weakly to f in $G_R(a,b;X)$ if and only if*

$$\int_a^b \langle g_n(t), df_n(t)\rangle \to \int_a^b \langle g(t), df(t)\rangle$$

for every sequence $\{g_n\}$ in $BV(a,b;X)$ such that

$$|g_n(a) - g(a)| \to 0\,, \quad \operatorname*{Var}_{[a,b]}(g_n - g) \to 0.$$

(ii) *f_n wbo-converges to f in $G_R(a,b;X)$ if and only if*

$$\int_a^b \langle f_n(t), dg_n(t)\rangle \to \int_a^b \langle f(t), dg(t)\rangle$$

for every sequence $\{g_n\}$ in $BV(a,b;X)$ such that

$$\operatorname*{Var}_{[a,b]} g_n \le 1\,, \quad |g_n(a) - g(a)| \to 0\,, \quad \|g_n - g\|_{[a,b]} \to 0.$$

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