

A BOTTLE IN A FREEZER*

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Abstract. We propose here a model for solidification of the liquid content of an elastic bottle in a freezer. The main goal is to explain the occurrence of high stresses inside the bottle. As a by-product, we derive a formula for the undercooling coefficient in terms of the elasticity constants, latent heat, and the phase expansion coefficient. We investigate the well-posedness of the three-dimensional model: we prove the existence and uniqueness of a solution for the corresponding initial-boundary value problem which couples a PDE with an integrodifferential equation and an ordinary differential inclusion ruling the evolution of the phase parameter. Finally, we prove some results on the long time behavior of solutions.

Key words. phase transitions, well-posedness, Moser iteration schemes, long time dynamics

AMS subject classifications. 80A22, 35K50, 35B40

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1. Introduction. We derive a simple model for solid-liquid phase transition of a medium inside a deformable elastic container. The main goal is to give a qualitative and quantitative description of the interaction between volume, pressure, phase, and temperature changes in the situation that the specific volume of the solid phase exceeds the specific volume of the liquid phase. We compute the undercooling coefficient for the special case of water and ice.

There is an abundant classical literature on the study of phase transition processes; see, e.g., the monographs [3], [5], [21] and the references therein. It seems, however, that only a few publications take into account different mass densities/specific volumes of the phases. In [6], the authors proposed to interpret a phase transition process in terms of a balance equation for macroscopic motions, and to include the possibility of voids. Well-posedness of an initial-boundary value problem associated with the resulting PDE system is proved there.

The microscopic approach has been pursued in [7] in the case of two different densities ϱ_1 and ϱ_2 for the two substances undergoing phase transitions. The evolution of a liquid substance, e.g., water, in a rigid container subject to freezing is described by a mass balance in Eulerian coordinates, an entropy balance, and a phase field equation. The flow is governed by a counterpart of the Darcy law. Since the density ϱ_2 of ice is lower than the density ϱ_1 of water, experiments—for instance, the freezing of a glass bottle filled with water—show that the water pressure increases up to the rupture of the bottle. When the container is not impermeable, freezing may produce a nonhomogeneous material, for instance, water, ice, or sorbet. This effect is mainly produced by the choice of boundary conditions. This particular example is treated

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in [7], where the model is presented and a suitable variational formulation of the resulting nonlinear and singular PDE system is solved. In the present paper, we have also other applications in mind.

Let us also mention the papers [18] and [19] dealing with macroscopic stresses in phase transitions models, where the different properties of the viscous (liquid) and elastic (solid) phases are taken into account and the coexisting viscous and elastic properties of the system are given a distinguished role, under the working assumption that they indeed influence the phase transition process. The model there includes inertia, viscous, and shear viscosity effects (depending on the phases), while thermal and phase expansions of the substance are neglected. This is reflected in the analytical expressions of the associated PDEs for the strain $\mathbf{u}$ and the phase parameter χ : the χ -dependence, e.g., in the stress-strain relation, leads to the possible degeneracy of the elliptic operator therein. In [18] and [19], respectively, local existence (in the three-dimensional case) and well-posedness (in the one-dimensional case) for the corresponding initial-boundary value problems are proved. Finally, we can quote in this framework the model analyzed in [14] and [15], which pertains to nonlinear thermoviscoplasticity: in the one-dimensional (in space) case, the authors prove the global well-posedness of a PDE system, incorporating both hysteresis effects and modeling phase change, which, however, does not display a degenerating character.

Another coupled system for temperature, displacement, and phase parameter has been derived in order to model the full thermomechanical behavior of shape memory alloys. The first mathematical results were published in [4], while a long list of references for further developments can be found in the monographs [5] and [21].

Here, in section 2, we derive a completely different model without referring to any microscopic balance laws, and we deal exclusively with physically measurable quantities. We assume that the displacements are small. This enables us to state the system in Lagrangian coordinates. The main difference with respect to the Eulerian framework in [7] is that in Lagrangian coordinates, the mass conservation law means that the mass density is constant and does not depend on the phase, while the specific volumes of the liquid and solid phases are possibly different. For simplicity, we assume that the speed of sound, specific heat, heat conductivity, viscosity, and thermal expansion coefficient do not depend on the phase; the evolution is slow; and the shear viscosity, shear stresses, and inertia effects are negligible. In particular, if the heat conductivity and the speed of sound are allowed to depend on the phase, the resulting mathematical problem will no longer be accessible to our method.

The process is driven by energy balance, quasi-static momentum balance, and a phase dynamics equation. Continuing in section 2, we verify the thermodynamic consistency of the model, and in section 3 we study the equilibria. We observe there that a pure solid state can be reached only if the external temperature is below a certain threshold, which is lower than the freezing point and depends in particular on the elasticity of the boundary. For water and ice, we explicitly compute the undercooling rate, which turns out to be around 5% of the absolute melting temperature if the container is rigid. For intermediate temperatures between freezing point and undercooling limit, there exists a continuum of distinct equilibria with mixtures of solid and liquid. If in this situation the bottle breaks, an instantaneous solidification takes place. The quantitative values are indeed related to our simplified model and we do not pretend that they describe the physical process with high accuracy. They are intended only to give a rough idea about the orders of magnitude.

The main results of this paper are stated in the last two sections. The well-

posedness of the three-dimensional model is established in section 4, and the asymptotic stabilization of the process is proved in section 5.

2. The model. As the reference states, we consider a liquid substance contained in a bounded connected bottle $\Omega \subset \mathbf{R}^3$ with boundary of class $C^{1,1}$. The state variables are the absolute temperature $\theta > 0$, the displacement $\mathbf{u} \in \mathbf{R}^3$, and the phase variable $\chi \in [0, 1]$. The value $\chi = 0$ means solid, $\chi = 1$ means liquid, and $\chi \in (0, 1)$ is a mixture of the two.

We make the following modeling hypotheses.

(A1) The displacements are small. Therefore, we state the problem in *Lagrangian coordinates*, in which the mass conservation is equivalent to the condition of a constant mass density $\varrho_0 > 0$.

(A2) The substance is isotropic and compressible, and the speed of sound does not depend on the phase.

(A3) The evolution is slow, and we neglect shear viscosity and inertia effects.

(A4) We neglect shear stresses and gravity effects.

In agreement with (A1), we define the strain ε as an element of the space $\mathbf{T}_{\text{sym}}^{3 \times 3}$ of symmetric tensors by the formula

$$(2.1) \quad \varepsilon = \nabla_s \mathbf{u} := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

Let $\delta \in \mathbf{T}_{\text{sym}}^{3 \times 3}$ denote the Kronecker tensor. By (A4), the elasticity matrix $\mathbf{A}$ has the form

$$(2.2) \quad \mathbf{A}\varepsilon = \lambda(\varepsilon : \delta) \delta,$$

where “ $:$ ” is the canonical scalar product in $\mathbf{T}_{\text{sym}}^{3 \times 3}$, and $\lambda > 0$ is the Lamé constant (or *bulk elasticity modulus*), which we assume to be independent of χ by virtue of (A2). Note that λ is related to the speed of sound v_0 by the formula $v_0 = \sqrt{\lambda/\varrho_0}$.

We want to model the situation where the specific volume V_{solid} of the solid phase is larger than the specific volume V_{liquid} of the liquid phase. Considering the liquid phase as the reference state, we introduce the dimensionless phase expansion coefficient $\alpha = (V_{\text{solid}} - V_{\text{liquid}})/V_{\text{liquid}} > 0$, and we define the phase expansion strain $\tilde{\varepsilon}$ by

$$(2.3) \quad \tilde{\varepsilon}(\chi) = \frac{\alpha}{3}(1 - \chi)\delta.$$

We fix positive constants c_0 (specific heat), L_0 (latent heat), θ_c (freezing point at standard atmospheric pressure), γ_0 (phase relaxation coefficient), and β (thermal expansion coefficient), and we assume the specific free energy f in the form

$$(2.4) \quad f = c_0 \theta \left(1 - \log \left(\frac{\theta}{\theta_c} \right) \right) + \frac{\lambda}{2\varrho_0} ((\varepsilon - \tilde{\varepsilon}(\chi)) : \delta)^2 - \frac{\beta}{\varrho_0} (\theta - \theta_c) \varepsilon : \delta \\ + L_0 \left(\chi \left(1 - \frac{\theta}{\theta_c} \right) + I(\chi) \right),$$

where I is the indicator function of the interval $[0, 1]$. We show in (2.32)–(2.34) that this choice is compatible with the principles of thermodynamics.

To derive the balance equations, we first proceed formally, assuming that the temperature is positive. This assumption will be justified in the subsequent sections.

The stress tensor σ is decomposed into the sum $\sigma^v + \sigma^e$ of the viscous component σ^v and elastic component σ^e . The state functions σ^v , σ^e , s (specific entropy), and e (specific internal energy) are given by the formulas

$$(2.5) \quad \sigma^v = \nu(\varepsilon_t : \delta)\delta,$$

$$(2.6) \quad \sigma^e = \varrho_0 \frac{\partial f}{\partial \varepsilon} = (\lambda(\varepsilon : \delta - \alpha(1 - \chi)) - \beta(\theta - \theta_c))\delta,$$

$$(2.7) \quad s = -\frac{\partial f}{\partial \theta} = c_0 \log\left(\frac{\theta}{\theta_c}\right) + \frac{L_0}{\theta_c}\chi + \frac{\beta}{\varrho_0}\varepsilon : \delta,$$

$$(2.8) \quad e = f + \theta s = c_0\theta + \frac{\lambda}{2\varrho_0}(\varepsilon : \delta - \alpha(1 - \chi))^2 + \frac{\beta}{\varrho_0}\theta_c\varepsilon : \delta + L_0(\chi + I(\chi)),$$

where $\nu > 0$ is the volume viscosity coefficient. The scalar quantity

$$(2.9) \quad p := -\nu\varepsilon_t : \delta - \lambda(\varepsilon : \delta - \alpha(1 - \chi)) + \beta(\theta - \theta_c)$$

is the *pressure*, and the stress has the form $\sigma = -p\delta$. The process is governed by the balance equations

$$(2.10) \quad \operatorname{div} \sigma = 0 \quad (\text{mechanical equilibrium}),$$

$$(2.11) \quad \varrho_0 e_t + \operatorname{div} \mathbf{q} = \sigma : \varepsilon_t \quad (\text{energy balance}),$$

$$(2.12) \quad -\gamma_0 \chi_t \in \partial_\chi f \quad (\text{phase relaxation law}),$$

where ∂_χ is the partial subdifferential with respect to χ , and $\mathbf{q}$ is the heat flux vector that we assume, for simplicity, is in the form

$$(2.13) \quad \mathbf{q} = -\kappa \nabla \theta$$

with a constant heat conductivity $\kappa > 0$. The equilibrium equation (2.10) can be rewritten in the form $\nabla p = 0$; hence

$$(2.14) \quad p(x, t) = P(t),$$

where P is a function of time only, which is to be determined. Recall that in the reference state $\varepsilon : \delta = \varepsilon_t : \delta = 0$, $\chi = 1$, and at standard pressure P_{stand} , the freezing temperature is θ_c . We thus see from (2.9) that $P(t)$ is in fact the deviation from the standard pressure. We assume also that the external pressure is in the form $P_{ext} = P_{stand} + p_0$ with a constant deviation p_0 . The normal force acting on the boundary is $(P(t) - p_0)\mathbf{n}$, where $\mathbf{n}$ denotes the unit outward normal vector. We assume an elastic response of the boundary, and a heat transfer proportional to the inner and outer temperature difference. On $\partial\Omega$, we thus prescribe boundary conditions for $\mathbf{u}$ and θ in the form

$$(2.15) \quad (P(t) - p_0)\mathbf{n} = \mathbf{k}(x)\mathbf{u},$$

$$(2.16) \quad \mathbf{q} \cdot \mathbf{n} = h(x)(\theta - \theta_\Gamma)$$

with a given symmetric positive definite matrix $\mathbf{k}$ (elasticity of the boundary), a positive function h (heat transfer coefficient), and a constant $\theta_\Gamma > 0$ (external temperature). This enables us to find an explicit relation between $\operatorname{div} \mathbf{u}$ and P . Indeed,

on $\partial\Omega$ we have by (2.15) that $\mathbf{u} \cdot \mathbf{n} = (P(t) - p_0)\mathbf{k}^{-1}(x)\mathbf{n}(x) \cdot \mathbf{n}(x)$. Assuming that $\mathbf{k}^{-1}\mathbf{n} \cdot \mathbf{n}$ belongs to $L^1(\partial\Omega)$, we set

$$(2.17) \quad \frac{1}{K_\Gamma} = \int_{\partial\Omega} \mathbf{k}^{-1}(x)\mathbf{n}(x) \cdot \mathbf{n}(x) \, ds(x)$$

and obtain by Gauss's theorem that

$$(2.18) \quad U_\Omega(t) := \int_\Omega \operatorname{div} \mathbf{u}(x, t) \, dx = \frac{1}{K_\Gamma} (P(t) - p_0).$$

Under the small strain hypothesis, the function $\operatorname{div} \mathbf{u}$ describes the local relative volume increment. Hence, (2.18) establishes a linear relation between the total relative volume increment $U_\Omega(t)$ and the relative pressure $P(t) - p_0$. We have $\varepsilon : \delta = \operatorname{div} \mathbf{u}$, and thus the mechanical equilibrium equation (2.14), due to (2.9) and 2.18, reads

$$(2.19) \quad \nu \operatorname{div} \mathbf{u}_t + \lambda(\operatorname{div} \mathbf{u} - \alpha(1 - \chi)) - \beta(\theta - \theta_c) = -p_0 - K_\Gamma U_\Omega(t).$$

As a consequence of (2.4), the energy balance and the phase relaxation equation in (2.11)–(2.12) have the form

$$(2.20) \quad \varrho_0 c_0 \theta_t - \kappa \Delta \theta = \nu(\operatorname{div} \mathbf{u}_t)^2 - \beta \theta \operatorname{div} \mathbf{u}_t - (\alpha \lambda(\operatorname{div} \mathbf{u} - \alpha(1 - \chi)) + \varrho_0 L_0) \chi_t,$$

$$(2.21) \quad -\varrho_0 \gamma_0 \chi_t \in \alpha \lambda(\operatorname{div} \mathbf{u} - \alpha(1 - \chi)) + \varrho_0 L_0 \left(1 - \frac{\theta}{\theta_c} + \partial I(\chi) \right),$$

where ∂ denotes the subdifferential. For simplicity, we now set

$$(2.22) \quad c := \varrho_0 c_0, \quad \gamma := \varrho_0 \gamma_0, \quad L := \varrho_0 L_0.$$

The system now reduces to the following three scalar equations: one PDE and two “ODEs” for three unknown functions θ , χ , and $U = \operatorname{div} \mathbf{u}$. Note that, mathematically, $\partial I(\chi)$ is the same as $L \partial I(\chi)$:

$$(2.23) \quad c \theta_t - \kappa \Delta \theta = \nu U_t^2 - \beta \theta U_t - (\alpha \lambda(U - \alpha(1 - \chi)) + L) \chi_t,$$

$$(2.24) \quad \nu U_t + \lambda U = \alpha \lambda(1 - \chi) + \beta(\theta - \theta_c) - p_0 - K_\Gamma U_\Omega(t),$$

$$(2.25) \quad -\gamma \chi_t \in \alpha \lambda(U - \alpha(1 - \chi)) + L \left(1 - \frac{\theta}{\theta_c} \right) + \partial I(\chi),$$

with $U_\Omega(t) = \int_\Omega U(x, t) \, dx$, and with boundary conditions (2.16), (2.13). To find the vector function $\mathbf{u}$, we first define Φ as a solution of the Poisson equation $\Delta \Phi = U$ with the Neumann boundary condition $\nabla \Phi \cdot \mathbf{n} = K_\Gamma U_\Omega(t) \mathbf{k}^{-1}(x) \mathbf{n}(x) \cdot \mathbf{n}(x)$. With this Φ , we find $\tilde{\mathbf{u}}$ as a solution to the problem

$$(2.26) \quad \operatorname{div} \tilde{\mathbf{u}} = 0 \quad \text{in } \Omega \times \infty,$$

$$(2.27) \quad \tilde{\mathbf{u}} \cdot \mathbf{n} = 0, \quad (\tilde{\mathbf{u}} + \nabla \Phi - K_\Gamma U_\Omega \mathbf{k}^{-1} \mathbf{n}) \times \mathbf{n} = 0 \quad \text{on } \partial\Omega \times (0, \infty),$$

and set $\mathbf{u} = \tilde{\mathbf{u}} + \nabla \Phi$. Then $\mathbf{u}$ satisfies a.e. in Ω the equation $\operatorname{div} \mathbf{u} = U$, together with the boundary condition (2.15), that is, $\mathbf{u} = K_\Gamma U_\Omega \mathbf{k}^{-1} \mathbf{n}$ on $\partial\Omega$.

For the solution to (2.26)–(2.27), we refer to [9, Lemma 2.2] which states that for each $\mathbf{g} \in H^{1/2}(\partial\Omega)^3$ satisfying $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n} \, ds(x) = 0$ there exists a function $\tilde{\mathbf{u}} \in H^1(\Omega)^3$, unique up to an additive function $\mathbf{v}$ from the set V of divergence-free $H^1(\Omega)$ functions

vanishing on $\partial\Omega$ such that $\operatorname{div} \tilde{\mathbf{u}} = 0$ in Ω , $\tilde{\mathbf{u}} = \mathbf{g}$ on $\partial\Omega$. In terms of the system (2.26)–(2.27), it suffices to set $\mathbf{g} = ((\nabla\Phi - K_\Gamma U_\Omega \mathbf{k}^{-1} \mathbf{n}) \times \mathbf{n}) \times \mathbf{n}$ and use the identity $(\mathbf{b} \times \mathbf{n}) \times \mathbf{n} = (\mathbf{b} \cdot \mathbf{n}) \mathbf{n} - \mathbf{b}$ for every vector $\mathbf{b}$. Moreover, the estimate

$$(2.28) \quad \inf_{\mathbf{v} \in V} \|\tilde{\mathbf{u}} + \mathbf{v}\|_{H^1(\Omega)} \leq C \|\mathbf{g}\|_{H^{1/2}(\partial\Omega)} \leq \tilde{C} \|\Phi\|_{H^2(\Omega)}$$

holds with some constants $C, \tilde{C}$. The required regularity is available here by virtue of the assumption that Ω is of class $C^{1,1}$, provided $\mathbf{k}^{-1}$ belongs to $H^{1/2}(\partial\Omega)$. Note that a weaker formulation of problem (2.26)–(2.27) can be found in [1, section 4].

Due to our hypotheses (A3), (A4), we thus lose any control on possible volume preserving turbulences $\mathbf{v} \in V$. This, however, has no influence on the system (2.23)–(2.25), which is the subject of our interest here. Inequality (2.28) shows that if U is small in agreement with hypothesis (A1), then also $\mathbf{v}$ can be chosen in such a way that hypothesis (A1), interpreted in terms of H^1 , is not violated.

The reader may notice indeed some similarity of the above system to the thermomechanical model for shape memory alloys proposed in [4]. Our system differs, however, by the occurrence of the quadratic terms on the right-hand side of the energy balance equation.

In terms of the new variables θ, U, χ , the energy e and entropy s can be written as

$$(2.29) \quad e = c_0 \theta + \frac{\lambda}{2\varrho_0} (U - \alpha(1 - \chi))^2 + \frac{\beta}{\varrho_0} \theta_c U + L_0(\chi + I(\chi)),$$

$$(2.30) \quad s = c_0 \log\left(\frac{\theta}{\theta_c}\right) + \frac{L_0}{\theta_c} \chi + \frac{\beta}{\varrho_0} U.$$

The energy functional has to be supplemented with the boundary energy term

$$(2.31) \quad E_\Gamma(t) = \frac{K_\Gamma}{2} \left(U_\Omega(t) + \frac{p_0}{K_\Gamma} \right)^2.$$

The energy and entropy balance equations now read

$$(2.32) \quad \frac{d}{dt} \left(\int_\Omega \varrho_0 e(x, t) \, dx + E_\Gamma(t) \right) = \int_{\partial\Omega} h(x) (\theta_\Gamma - \theta) \, ds(x),$$

$$(2.33) \quad \varrho_0 s_t + \operatorname{div} \frac{\mathbf{q}}{\theta} = \frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\gamma}{\theta} \chi_t^2 + \frac{\nu}{\theta} U_t^2 \geq 0,$$

$$(2.34) \quad \begin{aligned} \frac{d}{dt} \int_\Omega \varrho_0 s(x, t) \, dx &= \int_{\partial\Omega} \frac{h(x)}{\theta} (\theta_\Gamma - \theta) \, ds(x) \\ &\quad + \int_\Omega \left(\frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\gamma}{\theta} \chi_t^2 + \frac{\nu}{\theta} U_t^2 \right) \, dx. \end{aligned}$$

The entropy balance (2.33) says that the entropy production on the right-hand side is nonnegative in agreement with the second principle of thermodynamics. The system is not closed, and the energy supply through the boundary is given by the right-hand side of (2.32).

We prescribe the initial conditions

$$(2.35) \quad \theta(x, 0) = \theta^0(x),$$

$$(2.36) \quad U(x, 0) = U^0(x),$$

$$(2.37) \quad \chi(x, 0) = \chi^0(x)$$

for $x \in \Omega$, and compute from (2.29)–(2.30) the corresponding initial values e^0 , E_Γ^0 , and s^0 for specific energy, boundary energy, and entropy, respectively. Let $E^0 = \int_\Omega \varrho_0 e^0 dx$, $S^0 = \int_\Omega \varrho_0 s^0 dx$ denote the total initial energy and entropy, respectively. From the energy and entropy balance equations (2.32), (2.34), we derive the following crucial (formal for the moment) balance equation for the “extended” energy $\varrho_0(e - \theta_\Gamma s)$:

$$(2.38) \quad \int_\Omega \left(c\theta + \frac{\lambda}{2}(U - \alpha(1 - \chi))^2 + \beta\theta_c U + L\chi \right) (x, t) dx + \frac{K_\Gamma}{2} \left(U_\Omega(t) + \frac{p_0}{K_\Gamma} \right)^2 \\ + \theta_\Gamma \int_0^t \int_\Omega \left(\frac{\kappa |\nabla \theta|^2}{\theta^2} + \frac{\gamma}{\theta} \chi_t^2 + \frac{\nu}{\theta} U_t^2 \right) (x, \tau) dx d\tau \\ + \int_0^t \int_{\partial\Omega} \frac{h(x)}{\theta} (\theta_\Gamma - \theta)^2(x, \tau) ds(x) d\tau \\ = E^0 + E_\Gamma^0 - \theta_\Gamma S^0 + \theta_\Gamma \int_\Omega \left(c \log \left(\frac{\theta}{\theta_c} \right) + \frac{L}{\theta_c} \chi + \beta U \right) (x, t) dx.$$

We have $\log(\theta/\theta_c) = \log(\theta/2\theta_\Gamma) - \log(\theta_c/2\theta_\Gamma) \leq (\theta/2\theta_\Gamma) - 1 - \log(\theta_c/2\theta_\Gamma)$, hence there exists a constant $C > 0$ independent of t such that for all $t > 0$ we have

$$(2.39) \quad \int_\Omega (\theta + U^2) (x, t) dx + \int_0^t \int_\Omega \left(\frac{|\nabla \theta|^2}{\theta^2} + \frac{\chi_t^2}{\theta} + \frac{U_t^2}{\theta} \right) (x, \tau) dx d\tau \\ + \int_0^t \int_{\partial\Omega} \frac{h(x)}{\theta} (\theta_\Gamma - \theta)^2(x, \tau) ds(x) d\tau \leq C.$$

3. Equilibria. It follows from (2.16) and (2.23) that the only possible equilibrium temperature is $\theta = \theta_\Gamma$, and the equilibrium configurations U_∞, χ_∞ for U, χ satisfy for a.e. $x \in \Omega$ the equations

$$(3.1) \quad \lambda U_\infty(x) - \alpha\lambda(1 - \chi_\infty(x)) = \beta(\theta_\Gamma - \theta_c) - p_0 - K_\Gamma \int_\Omega U_\infty(x') dx',$$

$$(3.2) \quad -\lambda U_\infty(x) + \alpha\lambda(1 - \chi_\infty(x)) \in \frac{L}{\alpha} \left(1 - \frac{\theta_\Gamma}{\theta_c} \right) + \partial I(\chi_\infty(x)),$$

as a consequence of (2.24), (2.25); hence

$$(3.3) \quad \frac{L}{\alpha} \left(\frac{\theta_\Gamma}{\theta_c} - 1 \right) - \beta(\theta_\Gamma - \theta_c) + p_0 + K_\Gamma \int_\Omega U_\infty(x') dx' \in \partial I(\chi_\infty(x)) \quad \text{a.e.}$$

The equilibrium pressure P_∞ is given by (2.18), that is,

$$(3.4) \quad P_\infty = p_0 + K_\Gamma \int_\Omega U_\infty(x') dx'.$$

Integrating (3.1) over Ω yields

$$(3.5) \quad (\lambda + K_\Gamma |\Omega|) \int_\Omega U_\infty(x') dx' = |\Omega|(\beta(\theta_\Gamma - \theta_c) - p_0) + \alpha\lambda \int_\Omega (1 - \chi_\infty(x')) dx'.$$

Hence, a necessary and sufficient condition for $\chi_\infty(x)$ to be an equilibrium phase distribution reads

$$(3.6) \quad \frac{L}{\alpha\lambda} \left(\frac{\theta_\Gamma}{\theta_c} - 1 \right) - \frac{\beta(\theta_\Gamma - \theta_c) - p_0}{\lambda + K_\Gamma |\Omega|} + \frac{\alpha K_\Gamma}{\lambda + K_\Gamma |\Omega|} \int_\Omega (1 - \chi_\infty(x')) dx' \in \partial I(\chi_\infty(x)) \quad \text{a.e.}$$

Let us introduce a positive dimensionless parameter

$$(3.7) \quad d := \frac{\alpha^2 \lambda K_\Gamma |\Omega|}{L(\lambda + K_\Gamma |\Omega|)}.$$

Assume first that $\beta/(\lambda + K_\Gamma |\Omega|)$ and $p_0/(\lambda + K_\Gamma |\Omega|)$ are negligible with respect to the other terms. We then rewrite (3.6) in a simpler form:

$$(3.8) \quad \frac{\theta_\Gamma}{\theta_c} - 1 + \frac{d}{|\Omega|} \int_\Omega (1 - \chi_\infty(x')) \, dx' \in \partial I(\chi_\infty(x)) \quad \text{a.e.}$$

We distinguish three cases.

Case 1. $\theta_\Gamma \geq \theta_c$. Then (3.8) can be satisfied only if $\chi_\infty = 1$ a.e.; hence, by (3.5), $U_\infty = 0$ a.e., and by (3.4), the pressure P_∞ is in equilibrium with the external pressure. We only have the liquid phase in Ω and the system is stress-free.

Case 2. $d < 1$ and $\theta_\Gamma \leq (1 - d)\theta_c$. Then, similarly, (3.8) can be satisfied only if $\chi_\infty = 0$ a.e.; hence

$$\int_\Omega U_\infty(x') \, dx' = \frac{\alpha \lambda |\Omega|}{\lambda + K_\Gamma |\Omega|}, \quad U_\infty(x) = \alpha - \frac{\alpha K_\Gamma |\Omega|}{\lambda + K_\Gamma |\Omega|} = \frac{\alpha \lambda}{\lambda + K_\Gamma |\Omega|}.$$

We have only the solid phase subject to a balance between a positive volume expansion U_∞ and pressure $P_\infty - p_0 = K_\Gamma |\Omega| U_\infty$. In the limit case $K_\Gamma \rightarrow 0$ (stress-free boundary condition, i.e., infinitely soft bottle), we get $P_\infty \rightarrow p_0$, $U_\infty \rightarrow \alpha$, $d \rightarrow 0$. Hence, α indeed measures the relative volume expansion in the stress-free case. Similarly, in the limit case $K_\Gamma \rightarrow \infty$ (rigid bottle), we have $P_\infty - p_0 \rightarrow \alpha \lambda$, $U_\infty \rightarrow 0$. In this case, $\alpha \lambda$ is the pressure difference between inside and outside the bottle.

Case 3. $(1 - d)\theta_c < \theta_\Gamma < \theta_c$. Set $d_* = 1 - (\theta_\Gamma/\theta_c) < d$. Then every function χ_∞ with values in $[0, 1]$ satisfying the condition $(1/|\Omega|) \int_\Omega (1 - \chi_\infty(x')) \, dx' = d_*/d$ is an equilibrium. Hence, in this temperature range, we have a large number of possible equilibria.

We thus observe stable undercooled mushy regions in a nonzero temperature range, and full solidification takes place only if the temperature is below the value $(1 - d)\theta_c$. Theoretically, we cannot exclude the case $d \geq 1$, which would mean that the solid phase can never be achieved. We show now that in the case of water and ice, which is relevant for applications, the undercooling coefficient d is less than 1. Approximate values of the physical constants are listed in Table 3.1; see [8].

The maximum of d is achieved in a rigid bottle (i.e., $K_\Gamma \rightarrow \infty$). By Table 3.1 we have $\alpha = (V_{ice} - V_{water})/V_{water} = 0.09$, $\lambda \approx 2.25 \cdot 10^9 \, J/m^3$. Using (2.22) we obtain $L = \varrho_0 L_0 \approx 0.33 \cdot 10^9 \, J/m^3$; hence $d = \alpha^2 \lambda / L \approx 5.5\%$. Note that the standard atmospheric pressure is about $10^5 \, J/m^3$, while the pressure inside the bottle attains $\alpha \lambda \approx 2 \cdot 10^8 \, J/m^3$. This corresponds to a gravitational mass of 20 kilograms pressing on each square millimeter.

In reality, some values of the constants are different in water and in ice (the specific heat, for instance, is only $2 \cdot 10^3 \, J/(kg \, K)$ in the ice). A phase field model without mechanical effects for this situation was considered in [13]. Also the speed of sound in ice is about double that in water. We can in principle state the problem with coefficients depending on θ and χ here, too, but this would lead to serious technical difficulties that we want to avoid here. Moreover, in water and ice, the thermal expansion coefficient β is not constant and depends strongly on the temperature as well as on the phase. It may even become negative for temperatures in a right

TABLE 3.1
Physical constants for water.

Specific volume of water	$V_{water} = 1/\varrho_0$	10^{-3}	m^3/kg
Specific volume of ice	V_{ice}	$1.09 \cdot 10^{-3}$	m^3/kg
Speed of sound	$v_0 = \sqrt{\lambda/\varrho_0}$	$1.5 \cdot 10^3$	m/s
Freezing point	θ_c	273	K
Specific heat	c_0	$4.2 \cdot 10^3$	$J/(kg K)$
Latent heat	L_0	$3.3 \cdot 10^5$	J/kg
Thermal expansion coefficient	β/λ	$2.0 \cdot 10^{-4}$	K^{-1}

neighborhood of the freezing point. The values given in Table 3.1 are obtained by a rough linearization in order to have an idea about the orders of magnitude.

For the coefficient β we compute the estimate $\beta/\varrho_0 = v_0^2\beta/\lambda \approx 450 J/(kg K)$, while $L_0/(\alpha\theta_c) = L/(\varrho_0\alpha\theta_c) \approx 13400 J/(kg K)$. Let us define a new constant $L_\beta := L_0 - \beta\alpha\theta_c/\varrho_0$ as a small (3.3%) correction to the latent heat L_0 . Using (3.4), we may rewrite (3.3) as

$$(3.9) \quad \frac{\varrho_0 L_\beta}{\alpha} \left(\frac{\theta_\Gamma}{\theta_c} - 1 \right) + P_\infty \in \partial I(\chi_\infty(x)) \quad \text{a.e.}$$

We now show that (3.9) contains the Clausius–Clapeyron equation; cf. [10, Book 5, Chapter 5] or [22, pp. 124–126]. The pressure P_∞ is defined as the difference δP between the absolute pressure and the standard pressure. The phase transition takes place at temperature θ_Γ if the right-hand side of (3.9) vanishes. The temperature difference is $\delta\theta = \theta_\Gamma - \theta_c$, and we get the Clausius–Clapeyron relation in the form of equation (288) of [22], that is,

$$(3.10) \quad \frac{\delta P}{\delta\theta} = -\frac{\varrho_0 L_\beta}{\alpha\theta_c} = \frac{L_\beta}{\theta_c(V_{water} - V_{ice})}.$$

For general $\beta \geq 0$ and p_0 , we have an analogous classification as above. We introduce further dimensionless quantities

$$(3.11) \quad \tilde{\beta} = \frac{\alpha\lambda\beta\theta_c}{L(\lambda + K_\Gamma|\Omega|)}, \quad \omega = \frac{\alpha\lambda p_0}{L(\lambda + K_\Gamma|\Omega|)}.$$

The counterpart of (3.8) reads

$$(3.12) \quad (1 - \tilde{\beta}) \left(\frac{\theta_\Gamma}{\theta_c} - 1 \right) + \omega + \frac{d}{|\Omega|} \int_\Omega (1 - \chi_\infty(x')) dx' \in \partial I(\chi_\infty(x)).$$

Assuming that $\tilde{\beta} < 1$, we thus observe pure liquid for $\theta_\Gamma \geq \theta_c(1 - \omega/(1 - \tilde{\beta}))$, while pure solid corresponds to $\theta_\Gamma \leq \theta_c(1 - (\omega + d)/(1 - \tilde{\beta}))$. The dimensionless external pressure deviation ω can be assumed small. However, $\tilde{\beta}$ is a material constant, and the condition $\tilde{\beta} < 1$ might be restrictive. Again, for water and ice, the maximal value $\tilde{\beta} = (L_0 - L_\beta)/L_0 \approx 0.033$ corresponding to $K_\Gamma = 0$ shows that the influence of thermal expansion on the undercooling coefficient is negligible.

4. Existence and uniqueness of solutions. We construct the solution of (2.24)–(2.25) by the Banach contraction argument. The method of proof is independent of the actual values of the material constants, and we choose for simplicity

$$(4.1) \quad L = 2, \quad c = \theta_c = \alpha = \beta = \gamma = \kappa = \lambda = \nu = 1.$$

System (2.23)–(2.25) with boundary condition (2.16) then reads

(4.2)

$$\int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx = \int_{\Omega} (U_t^2 - \theta U_t - (U + \chi + 1)\chi_t) w(x) \, dx \\ - \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma}) w(x) \, ds(x),$$

$$(4.3) \quad U_t + U + \chi + K_{\Gamma} U_{\Omega}(t) = \theta - p_0,$$

$$(4.4) \quad \chi_t + U + \chi + \partial I(\chi) \ni 2\theta - 1,$$

where (4.2) is to be satisfied for all test functions $w \in W^{1,2}(\Omega)$ and a.e. $t > 0$, while (4.3)–(4.4) are assumed to hold a.e. in $\Omega_{\infty} := \Omega \times (0, \infty)$.

In this section we prove the following existence and uniqueness result.

THEOREM 4.1. *Let $0 < \theta_* \leq \theta_{\Gamma} \leq \theta^*$, $h \geq 0$, $h \in L^{\infty}(\partial\Omega)$, and $p_0 \in \mathbf{R}$ be given constants, and let the data satisfy the conditions*

$$\begin{array}{llll} \theta^0 & \in & W^{1,2}(\Omega) \cap L^{\infty}(\Omega), & \theta_* \leq \theta^0(x) \leq \theta^* \quad \text{a.e.}, \\ U^0, \chi^0 & \in & L^{\infty}(\Omega), & 0 \leq \chi^0(x) \leq 1 \quad \text{a.e.} \end{array}$$

Then there exists a unique solution (θ, U, χ) to (4.2)–(4.4) and (2.35)–(2.37) such that $\theta > 0$ a.e., $\chi \in [0, 1]$ a.e., $U, U_t, \chi_t, \theta, 1/\theta \in L^{\infty}(\Omega_{\infty})$, $\theta_t, \Delta\theta \in L^2(\Omega_{\infty})$, and $\nabla\theta \in L^{\infty}(0, T; L^2(\Omega)) \cap L^2(\Omega_{\infty})$.

Remark. For existence and uniqueness alone, we might allow the external temperature θ_{Γ} to depend on x and t , and assume only that it belongs to the space $W_{\text{loc}}^{1,2}(0, \infty; L^2(\partial\Omega)) \cap L_{\text{loc}}^{\infty}(\partial\Omega \times (0, \infty))$. For the global bounds, the assumption that θ_{Γ} is constant plays a substantial role.

The proof of Theorem 4.1 will be carried out in the following subsections. Notice first that the term $U_t^2 - \theta U_t - (U + \chi + 1)\chi_t$ on the right-hand side of (4.2) can be rewritten alternatively, using (4.4) and (4.3), as

$$(4.5) \quad U_t^2 - \theta U_t - (U + \chi + 1)\chi_t = U_t^2 - \theta U_t + \chi_t^2 - 2\theta\chi_t \\ = -(\chi + U + p_0 + K_{\Gamma} U_{\Omega})U_t - (U + \chi + 1)\chi_t.$$

We now fix some constant $R > 0$ and construct the solution for the truncated system

(4.6)

$$\int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx = \int_{\Omega} (U_t^2 + \chi_t^2 - Q_R(\theta)(U_t + 2\chi_t)) w(x) \, dx \\ - \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma}) w(x) \, ds(x) \quad \forall w \in W^{1,2}(\Omega),$$

$$(4.7) \quad U_t + U + \chi + K_{\Gamma} U_{\Omega}(t) = Q_R(\theta) - p_0,$$

$$(4.8) \quad \chi_t + U + \chi + \partial I(\chi) \ni 2Q_R(\theta) - 1,$$

first in a bounded domain $\Omega_T := \Omega \times (0, T)$ for any given $T > 0$, where Q_R is the cutoff function $Q_R(z) = \min\{z^+, R\}$. We then derive upper and lower bounds for θ independent of R and T , so that the local solution of (4.6)–(4.8) is also a global solution of (4.2)–(4.4) if R is sufficiently large.

4.1. A gradient flow. In a separable Hilbert space H with norm $|\cdot|$, consider a gradient flow

$$(4.9) \quad \dot{v}(t) + \partial\psi(v(t)) \ni f(t), \quad v(0) = v^0,$$

where $\psi : H \rightarrow [0, \infty]$ is a proper convex lower semicontinuous functional such that $\lim_{|v| \rightarrow \infty} \psi(v) = +\infty$, $\partial\psi$ is its subdifferential, and $v^0 \in \text{Dom } \psi$, $f \in L^2(0, \infty; H)$ are given. A classical existence and uniqueness result in [2, Théorème 3.6] states that for every $T > 0$ there exists a unique solution $v \in C([0, T]; H)$ to (4.9) such that $\dot{v} \in L^2(0, T; H)$, and

$$\left(\int_0^T |\dot{v}(\tau)|^2 d\tau \right)^{1/2} \leq \psi(v^0) + \left(\int_0^T |f(\tau)|^2 d\tau \right)^{1/2}.$$

We prove here the following lemma.

LEMMA 4.2. *Let $f, \dot{v}$ belong to $L^2(0, \infty; H)$. Then $\lim_{t \rightarrow \infty} \dot{v}(t) = 0$.*

Proof. For each $h > 0$ and a.e. $t > 0$ we have

$$\frac{1}{2} \frac{d}{dt} \left| \frac{v(t+h) - v(t)}{h} \right|^2 \leq \left| \frac{f(t+h) - f(t)}{h} \right| \left| \frac{v(t+h) - v(t)}{h} \right|;$$

hence

$$(4.10) \quad |\dot{v}(t)|^2 - |\dot{v}(s)|^2 \leq 2 \int_s^t |\dot{f}(\tau)| |\dot{v}(\tau)| d\tau$$

for almost all $0 < s < t$. Hence, the function $t \mapsto 2 \int_0^t |\dot{f}(\tau)| |\dot{v}(\tau)| d\tau - |\dot{v}(t)|^2$ is a.e. equal to a nondecreasing function in $(0, \infty)$. We are thus in the situation of [16, Proposition 5.2], which gives the desired statement. $\square$

We apply the above result to the case $H = L^2(\Omega) \times L^2(\Omega)$, and

$$(4.11) \quad v = \begin{pmatrix} U \\ \chi \end{pmatrix},$$

$$(4.12) \quad \begin{aligned} \psi(v) = & \int_{\Omega} \left(\frac{1}{2} (U + \chi - 1)^2 + (U + 2\chi)(1 - \theta_{\Gamma}) + I(\chi) \right) dx \\ & + \frac{K_{\Gamma}}{2} \left(\int_{\Omega} U dx + \frac{p_0}{K_{\Gamma}} \right)^2 + C_{\psi}, \end{aligned}$$

$$(4.13) \quad f = \begin{pmatrix} Q_R(\hat{\theta}) - \theta_{\Gamma} \\ 2(Q_R(\hat{\theta}) - \theta_{\Gamma}) \end{pmatrix},$$

where C_{ψ} is a suitable constant such that $\psi(v) \geq 0$ for all v , and $\hat{\theta}$ is a given function. The initial condition v^0 is given by (2.36), (2.37). We have

$$(4.14) \quad \begin{pmatrix} \eta \\ \zeta \end{pmatrix} \in \partial\psi(v) \iff \begin{cases} \eta = U + \chi - \theta_{\Gamma} + K_{\Gamma} \int_{\Omega} U dx + p_0, \\ \zeta \in U + \chi + 1 - 2\theta_{\Gamma} + \partial I(\chi), \end{cases}$$

and we see that (4.7)–(4.8) with θ replaced by $\hat{\theta}$ can be equivalently written as the gradient flow (4.9), (4.11)–(4.13). For this system, we prove the following result.

PROPOSITION 4.3. *Let the hypotheses of Theorem 4.1 hold, and let a function $\hat{\theta} \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ be given. Let (U, χ) be the solution of (4.9), (4.11)–(4.13). Then there exists a constant C_0 , independent of x, t , and R such that a.e. in Ω_∞ we have*

$$(4.15) \quad |U(x, t)| + |U_t(x, t)| + |\chi_t(x, t)| \leq C_0(1 + R).$$

Let furthermore $\hat{\theta}_1, \hat{\theta}_2 \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ be two functions, and let $(U_1, \chi_1), (U_2, \chi_2)$ be the corresponding solutions of (4.9), (4.11)–(4.13). Then the differences $\hat{\theta}_d = \hat{\theta}_1 - \hat{\theta}_2$, $U_d = U_1 - U_2$, $\chi_d = \chi_1 - \chi_2$ satisfy for every $t \geq 0$ and a.e. $x \in \Omega$ the inequality

$$(4.16) \quad \int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x, \tau) \, d\tau \leq C_0(1 + t) \int_0^t (|\hat{\theta}_d(x, \tau)| + t|\hat{\theta}_d(\tau)|_2) \, d\tau,$$

where the symbol $|\cdot|_2$ stands for the norm in $L^2(\Omega)$.

In what follows, we denote by $C_1, C_2, \dots$ any constant independent of x, t , and R .

Proof. Put $X_\Omega(t) = \int_\Omega (1 - \chi(x', t)) \, dx'$. Integrating (4.7) with θ replaced by $\hat{\theta}$ over Ω yields

$$\dot{U}_\Omega + (1 + K_\Gamma|\Omega|)U_\Omega = X_\Omega + \int_\Omega (Q_R(\hat{\theta}) - 1) \, dx - p_0|\Omega| \quad \text{a.e.}$$

Since χ attains values in $[0, 1]$, we easily obtain $|\dot{U}_\Omega| + |U_\Omega| \leq C_1(1 + R)$ a.e. Equation (4.7) now has a right-hand side bounded by a multiple of $1 + R$; hence $|U_t| + |U| \leq C_2(1 + R)$ a.e. To obtain the same bound for $|\chi_t|$, it suffices to multiply (4.8) by χ_t . This completes the proof of (4.15).

To prove (4.16), we rewrite (4.9), (4.11)–(4.13) as two scalar gradient flows

$$(4.17) \quad U_t + \partial\psi_1(U) = a,$$

$$(4.18) \quad \chi_t + \partial\psi_2(\chi) \ni b,$$

where $\psi_1(U) = \frac{1}{2}U^2$, $\psi_2 = \frac{1}{2}\chi^2 + I(\chi)$, $a = Q_R(\hat{\theta}) - \chi - p_0 - K_\Gamma U_\Omega$, $b = 2Q_R(\hat{\theta}) - 1 - U$. Consider now two different inputs. As above, we denote the differences $\{\cdot\}_1 - \{\cdot\}_2$ by $\{\cdot\}_d$ for all symbols $\{\cdot\}$. By [11, Theorem 1.12], we have for all $t > 0$ and a.e. $x \in \Omega$ that

$$(4.19) \quad \int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x, \tau) \, d\tau \leq 2 \int_0^t (|a_d| + |b_d|)(x, \tau) \, d\tau.$$

We multiply the difference of (4.17) by U_d , the difference of (4.18) by χ_d , and sum them up to obtain that

$$(4.20) \quad (U_d)_t U_d + (\chi_d)_t \chi_d + (U_d + \chi_d)^2 + K_\Gamma U_{\Omega d} U_d \leq |\hat{\theta}_d|(|U_d| + 2|\chi_d|) \quad \text{a.e.}$$

We first integrate (4.20) over Ω . Using the symbol $|\cdot|_2$ for the norm in $L^2(\Omega)$, we get for a.e. $t > 0$ that

$$(4.21) \quad \frac{1}{2} \frac{d}{dt} (|U_d|_2^2 + |\chi_d|_2^2) + K_\Gamma U_{\Omega d}^2 \leq |\hat{\theta}_d|_2 (|U_d|_2 + 2|\chi_d|_2) \leq \sqrt{5} |\hat{\theta}_d|_2 (|U_d|_2^2 + |\chi_d|_2^2)^{1/2}.$$

Hence, $\frac{d}{dt} (|U_d|_2^2 + |\chi_d|_2^2)^{1/2} \leq \sqrt{5} |\hat{\theta}_d|_2$ a.e., and integrating over t , we find that

$$(4.22) \quad (|U_d|_2^2 + |\chi_d|_2^2)^{1/2}(t) \leq \sqrt{5} \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau.$$

This implies in particular that

$$(4.23) \quad |U_{\Omega d}(t)| \leq \sqrt{5|\Omega|} \int_0^t |\hat{\theta}_d(\tau)|_2 \, d\tau.$$

Using again (4.20), we find for a.e. $(x, t) \in \Omega_\infty$ the inequality

$$(4.24) \quad \frac{1}{2} \frac{\partial}{\partial t} (|U_d|^2 + |\chi_d|^2)(x, t) \leq \left(K_\Gamma |U_{\Omega d}(t)| + |\hat{\theta}_d(x, t)| \right) (|U_d| + 2|\chi_d|)(x, t).$$

This is for almost all $x \in \Omega$ an inequality of the form $(d/dt)(Y^2(t)) \leq 2c(t)Y(t)$, $Y(0) = 0$, with $Y = (|U_d|^2 + |\chi_d|^2)^{1/2}$, which implies $Y(t) \leq \int_0^t c(\tau) \, d\tau$ for all $t > 0$. Hence,

$$(4.25) \quad (|U_d|^2 + |\chi_d|^2)^{1/2}(x, t) \leq C_1 \int_0^t \left(|\hat{\theta}_d(x, \tau)| + t|\hat{\theta}_d(\tau)|_2 \right) \, d\tau \quad \text{a.e.}$$

This enables us to estimate the right-hand side of (4.19) and obtain the bound

$$(4.26) \quad \int_0^t (|(U_d)_t| + |(\chi_d)_t|)(x, \tau) \, d\tau \leq C_2 \int_0^t \left((1+t)|\hat{\theta}_d(x, \tau)| + t(1+t)|\hat{\theta}_d(\tau)|_2 \right) \, d\tau$$

for a.e. $x \in \Omega$ and all $t \geq 0$. This completes the proof. $\square$

4.2. Existence of solutions for the truncated problem. We construct the solution of (4.6)–(4.8) for every $R > 0$ by the Banach contraction argument on a fixed time interval $(0, T)$.

LEMMA 4.4. *Let the hypotheses of Theorem 4.1 hold, and let $T > 0$ and $R > 0$ be given. Then there exists a unique solution (θ, U, χ) to (4.6)–(4.8), (2.35)–(2.37) such that $U \in W^{1,\infty}(\Omega_T)$, $\theta > 0$ a.e., $\chi_t, \theta, 1/\theta \in L^\infty(\Omega_T)$, $\theta_t, \Delta\theta \in L^2(\Omega_T)$, and $\nabla\theta \in L^\infty(0, T; L^2(\Omega))$.*

Proof. Let $\hat{\theta} \in L^2(\Omega_T)$ be a given function, and consider the system

$$(4.27) \quad \begin{aligned} \int_\Omega \theta_t w(x) \, dx + \int_\Omega \nabla\theta \cdot \nabla w(x) \, dx &= \int_\Omega (U_t^2 + \chi_t^2 - Q_R(\hat{\theta})(U_t + 2\chi_t))w(x) \, dx \\ &\quad - \int_{\partial\Omega} h(x)(\theta - \theta_\Gamma)w(x) \, ds(x) \quad \forall w \in W^{1,2}(\Omega), \end{aligned}$$

$$(4.28) \quad U_t + U + \chi + K_\Gamma U_\Omega(t) = Q_R(\hat{\theta}) - p_0,$$

$$(4.29) \quad \chi_t + U + \chi + \partial I(\chi) \ni 2Q_R(\hat{\theta}) - 1.$$

Equations (4.28)–(4.29) are solved as the gradient flow problem from subsection 4.1, while (4.27) is a simple linear parabolic equation for θ . Testing (4.27) by θ_t , we obtain by Proposition 4.3 that

$$(4.30) \quad \begin{aligned} \int_0^T \int_\Omega \theta_t^2 \, dx \, dt + \sup_{t \in (0, T)} \text{ess} \left(\int_\Omega |\nabla\theta|^2 \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_\Gamma)^2 \, ds(x) \right) \\ \leq T|\Omega| (C_0(1+R)(2C_0(1+R) + 3R))^2 =: M_R. \end{aligned}$$

Hence, we can define the mapping that with $\hat{\theta}$ associates the solution θ of (4.27)–(4.29) with initial conditions (2.35)–(2.37). We now show that it is a contraction on the set

$$(4.31) \quad \Xi_{T,R} := \{\hat{\theta} \in L^2(\Omega_T) : \text{conditions (4.32)–(4.35) hold}\},$$

where

$$(4.32) \quad \hat{\theta}_t \in L^2(\Omega_T);$$

$$(4.33) \quad \nabla \hat{\theta} \in L^\infty(0, T; L^2(\Omega));$$

$$(4.34) \quad \int_0^T \int_\Omega \hat{\theta}_t^2 dx dt + \sup_{t \in (0, T)} \left(\int_\Omega |\nabla \hat{\theta}|^2 dx + \int_{\partial\Omega} h(x) (\hat{\theta} - \theta_\Gamma)^2 ds(x) \right) \leq M_R;$$

$$(4.35) \quad \hat{\theta}(x, 0) = \theta^0(x) \text{ a.e.}$$

Let $\hat{\theta}_1, \hat{\theta}_2$ be two functions in $\Xi_{T,R}$, and let $(\theta_1, U_1, \chi_1), (\theta_2, U_2, \chi_2)$ be the corresponding solutions to (4.27)–(4.29) with the same initial conditions θ^0, U^0, χ^0 . We see from (4.30) that θ_1, θ_2 belong to $\Xi_{T,R}$. Integrating (4.27) for θ_1 and θ_2 with respect to time and testing their difference by $w = \theta_d := \theta_1 - \theta_2$, we obtain, using Proposition 4.3, that

$$(4.36) \quad \begin{aligned} & \int_\Omega \theta_d^2(x, t) dx + \frac{d}{dt} \left(\int_\Omega \left| \nabla \int_0^t \theta_d(x, \tau) d\tau \right|^2 dx + \int_{\partial\Omega} h(x) \left| \int_0^t \theta_d(x, \tau) d\tau \right|^2 ds(x) \right) \\ & \leq C_3(1 + R) \int_\Omega \left(|(U_d)_t| + |(\chi_d)_t| + |\hat{\theta}_d| \right)(x, \tau) d\tau \theta_d(x, t) dx \quad \text{a.e.} \end{aligned}$$

From (4.16) and Minkowski's inequality, it follows that

$$\begin{aligned} \left| \int_0^t (|(U_d)_t| + |(\chi_d)_t|)(\cdot, \tau) d\tau \right|_2 & \leq C_4(1 + t)^2 \int_0^t |\hat{\theta}_d(\tau)|_2 d\tau \\ & \leq C_4(1 + t)^2 \left(t \int_0^t |\hat{\theta}_d(\tau)|_2^2 d\tau \right)^{1/2}. \end{aligned}$$

By Young's inequality, we rewrite (4.36) as

$$(4.37) \quad \begin{aligned} & \int_\Omega \theta_d^2(x, t) dx + \frac{d}{dt} \left(\int_\Omega \left| \nabla \int_0^t \theta_d(x, \tau) d\tau \right|^2 dx + \int_{\partial\Omega} h(x) \left| \int_0^t \theta_d(x, \tau) d\tau \right|^2 ds(x) \right) \\ & \leq C_5(1 + R^2)(1 + t)^5 \int_0^t |\hat{\theta}_d(\tau)|_2^2 d\tau \quad \text{a.e.} \end{aligned}$$

Set $\Theta^2(t) = \int_0^t |\theta_d(\tau)|_2^2 d\tau$, $\hat{\Theta}^2(t) = \int_0^t |\hat{\theta}_d(\tau)|_2^2 d\tau$. Integrating (4.37) with respect to time, we obtain

$$(4.38) \quad \Theta^2(t) \leq C_5(1 + R^2) \int_0^t (1 + \tau)^5 \hat{\Theta}^2(\tau) d\tau.$$

We set $C_R := (C_5(1 + R^2)/6)$ and introduce in $L^\infty(0, T)$ the norm

$$\|w\|_C := \sup_{\tau \in [0, T]} e^{-C_R(1+\tau)^6} |w(\tau)|.$$

Then $\|\Theta\|_C^2 \leq \frac{1}{2}\|\hat{\Theta}\|_C^2$, and hence the mapping $\hat{\theta} \mapsto \theta$ is a contraction in $L^2(\Omega_T)$ with respect to the norm induced by $\|\cdot\|_C$. The set $\Xi_{T,R}$ is a closed subset of $L^2(\Omega_T)$. This implies the existence of a fixed point $\theta \in \Xi_{T,R}$, which is indeed a solution to (4.6)–(4.8). The positive upper and lower bounds for θ follow from the maximum principle. Indeed, the right-hand side (4.5) of (4.6) is bounded from above by $C_6(1 + R)^2$ and from below by $-2(\theta^+)^2$. Let us define the functions

$$\theta^\sharp(t) = \theta^* + C_6(1 + R)^2 t, \quad \theta^\flat(t) = \frac{\theta_*}{1 + 2\theta_* t}.$$

For every nonnegative test function w and a.e. $t \in (0, T)$ we have

$$(4.39) \quad \int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})w(x) \, ds(x) \\ \leq C_6(1 + R)^2 \int_{\Omega} w(x) \, dx,$$

$$(4.40) \quad \int_{\Omega} \theta_t w(x) \, dx + \int_{\Omega} \nabla \theta \cdot \nabla w(x) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})w(x) \, ds(x) \\ \geq -2 \int_{\Omega} (\theta^+)^2 w(x) \, dx,$$

$$(4.41) \quad \int_{\Omega} \theta_t^\sharp w(x) \, dx + \int_{\Omega} \nabla \theta^\sharp \cdot \nabla w(x) \, dx + \int_{\partial\Omega} h(x)(\theta^\sharp - \theta_{\Gamma})w(x) \, ds(x) \\ \geq C_6(1 + R)^2 \int_{\Omega} w(x) \, dx,$$

$$(4.42) \quad \int_{\Omega} \theta_t^\flat w(x) \, dx + \int_{\Omega} \nabla \theta^\flat \cdot \nabla w(x) \, dx + \int_{\partial\Omega} h(x)(\theta^\flat - \theta_{\Gamma})w(x) \, ds(x) \\ \leq -2 \int_{\Omega} (\theta^\flat)^2 w(x) \, dx.$$

We now subtract (4.41) from (4.39) and test by $w = (\theta - \theta^\sharp)^+$, which yields the pointwise bound $\theta(x, t) \leq \theta^\sharp(t)$. Similarly, we subtract (4.40) from (4.42) and test by $w = (\theta^\flat - \theta)^+$. We thus have the inequalities

$$(4.43) \quad \theta^\flat(t) \leq \theta(x, t) \leq \theta^\sharp(t) \quad \text{a.e.,}$$

which complete the proof of Lemma 4.4. $\square$

4.3. Proof of Theorem 4.1. The unique solution (θ, U, χ) to the system (4.6)–(4.8), (2.35)–(2.37) exists globally in the whole domain Ω_∞ . We now derive uniform bounds independent of t and R . Take first, for instance, any $R > 2\theta^*$. By (4.43), we know that the solution component θ of (4.6)–(4.8) remains smaller than R in a nondegenerate interval $(0, T)$ with $T > \theta^*/(C_6(1 + R)^2)$. Let $(0, T_0)$ be the maximal interval in which θ is bounded by R . Then, in $(0, T_0)$, the solution given by Lemma 4.4

is also a solution of the original problem (4.2)–(4.4). Moreover, due to estimate (2.39), we know that θ admits a bound in $L^\infty(0, T_0; L^1(\Omega))$ independent of R . In order to prove that $T_0 = +\infty$ if R is sufficiently large, we need the following variant of the Moser iteration lemma.

PROPOSITION 4.5. *Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with Lipschitzian boundary. Given nonnegative functions $h \in L^1(\partial\Omega)$ and $r \in L^\infty(0, \infty; L^q(\Omega))$ with a fixed $q > N/2$, an initial condition $v^0 \in L^\infty(\Omega)$, and a boundary datum $v_\Gamma \in L^\infty(\partial\Omega \times (0, \infty))$, consider the problem*

$$(4.44) \quad v_t - \Delta v + v = r(x, t) \mathcal{H}[v] \quad \text{a.e. in } \Omega_\infty,$$

$$(4.45) \quad \nabla v \cdot \mathbf{n} = -h(x) (f(x, t, v(x, t)) - v_\Gamma(x, t)) \quad \text{a.e. on } \partial\Omega \times (0, \infty),$$

$$(4.46) \quad v(x, 0) = v^0 \quad \text{a.e. in } \Omega$$

under the assumption that there exist positive constants $H_0, C_f, V, V_\Gamma, E_0$ such that the following hold:

- (i) The mapping $\mathcal{H} : L^\infty_{\text{loc}}(\Omega_\infty) \rightarrow L^\infty_{\text{loc}}(\Omega_\infty)$ satisfies for every $v \in L^\infty_{\text{loc}}(\Omega_\infty)$ and a.e. $(x, t) \in \Omega_\infty$ the inequality

$$v(x, t) \mathcal{H}[v](x, t) \leq H_0 |v(x, t)| \left(1 + |v(x, t)| + \int_0^t \xi(t - \tau) |v(x, \tau)| d\tau \right),$$

where $\xi \in W^{1,1}(0, \infty)$ is a given nonnegative function such that

$$(4.47) \quad \dot{\xi}(t) \leq -\xi(t) \quad \text{a.e.}$$

- (ii) f is a Carathéodory function on $\Omega \times (0, \infty) \times \mathbf{R}$ such that $f(x, t, v) v \geq C_f v^2$ a.e. for all $v \in \mathbf{R}$.
 (iii) $|v^0(x)| \leq V$ a.e. in Ω .
 (iv) $|v_\Gamma(x, t)| \leq V_\Gamma$ a.e. on $\partial\Omega \times (0, \infty)$.
 (v) System (4.44)–(4.46) admits a solution

$$v \in W^{1,2}_{\text{loc}}(0, \infty; (W^{1,2})'(\Omega)) \cap L^2_{\text{loc}}(0, \infty; W^{1,2}(\Omega)) \cap L^\infty_{\text{loc}}(\Omega_\infty)$$

satisfying the estimate

$$\int_\Omega |v(x, t)| dx \leq E_0 \quad \text{a.e. in } (0, \infty).$$

Then there exists a positive constant C^* depending only on $|h|_{L^1(\partial\Omega)}$, C_f , H_0 , $\xi(0)$, and $r^* := |r|_{L^\infty(0, \infty; L^q(\Omega))}$ such that

$$(4.48) \quad |v(t)|_{L^\infty(\Omega)} \leq C^* \max \{1, V, V_\Gamma, E_0\} \quad \text{for a.e. } t > 0.$$

Remark. As a consequence of (4.47), we have $\xi(t) \leq \xi(0)e^{-\xi(0)t}$ for all $t \geq 0$; hence $\int_0^\infty \xi(t) dt \leq 1$. As a typical function satisfying (4.47), let us mention, for example,

$$(4.49) \quad \xi(t) = \frac{m_1}{\sum_{k=1}^n r_k} \sum_{k=1}^n r_k e^{-m_k t}$$

with any $0 < m_1 \leq \dots \leq m_n$ and $r_k > 0$, $k = 1, \dots, n$.

We split the proof of Proposition 4.5 into several steps.

LEMMA 4.6. *Let ξ be as in Proposition 4.5, let $x \in W_{\text{loc}}^{1,1}(0, \infty)$ and $y \in L_{\text{loc}}^1(0, \infty)$ be nonnegative functions, and let $a > 0$, $C > 0$, $\delta \in (0, 1)$ be given constants. Set $\mu = \min\{a, \xi(0)(1 - \delta)\}$, and assume that for a.e. $t > 0$ we have*

$$(4.50) \quad \dot{x}(t) + ax(t) + y(t) \leq C + \delta \int_0^t \xi(t - \tau) y(\tau) d\tau.$$

Then $x(t) \leq \max\{x(0), C/\mu\}$ for all $t > 0$.

Proof. Set $z(t) = \int_0^t \xi(t - \tau) y(\tau) d\tau$. Then $(1/\xi(0))\dot{z}(t) + z(t) \leq y(t)$ a.e.; hence

$$\dot{x}(t) + ax(t) + \frac{1}{\xi(0)}\dot{z}(t) + (1 - \delta)z(t) \leq C \quad \text{a.e.}$$

With μ as above, we have

$$\left(\dot{x}(t) + \frac{1}{\xi(0)}\dot{z}(t)\right) + \mu \left(x(t) + \frac{1}{\xi(0)}z(t)\right) \leq C \quad \text{a.e.},$$

which yields

$$x(t) + \frac{1}{\xi(0)}z(t) \leq \max \left\{ x(0) + \frac{1}{\xi(0)}z(0), \frac{C}{\mu} \right\},$$

and the desired inequality follows easily. $\square$

LEMMA 4.7. *Let $\mathcal{H}$ be as in Proposition 4.5, and let $|\cdot|_p$ denote the norm in $L^p(\Omega)$ for $1 \leq p \leq \infty$. Let $v \in L_{\text{loc}}^\infty(\Omega_\infty)$ and $p, s \geq 1$ be arbitrary. For $(x, t) \in \Omega_\infty$ set $h_p = \mathcal{H}[v]v|v|^{p-2}$. Then, for all $t > 0$ we have*

$$|h_p(t)|_s \leq H_0 \left(\frac{1}{p} + 3|v(t)|_{ps}^p + \frac{1}{p} \int_0^t \xi(t - \tau) |v(\tau)|_{ps}^p d\tau \right),$$

where ξ is as in (4.47).

Proof. We have for a.e. $(x, t) \in \Omega_\infty$ that

$$|h_p(x, t)| \leq H_0 (|v(x, t)|^{p-1} + 2|v(x, t)|^p + w_p(x, t)),$$

where

$$\begin{aligned} w_p(x, t) &= \frac{1}{p} \left(\int_0^t \xi(t - \tau) |v(x, \tau)| d\tau \right)^p \\ &\leq \frac{1}{p} \left(\left(\int_0^t \xi(t - \tau) d\tau \right)^{1/p'} \left(\int_0^t \xi(t - \tau) |v(x, \tau)|^p d\tau \right)^{1/p} \right)^p \\ &\leq \frac{1}{p} \int_0^t \xi(t - \tau) |v(x, \tau)|^p d\tau. \end{aligned}$$

Here, we have used Hölder's inequality with conjugate exponents p, p' and the remark after Proposition 4.5. The assertion now follows from Minkowski's inequality

$$\left| \int_0^t \xi(t - \tau) |v(\cdot, \tau)|^p d\tau \right|_s \leq \int_0^t \xi(t - \tau) |v(\tau)|_{ps}^p d\tau. \quad \square$$

LEMMA 4.8. *Let the hypotheses of Proposition 4.5 hold, and let $|\cdot|_{\infty, p}$ denote the norm in $L^\infty(0, \infty; L^p(\Omega))$. Let $v \in W_{\text{loc}}^{1,2}(0, \infty; (W^{1,2})'(\Omega)) \cap L_{\text{loc}}^2(0, \infty; W^{1,2}(\Omega)) \cap$*

$L_{\text{loc}}^\infty(\Omega_\infty)$ be a solution to (4.44)–(4.46) such that $|v|_{\infty,p} < \infty$ for some $p \geq 1$. Then there exists a constant $\bar{C} > 0$ independent of v and p such that

$$(4.51) \quad |v|_{\infty,2p} \leq (\bar{C}p^{1+b})^{1/2p} \max\{1, V, V_\Gamma/C_f, |v|_{\infty,p}\},$$

where

$$(4.52) \quad b = \frac{N(q+1)}{2q-N}.$$

Proof. Let $r^* = |r|_{L^\infty(0,\infty;L^q(\Omega))}$ be as above. Test (4.44) by $v|v|^{2p-2}$ to obtain, using Lemma 4.7, that

$$(4.53) \quad \begin{aligned} & \frac{1}{2p} \frac{d}{dt} \int_\Omega |v(x,t)|^{2p} dx + \frac{2p-1}{p^2} \int_\Omega |\nabla |v|^p|^2(x,t) dx + \int_\Omega |v(x,t)|^{2p} dx \\ & \quad + \int_{\partial\Omega} h(x) C_f |v|^{2p} - v_\Gamma v |v|^{2p-2} ds(x) \\ & \leq r^* H_0 \left(\frac{1}{2p} + 3|v(t)|_{2pq'}^{2p} + \frac{1}{2p} \int_0^t \xi(t-\tau) |v(\tau)|_{2pq'}^{2p} d\tau \right). \end{aligned}$$

We estimate the boundary integral using Young's inequality

$$v_\Gamma v |v|^{2p-2} \leq \frac{2p-1}{2p} C_f |v|^{2p} + \frac{1}{2p} C_f^{1-2p} V_\Gamma^{2p}.$$

Set $v_p(x,t) = |v(x,t)|^p$. Then

$$(4.54) \quad \begin{aligned} & \frac{d}{dt} |v_p(t)|_2^2 + 2|\nabla v_p(t)|_2^2 + 2p|v_p(t)|_2^2 \\ & \leq C_f^{1-2p} |h|_{L^1(\partial\Omega)} V_\Gamma^{2p} + r^* H_0 \left(1 + 6p|v_p(t)|_{2q'}^2 + \int_0^t \xi(t-\tau) |v_p(\tau)|_{2q'}^2 d\tau \right) \text{ a.e.} \end{aligned}$$

By the Gagliardo–Nirenberg inequality [17], there exists a constant G such that for all $\delta > 0$ and $t > 0$ we have

$$(4.55) \quad |v_p(t)|_{2q'}^2 \leq G \left(\delta |\nabla v_p(t)|_2^2 + \delta^{-b} |v_p(t)|_1^2 \right) \leq G \left(\delta |\nabla v_p(t)|_2^2 + \delta^{-b} |v|_{\infty,p}^{2p} \right),$$

with b given by (4.52). We now choose δ such that

$$6pr^* H_0 G \delta = 1,$$

and obtain

$$(4.56) \quad \begin{aligned} & \frac{d}{dt} |v_p(t)|_2^2 + |\nabla v_p(t)|_2^2 + 2p|v_p(t)|_2^2 \leq C_7 \left(1 + \left(\frac{V_\Gamma}{C_f} \right)^{2p} + p^{1+b} |v|_{\infty,p}^{2p} \right) \\ & \quad + \frac{1}{6p} \int_0^t \xi(t-\tau) |\nabla v_p(\tau)|_2^2 d\tau \quad \text{a.e.}, \end{aligned}$$

with a constant C_7 depending only on $\xi(0)$, C_f , $|h|_{L^1(\partial\Omega)}$, r^* , H_0 , and G . We now use Lemma 4.6 with $x(t) = |v_p(t)|_2^2$, $y(t) = |\nabla v_p(t)|_2^2$, $a = 2p$, $\delta = 1/(6p)$, $\mu \geq \min\{2, (5/6)\xi(0)\}$, and $C = C_7(1 + (V_\Gamma/C_f)^{2p} + p^{1+b}|v|_{\infty,p}^{2p})$, which yields that

$$(4.57) \quad |v_p(t)|_2^2 \leq C_8 (1 + V^{2p} + (V_\Gamma/C_f)^{2p} + p^{1+b}|v|_{\infty,p}^{2p})$$

with a constant C_8 independent of p and t , and (4.51) immediately follows. $\square$

We are now ready to finish the proof of Proposition 4.5.

Proof of Proposition 4.5. For $k = 0, 1, 2, \dots$ set $y_k = \max\{1, V, V_\Gamma/C_f, |v|_{\infty,2^k}\}$. We have $y_0 \leq \max\{1, V, V_\Gamma/C_f, E_0\}$, and, as a consequence of Lemma 4.8,

$$y_{k+1} \leq (\bar{C}2^{k(1+b)})^{2^{-(k+1)}} y_k.$$

This yields

$$\log y_{k+1} \leq \log y_k + 2^{-(k+1)}(\log \bar{C} + k(1+b)\log 2).$$

Hence,

$$(4.58) \quad \log y_n \leq \log y_0 + \sum_{k=1}^n (2^{-k}(\log \bar{C} + (k-1)(1+b)\log 2)).$$

The sum on the right-hand side of (4.58) is convergent, and we easily complete the proof. $\square$

We now finish the proof of Theorem 4.1 by showing that T_0 introduced at the beginning of this subsection is $+\infty$ if R is sufficiently large. In (4.3), set again $U_\Omega(t) = \int_\Omega U(x', t) dx'$. Then

$$\dot{U}_\Omega(t) + (1 + K_\Gamma|\Omega|)U_\Omega(t) = \int_\Omega (\theta - \chi)(x', t) dx' - |\Omega|p_0 \quad \text{a.e.}$$

By (2.39), the right-hand side of this ODE is uniformly bounded independently of R ; hence $|U_\Omega(t)| \leq C_9$ in $(0, T_0)$. Using (4.3) once again, we obtain that

$$(4.59) \quad |U(x, t)| \leq C_{10} \left(1 + \int_0^t e^{\tau-t} \theta(x, \tau) d\tau \right) \quad \text{a.e.},$$

$$(4.60) \quad |U_t(x, t)| \leq C_{11} \left(1 + \theta(x, t) + \int_0^t e^{\tau-t} \theta(x, \tau) d\tau \right) \quad \text{a.e.};$$

hence also (cf. 4.4)

$$(4.61) \quad |\chi_t(x, t)| \leq C_{12} \left(1 + \theta(x, t) + \int_0^t e^{\tau-t} \theta(x, \tau) d\tau \right) \quad \text{a.e.}$$

As in (4.5), we rewrite the right-hand side of (4.2) as

$$-(\chi + U + p_0 + K_\Gamma U_\Omega)U_t - (U + \chi + 1)\chi_t.$$

By (2.39), the function U is in $L^\infty(0, \infty; L^2(\Omega))$ and the bound does not depend on R . Equation (4.2), with θ added to both the left- and the right-hand side, thus satisfies the hypotheses of Proposition 4.5 for $N = 3$ and $q = 2$. This enables us to conclude that $\theta(x, t)$ is uniformly bounded from above by a constant, independently of R , so

that θ never reaches the value R if R is sufficiently large, which we wanted to prove. By (4.59)–(4.61), also U , U_t , and χ_t are uniformly bounded by a constant.

We proceed similarly to prove a uniform positive lower bound for θ . Set $R_0 := \sup \theta$, and in (4.6) with $R > R_0$ put $w = -\tilde{w}/\theta$, $\tilde{w} \in W^{1,2}(\Omega)$. For a new (nonnegative) variable $v(x, t) := \log R_0 - \log \theta(x, t)$ we obtain the equation

$$(4.62) \quad \int_{\Omega} v_t \tilde{w}(x) \, dx + \int_{\Omega} \nabla v \cdot \nabla \tilde{w}(x) \, dx + \int_{\partial\Omega} h(x) \left(\frac{\theta_{\Gamma}}{\theta} - 1 \right) \tilde{w}(x) \, ds(x) \\ = \int_{\Omega} \left(-\frac{U_t^2 + \chi_t^2}{\theta} - \frac{|\nabla \theta|^2}{\theta^2} + U_t + 2\chi_t \right) \tilde{w}(x) \, dx.$$

We now set

$$\mathcal{H}[v] = \text{sign}(v) \left(-\frac{U_t^2 + \chi_t^2}{\theta} - \frac{|\nabla \theta|^2}{\theta^2} + U_t + 2\chi_t \right)$$

and check that the hypotheses of Proposition 4.5 are satisfied with the choice $f(v) = (\theta_{\Gamma}/R_0)(e^v - 1)$, $v_{\Gamma} = (R_0 - \theta_{\Gamma})/R_0$, $r \equiv 1$, and $v\mathcal{H}[v] \leq 2C|v|$, where C is a common upper bound for U_t and χ_t . Hence, v is bounded above by some v^* , which entails $\theta \geq R_0 e^{-v^*}$. This concludes the proof of Theorem 4.1.

5. Long time behavior. In order to emphasize the relation to section 3, we keep the original physical constants as in (2.23)–(2.25). We prove the following statement.

PROPOSITION 5.1. *Let the hypotheses of Theorem 4.1 hold, and let the constants $\tilde{\beta}, \omega$ introduced in (3.11) satisfy the condition $1 - \tilde{\beta} > \max\{0, \omega\}$. Then we have*

$$(5.1) \quad \int_0^{\infty} \left(\int_{\Omega} (\theta_t^2 + U_t^2 + \chi_t^2 + |\nabla \theta|^2) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})^2 \, ds(x) \right) dt < \infty,$$

$$(5.2) \quad \lim_{t \rightarrow \infty} \left(\int_{\Omega} (U_t^2 + \chi_t^2 + |\nabla \theta|^2)(x, t) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})^2(x, t) \, ds(x) \right) = 0.$$

Furthermore, the functions $U_{\Omega}(t) = \int_{\Omega} U(x, t) \, dx$, $X_{\Omega}(t) = \int_{\Omega} (1 - \chi(x, t)) \, dx$ converge to their equilibrium values as $t \rightarrow \infty$.

The condition $1 - \tilde{\beta} > \max\{0, \omega\}$ is certainly satisfied if $|p_0| + \beta\theta_c < L/\alpha$; hence it is not restrictive in standard situations.

The function on the left-hand side of (5.2) is a.e. equal to a function of bounded variation. The limit is to be understood in this sense.

We see in particular that the temperature converges strongly in $W^{1,2}(\Omega)$ to its equilibrium value θ_{Γ} , and the total ice content X_{Ω} , as well as the pressure, converges to a constant as $t \rightarrow \infty$. For temperatures θ_{Γ} above the freezing point or below the undercooling limit, this means, in agreement with the discussion in section 3, that also both $\chi(x, t)$ and $U(x, t)$ converge strongly in $L^1(\Omega)$ (hence, strongly in every $L^p(\Omega)$ for $p < \infty$) to their respective equilibrium values. For intermediate temperatures, only the limit total ice content can be identified, but we are not able to decide about the convergence of the individual trajectories to some of the equilibria.

Proof. By (2.39), since θ is uniformly bounded from above, we have

$$(5.3) \quad \int_0^{\infty} \int_{\Omega} (U_t^2 + \chi_t^2 + |\nabla \theta|^2)(x, t) \, dx \, dt + \int_0^{\infty} \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})^2(x, t) \, ds(x) \, dt < \infty.$$

We rewrite (2.23) in the form

$$(5.4) \quad c\theta_t - \kappa\Delta\theta = \nu(U_t)^2 - \beta\theta U_t + \gamma\chi_t^2 - \frac{L}{\theta_c}\theta\chi_t.$$

Due to the uniform L^∞ upper bounds for θ and U_t , we can test (5.4) by θ_t and obtain

$$(5.5) \quad \int_{\Omega} \theta_t^2(x, t) \, dx + \frac{d}{dt} \left(\int_{\Omega} |\nabla\theta|^2(x, t) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})^2 \, ds(x) \right) \\ \leq C_{13} \int_{\Omega} (U_t^2 + \chi_t^2) \, dx \quad \text{a.e.},$$

with C_{13} independent of t , and (5.3) with (5.5) together with [20, Lemma 3.1] yield

$$(5.6) \quad \lim_{t \rightarrow \infty} \int_{\Omega} |\nabla\theta|^2(x, t) \, dx + \int_{\partial\Omega} h(x)(\theta - \theta_{\Gamma})^2(x, t) \, ds(x) = 0.$$

System (2.24)–(2.25) can be again considered as a gradient flow of the form (4.9), with v and $\psi(v)$ analogous to (4.11)–(4.12); more precisely

$$(5.7) \quad v = \begin{pmatrix} \nu U \\ \gamma\chi \end{pmatrix},$$

$$(5.8) \quad \psi(v) = \int_{\Omega} \left(\frac{\lambda}{2} (U - \alpha(1 - \chi))^2 + (L\chi + \beta\theta_c U) \left(1 - \frac{\theta_{\Gamma}}{\theta_c} \right) + I(\chi) \right) \, dx \\ + \frac{K_{\Gamma}}{2} \left(\int_{\Omega} U \, dx + \frac{p_0}{K_{\Gamma}} \right)^2 + C_{\psi},$$

$$(5.9) \quad f = \begin{pmatrix} \beta(\theta - \theta_{\Gamma}) \\ (L/\theta_c)(\theta - \theta_{\Gamma}) \end{pmatrix}.$$

We have $f \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ by (5.3), and $\dot{f} \in L^2(0, \infty; L^2(\Omega) \times L^2(\Omega))$ by (5.5). From Lemma 4.2 we conclude that

$$(5.10) \quad \lim_{t \rightarrow \infty} \int_{\Omega} (U_t^2 + \chi_t^2)(x, t) \, dx = 0.$$

Set $\theta_{\Omega}(t) = \int_{\Omega} \theta(x, t) \, dx$. The equation for U_{Ω} now reads

$$(5.11) \quad \nu \dot{U}_{\Omega} + (\lambda + K_{\Gamma}|\Omega|)U_{\Omega} = \alpha\lambda X_{\Omega} + \beta\theta_{\Omega} - (p_0 + \beta\theta_c)|\Omega|;$$

hence

$$(5.12) \quad \lim_{t \rightarrow \infty} ((\lambda + K_{\Gamma}|\Omega|)U_{\Omega}(t) - \alpha\lambda X_{\Omega}(t)) = (\beta(\theta_{\Gamma} - \theta_c) - p_0)|\Omega|.$$

From (2.24) and (2.25) we obtain for a.e. $(x, t) \in \Omega_{\infty}$ that

$$(5.13) \quad \lambda(U - \alpha(1 - \chi)) = -\nu U_t + \beta(\theta - \theta_c) - p_0 - K_{\Gamma}U_{\Omega},$$

$$(5.14) \quad -\gamma\chi_t \in \alpha(-\nu U_t + \beta(\theta_{\Gamma} - \theta_c) - p_0 - K_{\Gamma}U_{\Omega}) + L \left(1 - \frac{\theta_{\Gamma}}{\theta_c} \right) + \left(\alpha\beta - \frac{L}{\theta_c} \right) (\theta - \theta_{\Gamma}) + \partial I(\chi).$$

We define an auxiliary function

$$(5.15) \quad A(x, t) := -\gamma\chi_t(x, t) + \alpha\nu U_t(x, t) - \left(\alpha\beta - \frac{L}{\theta_c}\right)(\theta(x, t) - \theta_\Gamma)$$

$$(5.16) \quad + \alpha K_\Gamma U_\Omega(t) - \frac{\alpha^2 \lambda K_\Gamma X_\Omega(t) + \alpha(\beta(\theta_\Gamma - \theta_c) - p_0)K_\Gamma|\Omega|}{\lambda + K_\Gamma|\Omega|}.$$

With the notation of (3.7), (3.11) we rewrite (5.14) in the form

$$(5.17) \quad \frac{1}{L}A(x, t) + \frac{d}{|\Omega|}X_\Omega(t) + (1 - \tilde{\beta})\left(\frac{\theta_\Gamma}{\theta_c} - 1\right) + \omega \in \partial I(\chi(x, t)) \quad \text{a.e.},$$

as an evolution counterpart of the equilibrium condition (3.12). The above computations show that $\lim_{t \rightarrow \infty} |A(t)|_2 = 0$. We now prove the following implications:

- (i) If $\theta_\Gamma \geq \theta_c(1 - \omega/(1 - \tilde{\beta}))$, then $\int_\Omega |1 - \chi(x, t)| dx \rightarrow 0$ as $t \rightarrow \infty$.
- (ii) If $\theta_\Gamma \leq \theta_c(1 - (\omega + d)/(1 - \tilde{\beta}))$ then $\int_\Omega \chi(x, t) dx \rightarrow 0$ as $t \rightarrow \infty$.
- (iii) If $\theta_c(1 - (\omega + d)/(1 - \tilde{\beta})) < \theta_\Gamma < \theta_c(1 - \omega/(1 - \tilde{\beta}))$, then

$$X_\Omega(t) \rightarrow (|\Omega|/d)((1 - \tilde{\beta})(1 - (\theta_\Gamma/\theta_c)) - \omega)$$

as $t \rightarrow \infty$.

The corresponding convergence of U then follows from (5.12)–(5.13).

To prove the above statements (i)–(iii), set

$$\chi^* := \frac{1}{d} \left((1 - \tilde{\beta}) \left(1 - \frac{\theta_\Gamma}{\theta_c} \right) - \omega \right), \quad A^*(x, t) := \frac{1}{Ld} A(x, t).$$

Equation (5.17) reads

$$(5.18) \quad A^*(x, t) + \frac{1}{|\Omega|}X_\Omega(t) - \chi^* \in \partial I(\chi(x, t)) \quad \text{a.e.},$$

that is,

$$(5.19) \quad \left(A^*(x, t) + \frac{1}{|\Omega|}X_\Omega(t) - \chi^* \right) (\tilde{\chi} - \chi(x, t)) \leq 0 \quad \text{a.e.} \quad \forall \tilde{\chi} \in [0, 1].$$

Integrating over Ω , we obtain for every $\tilde{\chi} \in [0, 1]$ and a.e. $t > 0$ that

$$(5.20) \quad \left(\frac{1}{|\Omega|}X_\Omega(t) - \chi^* \right) \left(\frac{1}{|\Omega|}X_\Omega(t) - (1 - \tilde{\chi}) \right) \leq -\frac{1}{|\Omega|} \int_\Omega A^*(x, t) (\tilde{\chi} - \chi(x, t)) dx.$$

The right-hand side of (5.20) tends to 0 as t tends to ∞ . Hence,

$$(5.21) \quad \limsup_{t \rightarrow \infty} \left(\frac{1}{|\Omega|}X_\Omega(t) - \chi^* \right) \left(\frac{1}{|\Omega|}X_\Omega(t) - (1 - \tilde{\chi}) \right) \leq 0 \quad \forall \tilde{\chi} \in [0, 1].$$

Implications (i)–(iii) now follow easily from (5.21) by putting

$$\tilde{\chi} = \begin{cases} 1 & \text{if } \chi^* \leq 0 \quad (\text{case (i)}), \\ 0 & \text{if } \chi^* \geq 1 \quad (\text{case (ii)}), \\ 1 - \chi^* & \text{if } 0 < \chi^* < 1 \quad (\text{case (iii)}). \end{cases} \quad \square$$

Note that in all cases the difference $U - \alpha(1 - \chi)$ converges in $L^2(\Omega)$ to its equilibrium value as $t \rightarrow \infty$. The question of whether $\chi(x, t)$ and $U(x, t)$ separately converge in case (iii) is still open.

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