Polyhedral sweeping processes with oblique reflection in the space of regulated functions

Pavel Krejčí [1] , Alexander A. Vladimirov [2]

Abstract

The Lipschitz continuity of the solution operator of a sweeping process with moving polyhedral characteristic and oblique reflection in the space of left-continuous vector-valued regulated functions with respect to the supremum norm is proved under the hypothesis of ℓ-paracontractivity and transversality of the associated projection system.

Keywords: sweeping process, oblique projection, paracontraction, Lipschitz continuity

MSC 2000: 34A60, 47A30, 52B70, 90C25

Introduction

Sweeping processes have been introduced in [12] as a general mathematical description for the evolution of systems under time dependent convex constraints. Typically, such problems arise in elastoplasticity, nonsmooth mechanics, convex optimization, mathematical economics, queuing theory, and many others, see e. g. [11, 4], and the references therein.

Formally, the problem consists in determining the evolution of a vector $\xi(t)$ inside a moving convex set $Z(t)$ under the condition that the time derivative of ξ (possibly in a generalized sense) lies in a prescribed cone of admissible directions (the so-called reflection cone). The classical case of normal reflection is related to the theory of variational inequalities on the one hand, and to the theory of hysteresis operators on the other hand, and a survey can be found in [8] or [3]. Oblique reflections have been investigated in [4, 5, 9] in the case where $Z(t)$ at each time t is a shift $Z + u(t)$ of a fixed polyhedral set Z (the *Skorokhod problem*). The results of [4, 5] include the well-posedness of the Skorokhod problem in the space of (right-continuous) regulated functions (that is, functions which admit both one-sided limits at each point of their domain of definition), and the Lipschitz continuity of the solution map, under appropriate conditions on the reflection directions. The paper [9] has developed the idea of [14, 15] and applied the concept of *ℓ-paracontracting associated projection systems* introduced in [6] to the Skorokhod problem in the space of continuous and absolutely continuous functions.

In this paper, we extend these results to the case where the faces of the polyhedron $Z(t)$ are allowed to move independently of each other under preservation of the normal and the reflection directions, the motion being described by regulated functions. To keep our approach consistent with the variational one in [10], we restrict ourselves to *left-continuous* regulated functions. The main result consists in proving that if the associated projection

[1]Mathematical Institute, Academy of Sciences of the Czech Republic, Žitná 25, CZ-11567 Praha 1, Czech Republic, E-Mail: `krejci@math.cas.cz`.

[2]Institute for Information Transmission, Bol'shoi Karetnyi 19, Moscow, Russia,
E-Mail: `vladim@iitp.ru, aav@redline.ru`.

system is ℓ*-paracontracting* and *transversal*, see Hypothesis 1.2 below, then the problem is well-posed in the space of left-continuous regulated functions, and the input-output mapping is Lipschitz continuous. Moreover, if the moving polyhedron contains a ball with a fixed positive radius and with a possibly moving center of bounded variation, then the outputs exhibit the same regularity feature as in the case of constant Z in [4, 5, 9], namely that they are of bounded variation even if the inputs are not.

The paper is organized as follows. In Section 1 we introduce the problem, list the hypotheses and state Theorems 1.3, 1.4 as the main results of the paper. Section 2 is devoted to a survey of results of [9] on invariant sets introduced in [4, 5] as a very efficient tool for the investigation of the Skorokhod problem. In Section 3 we establish the existence of a Lipschitz continuous oblique projection in the reflection direction onto a fixed convex set Z under the hypotheses of transversality and ℓ-paracontractivity. Theorem 1.3 is proved in Section 4 first for step functions, and then for any left-continuous regulated functions by density argument. Approximation by step functions is also used in the proof of the bounded variation result in Theorem 1.4. We also show in Corollary 4.4 that the output is continuous at each continuity point of the input. Finally, in Section 5, we give a few illustrating examples.

1 Statement of the problem

Let X be a finite-dimensional linear space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and with the norm $|x| = \langle x, x \rangle^{1/2}$ for $x \in X$.

We consider a moving polyhedral set $Z_t(\boldsymbol{c}) \subset X$ during a given time interval $t \in [0, T]$, defined in terms of a fixed system $n_1, \dots, n_p$ of unit outward normal vectors, as the intersection

$$Z_t(\boldsymbol{c}) := \bigcap_{j \in J} H_t^j(\boldsymbol{c}) \tag{1.1}$$

of half-spaces

$$H_t^j(\boldsymbol{c}) := \{z \in X \,;\, \langle z, n_j \rangle \le c_j(t)\} \quad \text{for } j \in J\,,\ t \in [0, T]\,, \tag{1.2}$$

where $J = \{1, \dots, p\}$ is the index set, and $\boldsymbol{c} = (c_1, \dots, c_p)$, $c_j : [0, T] \to [0, \infty[$ for $j \in J$, are given functions. The restriction to non-negative constraints c_j ensures that $0 \in Z_t(\boldsymbol{c})$, hence $Z_t(\boldsymbol{c})$ is non-empty for every $t \in [0, T]$.

Our principal framework will be the space $G(0, T; X)$ of *regulated functions* introduced in [1], that is, functions $w : [0, T] \to X$ which at each point $t \in [0, T]$ admit both one-sided limits $w(t+)$, $w(t-)$, with the convention $w(0-) := w(0)$, $w(T+) := w(T)$. Note that the set $BV(0, T; X)$ of functions $w : [0, T] \to X$ of bounded variation is contained in $G(0, T; X)$. A more complete information can be found e. g. in [7] or in [10] in connection with variational inequalities.

We endow the space $G(0, T; X)$ with the family of seminorms

$$|w|_{[t_1, t_2]} := \sup_{t \in [t_1, t_2]} |w(t)| \quad \text{for } w \in G(0, T; X)\,, \quad 0 \le t_1 < t_2 \le T\,.$$

Indeed, $|\cdot|_{[0,T]}$ is a norm. With respect to this norm, the set of functions of bounded variation as well as the set of step functions are dense in $G(0, T; X)$. We similarly denote by $G(0,T;\,\mathbb{R}^p_+)$ the space of componentwise non-negative regulated vector functions $\boldsymbol{c} = (c_1,\dots,c_p)$, and by $G_L(0,T;\,X)$, $G_L(0,T;\,\mathbb{R}^p_+)$ the corresponding spaces of left-continuous regulated functions.

For a vector $\boldsymbol{a} \in \mathbb{R}^p_+$, $\boldsymbol{a} = (a_1,\dots,a_p)$ we define the norm

$$|\boldsymbol{a}|' \;:=\; \max_{j\in J} a_j\,,$$

and for $\boldsymbol{c} \in G(0,T;\,\mathbb{R}^p_+)$, $0 \le t_1 < t_2 \le T$, we analogously put

$$|\boldsymbol{c}|'_{[t_1,t_2]} \;:=\; \sup_{t\in[t_1,t_2]} |\boldsymbol{c}(t)|'\,.$$

Let us assume that $\boldsymbol{c} \in G_L(0,T;\,\mathbb{R}^p_+)$ and $w \in G_L(0,T;\,X)$ are such that $w(t) \in Z_t(\boldsymbol{c})$ for every $t \in [0,T]$. The constraint c_j is said to be *active* at time t if $w(t) \in \partial H^j_t(\boldsymbol{c})$, that is, $\langle w(t), n_j\rangle = c_j(t)$. We extend this intuitive idea also to right limits and intervals and define the sets of 'active indices'

$$\begin{aligned}
\tilde J_t(\boldsymbol{c},w) \;&:=\; \{j\in J\,;\,\langle w(t),n_j\rangle = c_j(t)\}\ \text{ for }\ t\in[0,T]\,, && (1.3)\\
\tilde J_{t+}(\boldsymbol{c},w) \;&:=\; \{j\in J\,;\,\langle w(t+),n_j\rangle = c_j(t+)\}\ \text{ for }\ t\in[0,T[\ , && (1.4)\\
\tilde J_{[t_1,t_2]}(\boldsymbol{c},w) \;&:=\; \Bigg(\bigcup_{t\in[t_1,t_2]} \tilde J_t(\boldsymbol{c},w) \cup \bigcup_{t\in[t_1,t_2[} \tilde J_{t+}(\boldsymbol{c},w)\Bigg) && (1.5)
\end{aligned}$$

$$\text{for }\ [t_1,t_2]\subset[0,T]\,.$$

We will make substantial use of the following upper continuity result.

Lemma 1.1 *Let $\boldsymbol{c} \in G_L(0,T;\,\mathbb{R}^p_+)$ and $w \in G_L(0,T;\,X)$ be given, $w(t) \in Z_t(\boldsymbol{c})$ for every $t \in [0,T]$. Then for each $t\in[0,T]$ there exist $\varepsilon(t) > 0$ and $\delta(t) > 0$ such that for every $\widehat{\boldsymbol{c}} \in G_L(0,T;\,\mathbb{R}^p_+)$, $\hat w \in G_L(0,T;\,X)$, $\hat w(\tau) \in Z_\tau(\widehat{\boldsymbol{c}})$ for every $\tau \in [0,T]$, $|w-\hat w|_{[0,T]} < \varepsilon(t)$, $|\boldsymbol{c}-\widehat{\boldsymbol{c}}|'_{[0,T]} < \varepsilon(t)$, we have*

$$\begin{array}{llll}
\text{(i)} & \tilde J_\tau(\widehat{\boldsymbol{c}},\hat w)\ \subset\ \tilde J_t(\boldsymbol{c},w) & \forall\tau\in[t-\delta(t),t] & \textit{for }\ t>0\,,\\
\text{(ii)} & \tilde J_{\tau+}(\widehat{\boldsymbol{c}},\hat w)\ \subset\ \tilde J_t(\boldsymbol{c},w) & \forall\tau\in[t-\delta(t),t[& \textit{for }\ t>0\,,\\
\text{(iii)} & \tilde J_\tau(\widehat{\boldsymbol{c}},\hat w)\ \subset\ \tilde J_{t+}(\boldsymbol{c},w) & \forall\tau\in\,]t,t+\delta(t)] & \textit{for }\ t<T\,,\\
\text{(iv)} & \tilde J_{\tau+}(\widehat{\boldsymbol{c}},\hat w)\ \subset\ \tilde J_{t+}(\boldsymbol{c},w) & \forall\tau\in[t,t+\delta(t)[& \textit{for }\ t<T\,.
\end{array}$$

Proof. The numbers $\varepsilon_i(t) = c_i(t) - \langle w(t), n_i\rangle$ for $i \in J\setminus \tilde J_t(\boldsymbol{c},w)$, $\varepsilon_j^+(t) = c_j(t+) - \langle w(t+), n_j\rangle$ for $j \in J\setminus\tilde J_{t+}(\boldsymbol{c},w)$, are all positive. Putting

$$\varepsilon(t) \;:=\; \frac{1}{5}\min\{\varepsilon_i(t),\varepsilon_j^+(t)\,;\, i\in J\setminus\tilde J_t(\boldsymbol{c},w)\,,\ j\in J\setminus\tilde J_{t+}(\boldsymbol{c},w)\}\,,$$

we find $\delta(t) > 0$ such that $|c_i(\tau) - c_i(t)| < \varepsilon(t)$, $|w(\tau) - w(t)| < \varepsilon(t)$ for every $\tau \in [t-\delta(t),t]$ and every $i \in J\setminus\tilde J_t(\boldsymbol{c},w)$, $|c_j(\tau) - c_j(t+)| < \varepsilon(t)$, $|w(\tau) - w(t+)| < \varepsilon(t)$ for every $\tau \in]t, t+\delta(t)]$ and every $j \in J \setminus \tilde{J}_{t+}(\boldsymbol{c}, w)$. In case (i), we have for $i \in J \setminus \tilde{J}_t(\boldsymbol{c}, w)$, $t \in [t-\delta(t), t]$, and $|\hat{\boldsymbol{c}} - \boldsymbol{c}|'_{[0,T]} < \varepsilon(t)$, $|\hat{w} - w|_{[0,T]} < \varepsilon(t)$ that

$$\begin{aligned} \hat{c}_i(\tau) - \langle \hat{w}(\tau), n_i \rangle \ &\geq \ c_i(t) - \langle w(t), n_i \rangle - |\hat{c}_i(\tau) - c_i(t)| - |\hat{w}(\tau) - w(t)| \\ &\geq \ \varepsilon(t) \ > \ 0 \, , \end{aligned}$$

and arguing similarly in cases (ii) – (iv) we obtain the assertion. ■

We associate with the normal vectors $n_1, \dots, n_p$ another system $r_1, \dots, r_p$ of unit *reflection vectors*. For any subset J' of J we denote by $\mathcal{C}(J')$ the convex cone generated by vectors r_j with indices from J', that is

$$\mathcal{C}(J') := \left\{ y = \sum_{j \in J'} \alpha_j \, r_j \, , \ \alpha_j \geq 0 \ \text{ for } \ j \in J' \right\} .$$

In particular, for functions $\boldsymbol{c} \in G_L(0,T; \mathbb{R}^p_+)$, $w \in G_L(0,T; X)$, $w(t) \in Z_t(\boldsymbol{c})$ for every $t \in [0,T]$, we define the *reflection cone* at the point $w(t)$ by the formula

$$\mathcal{R}_t(\boldsymbol{c}, w) := \mathcal{C}(\tilde{J}_t(\boldsymbol{c}, w)) \, , \tag{1.6}$$

and similarly

$$\begin{aligned} \mathcal{R}_{t+}(\boldsymbol{c}, w) \ &:= \ \mathcal{C}(\tilde{J}_{t+}(\boldsymbol{c}, w)) \qquad && \text{for } \ t \in [0, T[\ , \\ \mathcal{R}_{[t_1,t_2]}(\boldsymbol{c}, w) \ &:= \ \mathcal{C}(\tilde{J}_{[t_1,t_2]}(\boldsymbol{c}, w)) \qquad && \text{for } \ [t_1, t_2] \subset [0,T] \, . \end{aligned} \tag{1.7}$$

We define the *polyhedral sweeping process* in the space of regulated functions as follows:

Problem PSP.

Given $u \in G_L(0,T; X)$, $\boldsymbol{c} \in G_L(0,T; \mathbb{R}^p_+)$, and $x_0 \in Z_0(\boldsymbol{c})$, find $\xi \in G_L(0,T; X)$ such that

$$\begin{array}{lll} \text{(i)} & x(t) := u(t) - \xi(t) \in Z_t(\boldsymbol{c}) & \text{for every } \ t \in [0,T] \, , \\ \text{(ii)} & x(0) = x_0 \, , & \\ \text{(iii)} & \xi(t+) - \xi(t) \ \in \ \mathcal{R}_{t+}(\boldsymbol{c}, x) & \text{for every } \ t \in [0, T[\ , \\ \text{(iv)} & \xi(t_2) - \xi(t_1) \ \in \ \mathcal{R}_{[t_1,t_2]}(\boldsymbol{c}, x) & \text{for every } \ 0 \leq t_1 < t_2 \leq T \, . \end{array}$$

This is an extension of the Skorokhod problem in the setting of [9] in two directions. First, we pass from continuous to regulated functions, and second, the constraints c_j are allowed to be non-constant regulated functions as well. Indeed, in the context of absolutely continuous functions, conditions (iii), (iv) can be replaced by a more natural one $\dot{\xi}(t) \in \mathcal{R}_t(\boldsymbol{c}, x)$ a. e., where the dot denotes derivative with respect to t. Note that our formulation assumes no a priori bound for the total variation of the inputs u, $\boldsymbol{c}$ or of the output ξ. Under the assumption of a 'uniformly non-empty interior', see Theorem 1.4 below, a finite bound for the total variation of ξ will however be obtained as an independent result.

We repeat here general hypotheses from [9] on the normal and reflection vectors which are assumed to be valid throughout the whole paper.

Hypothesis 1.2

(i) $X = \operatorname{span}\{n_1, \dots, n_p, r_1, \dots, r_p\}$.

(ii) $\langle n_j, r_j \rangle > 0$ *for every* $j \in J$.

(iii) *For a subset* $J' \subset J$ *we define the subspaces* $N_{J'} := \operatorname{span}\{n_j\,;\, j \in J'\}$, $R_{J'} := \operatorname{span}\{r_j\,;\, j \in J'\}$. *Denoting by* $N_{J'}^{\perp}$ *the orthogonal complement of* $N_{J'}$, *we assume that*

$$N_{J'}^{\perp} \cap R_{J'} = \{0\} \quad \forall\, J' \subset J\,. \tag{1.8}$$

(iv) *Let* $\mathcal{P}$ *be the system of all oblique projections* $P_j : X \to X$ *defined by the formula*

$$P_j = I - Q_j\,, \qquad Q_j x \;=\; \frac{\langle x, n_j \rangle}{\langle n_j, r_j \rangle} r_j\,, \qquad \textit{for}\ \ j \in J\,, \tag{1.9}$$

where I *is the identity. We assume that there exists a norm* $\|\cdot\|$ *in* X *and a constant* $\gamma > 0$ *such that for every* $x \in X$ *and every* $j \in J$ *we have*

$$\|x\| \;\geq\; \|P_j x\| + \gamma\, \|Q_j x\|\,. \tag{1.10}$$

Condition (iii) is called the *transversality*, and (iv) is the *ℓ–paracontractivity*. Hypotheses 1.2 (ii) – (iv) are automatically satisfied if $n_j = r_j$ for all $j \in J$ (the *normal reflection case*), and the norm in (1.10) can be constructed explicitly, see [2, 3, 15]. It was also shown in [9] that the property (iv) remains valid if the vectors r_j are replaced with sufficiently close vectors r_j'. The general situation is more difficult and is closely related to the problem of stability of discrete linear inclusions of the form $x_k = P_{j_k} x_{k-1}$, $k = 1, 2, \dots$, with $P_{j_k} \in \mathcal{P}$, see [15]. A geometric characterization of ℓ-paracontractivity in the case $p = 2$ is given in Example 5.1 below.

We first establish the existence, uniqueness, and Lipschitz continuity in Problem **PSP**.

Theorem 1.3 *Let Hypothesis 1.2 hold, and let*

$$\mathcal{D} \subset G_L(0,T;\, \mathbb{R}^p_+) \times G_L(0,T;\, X) \times X$$

be the set of all $(\boldsymbol{c}, u, x_0)$ *such that* $x_0 \in Z_0(\boldsymbol{c})$. *Then*

(i) *Problem* **PSP** *admits a unique solution* $\xi \in G_L(0,T;\, X)$ *for every* $(\boldsymbol{c}, u, x_0) \in \mathcal{D}$*;*

(ii) *there exists a constant* $C > 0$ *such that for every choice of* $(\boldsymbol{c}, u, x_0)$, $(\boldsymbol{c}', u', x_0') \in \mathcal{D}$, *the corresponding solutions* ξ, ξ' *to Problem* **PSP** *satisfy*

$$|\xi - \xi'|_{[0,T]} \;\leq\; C \left(|x_0 - x_0'| + |u - u'|_{[0,T]} + |\boldsymbol{c} - \boldsymbol{c}'|'_{[0,T]} \right)\,.$$

The bounded variation result reads as follows.

Theorem 1.4 *Let Hypothesis 1.2 hold, and let* $\varrho > 0$ *and* $v \in BV(0,T\,;X)$ *be given. Let* $U \subset G_L(0,T;\,X)$ *be a compact set and let* $V \subset G_L(0,T;\,\mathbb{R}^p_+)$ *be a set such that the ball*

$$B_\varrho(v(t)) \;=\; \{z \in X\,;\,|z - v(t)| \le \varrho\}$$

is contained in $Z_t(\boldsymbol{c})$ *for every* $t \in [0,T]$ *and every* $\boldsymbol{c} \in V$, *that is,*

$$\langle v(t), n_j\rangle \;\le\; c_j(t) - \varrho \qquad \forall \boldsymbol{c} \in V \quad \forall t \in [0,T] \quad \forall j \in J\,. \tag{1.11}$$

Then there exists a constant $C > 0$ *such that the total variation of* ξ *over* $[0,T]$ *denoted by* $\mathrm{Var}_{[0,T]}\,\xi$ *satisfies the inequality*

$$\mathop{\mathrm{Var}}_{[0,T]} \xi \;\le\; C \tag{1.12}$$

for every solution ξ *of Problem* **PSP** *independently of the choice of* $(\boldsymbol{c}, u, x_0) \in \mathcal{D}$ *with* $\boldsymbol{c} \in V$ *and* $u \in U$.

We do not need any special properties of compact subsets of the space $G(0,T;\,X)$. Note only that a compactness criterion of Arzelà-Ascoli type was established in [7].

The proofs will be given in Section 4. Before, we recall in Section 2 the concept of a $\mathcal{P}$-invariant set, and investigate oblique projections on convex polyhedral sets in the reflection direction in Section 3.

2 Invariant sets

Invariant sets (in a slightly different setting) were introduced in [4] as a powerful tool for the investigation of the Lipschitz continuity for the Skorokhod problem. We list here without proof some related results in the form given in Theorem 4.1, Corollaries 4.3, 4.4, 4.5, Proposition 4.10, and Theorem 5.5 of [9].

Theorem 2.1 *Let Hypothesis 1.2 (ii) hold, let* P_j, Q_j *be the projections defined by (1.9), and let* $B \subset X$ *be a closed convex set,* $0 \in B$. *Then the following two conditions are equivalent.*

($\mathcal{A}$) $\forall\, x \in B\,, \forall\, j \in J:\quad w := P_j\, x \pm Q_j\, n_j \,\in\, B\,,$

($\mathcal{B}$) $\forall\, x \in B\,, \forall\, y \in \mathcal{N}_B(x)\,, \forall\, j \in J:\; |\langle x, n_j\rangle| < 1 \;\Rightarrow\; \langle y, r_j\rangle = 0\,,$

where $\mathcal{N}_B(x)$ *denotes the outward normal cone to* B *at the point* x.

In the sequel, by a *$\mathcal{P}$-invariant set* we understand any convex closed set B containing the origin and satisfying ($\mathcal{A}$).

As an easy consequence of Theorem 2.1, we have

Corollary 2.2 *Let Hypothesis 1.2 (ii) hold, and let* B *be a* $\mathcal{P}$*-invariant set. We then have*

$$\forall\, z \in B \quad \forall\, y \in \mathcal{N}_B(z) \quad \forall\, j \in J: \quad \langle z, n_j\rangle\,\langle y, r_j\rangle \;\ge\; 0\,.$$

The following Theorem gives a rather general sufficient condition for the existence of a $\mathcal{P}$-invariant set, although it does not cover all possible situations, see Example 5.3 below.

Theorem 2.3 *Let Hypothesis 1.2 hold. Then there exists a $\mathcal{P}$-invariant set B which is bounded, symmetric, and has non-empty interior.*

Theorem 2.3 implies in particular that the Minkowski functional m_B of the set B defined by the formula

$$m_B(x) \;:=\; \inf\left\{s > 0\,;\, \frac{1}{s}x \in B\right\} \qquad \text{for } \; x \in X \tag{2.13}$$

is an equivalent norm in X.

3 Oblique convex projections

In this section, we construct projection operators $\Pi_t(\boldsymbol{c}) : X \to Z_t(\boldsymbol{c})$ in the reflection direction for each $\boldsymbol{c} \in G_L(0,T;\mathbb{R}^p_+)$ and $t \in [0,T]$. To this end, we keep $\boldsymbol{c}$ and t fixed, and consider a polyhedron

$$Z \;=\; \bigcap_{j\in J} H_j\,, \tag{3.14}$$

$$H_j \;:=\; \{z \in X\,;\, \langle z, n_j\rangle \,\le\, c_j\} \quad \text{for } \; j \in J\,, \tag{3.15}$$

where $c_j \ge 0$ are given numbers. Similarly as in (1.3), (1.6) we denote

$$\tilde{J}(z) := \{j \in J\,;\, \langle z, n_j\rangle = c_j\}\,, \quad \mathcal{R}(z) := \mathcal{C}(\tilde{J}(z)) \quad \text{for } \; z \in Z\,. \tag{3.16}$$

We start with an easy lemma.

Lemma 3.1 *Let Hypothesis 1.2 (i), (ii), (iv) hold. Then for every $j \in J$, $x \in X$, and $0 \le \delta \le 1$ we have*

$$\|x\| \;\ge\; \|(I - \delta Q_j)\,x\| + \delta\gamma\|Q_j\,x\|\,. \tag{3.17}$$

Proof. Multiplying (1.10) by δ and using the triangle inequality, we get

$$\delta\|x\| \ge \|\delta(I-Q_j)x\| + \delta\gamma\|Q_jx\| \ge \|(I-\delta Q_j)x\| - (1-\delta)\|x\| + \delta\gamma\|Q_jx\|$$

and (3.17) follows easily. ∎

Let us define nonlinear operators of oblique projection onto half-spaces H_j, $j \in J$, as

$$\pi_j(x) \;:=\; \begin{cases} x & \text{if } \;\langle x, n_j\rangle \le c_j\,, \\ P_j\,x \,+\, c_j\,Q_j\,n_j & \text{if } \;\langle x, n_j\rangle > c_j\,. \end{cases} \tag{3.18}$$

We will need the following two properties of operators π_j:

Lemma 3.2 *Let Hypothesis 1.2 (i), (ii), (iv) hold. Then for each index $j \in J$ the following inequalities hold.*

$$
\begin{aligned}
&\text{(i)} \qquad \forall x \in X\,,\ \forall z \in H_j\ :\ \gamma\,\|\pi_j(x) - x\| \ \le\ \|x - z\| - \|\pi_j(x) - z\|\,,\\
&\text{(ii)} \qquad \forall x_1, x_2 \in X\ :\ \|\pi_j(x_1) - \pi_j(x_2)\|\\
&\qquad\qquad \le\ \|x_1 - x_2\| - \gamma\,\|(x_1 - \pi_j(x_1)) - (x_2 - \pi_j(x_2))\|\,.
\end{aligned}
$$

Proof.

(i) Let us denote $v = x - z$, $w = \pi_j(x) - z$. We have

$$
w = (I - \delta Q_j)v, \quad \text{where} \quad \delta = \begin{cases} 0 & \text{if } \ \langle x, n_j\rangle \le c_j\,,\\ \dfrac{\langle x, n_j\rangle - c_j}{\langle x, n_j\rangle - \langle z, n_j\rangle} & \text{if } \ \langle x, n_j\rangle > c_j\,. \end{cases}
$$

We have $0 \le \delta \le 1$ as $\langle z, n_j\rangle \le c_j$; hence the assertion follows from Lemma 3.1.

(ii) If $\langle x_i, n_j\rangle \le c_j$ for one or both of x_1, x_2, it suffices to use (i). Otherwise, we have $\pi_j(x_1) - \pi_j(x_2) = P_j\,(x_1 - x_2)$ and the statement follows directly from (1.10). ■

We further define a mapping $\pi : X \to Z$ such that for every $x \in X$ close to a point $z \in Z$, the difference $x - \pi(x)$ lies in the reflection cone of z (a precise formulation will be given in Lemma 3.4 below).

We take a specific sequence $\{j_k\}_{k=1}^{\infty}$ of indices from J, namely

$$
j_k \ :=\ (k-1)\,[\mathrm{mod}\,p] + 1 \quad \text{ for } \ k \in \mathbb{N}\,, \tag{3.19}
$$

and for a given $x \in X$ we define recursively the sequence

$$
y_0 \ :=\ x\,, \quad y_k \ :=\ \pi_{j_k}(y_{k-1})\,, \quad k \in \mathbb{N}\,. \tag{3.20}
$$

For every $z \in Z$, it follows from Proposition 3.2 (i) that

$$
\gamma\,\|y_k - y_{k-1}\| \ \le\ \|y_{k-1} - z\| - \|y_k - z\|\,, \tag{3.21}
$$

hence (here, we use the assumption that Z is non-empty !)

$$
\gamma \sum_{k=1}^{\infty} \|y_k - y_{k-1}\| \le \|x - z\| \qquad \forall\, z \in Z\,.
$$

The sequence $\{y_k\}$ is therefore convergent and we may put

$$
\pi(x) \ :=\ \lim_{k\to\infty} y_k \qquad \text{for } \ x \in X\,. \tag{3.22}
$$

For every $k \in \mathbb{N}$ we have $y_k \in H_{j_k}$, hence $\pi(x) \in Z$ for every $x \in X$. The mapping $\pi : X \to Z$ defined by formula (3.22) is called a *quasiprojection operator*. By changing the ordering in Eq. (3.19), we would get a different quasiprojection, indeed.

We now list further properties of π.

Lemma 3.3 *Let Hypothesis 1.2 (i), (ii), (iv) hold. Then for every $x, x_1, x_2 \in X$ we have*

$$\text{(i)} \quad \|\pi(x) - z\| \;\leq\; \|x - z\| - \gamma\,\|x - \pi(x)\| \qquad \forall\, z \in Z\,,$$
$$\text{(ii)} \quad \|x - \pi(x)\| \;\leq\; 2\min_{z\in Z} \|x - z\|\,,$$
$$\text{(iii)} \quad \|\pi(x_1) - \pi(x_2)\| \leq \|x_1 - x_2\| - \gamma\,\|(x_1 - \pi(x_1)) - (x_2 - \pi(x_2))\|\,.$$

Proof.

(i) follows immediately from (3.21).

(ii) Let $z^* \in Z$ be such that $\|x - z^*\| = \min_{z\in Z} \|x - z\|$. From (3.21) we obtain $\|y_k - z^*\| \leq \|y_0 - z^*\|$, hence $\|x - y_k\| \leq \|x - z^*\| + \|y_k - z^*\| \leq 2\,\|x - z^*\|$ and (ii) follows.

(iii) Let $\{y_k^{(i)}\}$ for $i = 1, 2$ be the sequences (3.20) with initial conditions $y_0^{(i)} = x_i$. By Proposition 3.2 (ii), for all k we have

$$\|y_{k+1}^{(1)} - y_{k+1}^{(2)}\| \leq \|y_k^{(1)} - y_k^{(2)}\| - \gamma\,\|(y_k^{(1)} - y_{k+1}^{(1)}) - (y_k^{(2)} - y_{k+1}^{(2)})\|$$

and analogously to (i), a summation argument completes the proof. ∎

The following property of π plays a substantial role in our argument.

Lemma 3.4 *Let Hypothesis 1.2 (i), (ii), (iv) hold. Let $z \in Z$ be given and let $\varepsilon > 0$ be such that the implication*

$$\|x - z\| < \varepsilon \;\Rightarrow\; \langle x, n_j\rangle < c_j \qquad \forall\, j \in J \setminus \tilde{J}(z)$$

holds for every $x \in X$. We then have

$$x - \pi(x) \;\in\; \mathcal{R}(z) \qquad \forall\, x \in X\,,\ \|x - z\| < \varepsilon\,, \tag{3.23}$$

where $\mathcal{R}(z)$ is the reflection cone defined by (3.16).

Proof. Let $\{y_k\}$ be the sequence (3.20). By (3.21) we have $\|y_k - z\| \leq \|x - z\| < \varepsilon$ for every k, hence

$$\langle y_k, n_j\rangle \;<\; c_j \qquad \forall\, j \in J \setminus \tilde{J}(z)\ \ \forall k \in \mathbb{N}\,. \tag{3.24}$$

On the other hand, we have

$$\begin{aligned} x - \pi(x) \;&=\; \sum_{k=1}^{\infty}(y_{k-1} - y_k) \;=\; \sum_{k\in K} Q_{j_k}\,(y_{k-1} - c_{j_k}\,n_{j_k}) \\ &=\; \sum_{k\in K} \frac{\langle y_{k-1}, n_{j_k}\rangle - c_{j_k}}{\langle r_{j_k}, n_{j_k}\rangle}\; r_{j_k}\,, \end{aligned} \tag{3.25}$$

where $K = \{k\,;\,\langle y_{k-1}, n_{j_k}\rangle > c_{j_k}\}$. By (3.24) we therefore have $j_k \in \tilde{J}(z)$ for every $k \in K$, and from (3.25) we conclude that there exist coefficients $\alpha_j \geq 0$ such that

$$x - \pi(x) \;=\; \sum_{j\in\tilde{J}(z)} \alpha_j\, r_j\,, \tag{3.26}$$

which we wanted to prove. ■

The transition from quasiprojections to projections will be carried out in the following way.

Lemma 3.5 *Let Hypothesis 1.2 (i), (ii), (iv) hold, and let* π *be the quasiprojection (3.22). Then for every* $\sigma \in]0,1[$ *there exists a mapping* $\Pi_\sigma : X \to Z$ *such that* $\Pi_\sigma(x) = x$ *for* $x \in Z$, *and for every* $x, y \in X$ *we have*

$$\Pi_\sigma(x) \;=\; \pi(\Pi_\sigma(x) + \sigma(x - \Pi_\sigma(x)))\,, \tag{3.27}$$

$$\|\Pi_\sigma(x) - \Pi_\sigma(y)\| \;\le\; \|x-y\| - \gamma\,\|(x - \Pi_\sigma(x)) - (y - \Pi_\sigma(y))\|\,. \tag{3.28}$$

Proof. Let $\sigma \in]0,1[$ and $x \in X$ be given. For $z \in Z$ put $G(z) := \pi(z + \sigma(x-z))$. By Lemma 3.3 (iii) we have $\|G(z_1) - G(z_2)\| \le (1-\sigma)\,\|z_1 - z_2\|$ for every $z_1, z_2 \in Z$, hence $G : Z \to Z$ is a contraction. It suffices to denote by $\Pi_\sigma(x)$ its unique fixed point. If $x \in Z$, then $z + \sigma(x-z) \in Z$ for $z \in Z$, hence $\pi(z + \sigma(x-z)) = z + \sigma(x-z)$, and we immediately obtain $\Pi_\sigma(x) = x$. To check that (3.28) holds, we apply Lemma 3.3 (iii) with $x_1 = \Pi_\sigma(x) + \sigma(x - \Pi_\sigma(x))$, $x_2 = \Pi_\sigma(y) + \sigma(y - \Pi_\sigma(y))$, and the triangle inequality yields the assertion. ■

Lemma 3.6 *Let Hypothesis 1.2 (i), (ii), (iv) hold. Then for every* $x \in X$ *there exists* $z \in Z$ *such that* $x - z \in \mathcal{R}(z)$.

Proof. Let $x \in X$ be given, and for $\sigma \in]0,1[$ put $z(\sigma) := \Pi_\sigma(x)$. Inequality (3.28) for a fixed $y \in Z$ implies in particular that the one-parametric family $\{z(\sigma)\,;\, \sigma \in]0,1[\}$ is bounded, hence there exists a sequence $\{\sigma_k\,;\, k \in \mathbb{N}\}$ and an element $z \in Z$ such that $\sigma_k \to 0$, $z(\sigma_k) \to z$ as $k \to \infty$. Consequently,

$$x_k \;:=\; z(\sigma_k) + \sigma_k\,(x - z(\sigma_k)) \quad\to\; z \qquad \text{as } \; k \to \infty\,. \tag{3.29}$$

From Lemma 3.4 it follows that for k sufficiently large we have

$$x_k - \pi(x_k) \;=\; \sigma_k\,(x - z(\sigma_k)) \;\in\; \mathcal{R}(z)\,, \tag{3.30}$$

hence also $x - z(\sigma_k) \in \mathcal{R}(z)$, and letting k tend to infinity we obtain the assertion. ■

Lemma 3.7 *Let Hypothesis 1.2 hold, and let* $x_1, x_2 \in X$ *and* $z_1, z_2 \in Z$ *be such that* $\xi_i := x_i - z_i \in \mathcal{R}(z_i)$ *for* $i = 1,2$. *Let* B, m_B *be as in Theorem 2.3 and Eq. (2.13). Then we have*

$$m_B(\xi_1 - \xi_2) \;\le\; |x_1 - x_2|\,.$$

Proof. Put $\bar\psi := m_B(\xi_1 - \xi_2)$ and assume $\bar\psi > |x_1 - x_2|$. Then $z := (\xi_1 - \xi_2)/\bar\psi \in \partial B$, and for every $y \in \mathcal{N}_B(z)$ we have $\bar\psi\,\langle y, z\rangle = \langle y, \xi_1 - \xi_2\rangle > 0$. This implies the existence of either at least one $j \in \tilde J(z_1)$ such that $\langle y, r_j\rangle > 0$, or at least one $i \in \tilde J(z_2)$ such that $\langle y, r_i\rangle < 0$. Both cases are symmetric to each other, so let us choose the former one.

By Corollary 2.2 we have $\langle z, n_j\rangle \geq 0$, hence $\langle \xi_1 - \xi_2, n_j\rangle \geq 0$. On the other hand, since $j \in \tilde{J}(z_1)$, we have $\langle z_1 - z_2, n_j\rangle = c_j - \langle z_2, n_j\rangle \geq 0$. We conclude that

$$0 \leq \langle \xi_1 - \xi_2, n_j\rangle \leq \langle x_1 - x_2, n_j\rangle \leq |x_1 - x_2| < \bar{\psi}\,,$$

hence $|\langle z, n_j\rangle| < 1$, $\langle y, r_j\rangle > 0$, which contradicts Theorem 2.1. ■

Note that for the *existence* of an element z satisfying Lemma 3.6 we did not need the transversality condition in Hypothesis 1.2 (iii). It is however substantial for the boundedness of the $\mathcal{P}$-invariant set B, cf. Corollary 4.8 of [9], and for the *uniqueness* in Lemma 3.7, see Example 5.4 below.

The main result of this section reads as follows.

Theorem 3.8 *Let Z be given by (3.14), (3.15), and let Hypothesis 1.2 hold. Then there exists a mapping $\Pi : X \to Z$ such that $x - \Pi(x) \in \mathcal{R}(\Pi(x))$ for every $x \in X$, $\Pi(x) = x$ for $x \in Z$, and*

$$\|\Pi(x) - \Pi(y)\| \ \leq\ \|x - y\| - \gamma\, \|(x - \Pi(x)) - (y - \Pi(y))\| \qquad \forall x, y \in X\,. \tag{3.31}$$

Proof. By Lemmas 3.6 and 3.7, for every $x \in X$ there exists a unique $z \in Z$ such that $x - z \in \mathcal{R}(z)$. The argument of the proof of Lemma 3.6 implies that $z = \lim_{\sigma\to 0} \Pi_\sigma(x)$, and it suffices to put $\Pi(x) := z$. Passing to the limit as $\sigma \to 0$ in (3.28) we obtain the assertion. ■

4 Proofs of main results

We start with the case where both $\boldsymbol{c}$ and u are step functions of the form

$$u(t) \ = \ u_0\, \chi_{\{0\}}(t) + \sum_{k=1}^{m} u_k\, \chi_{]\tau_{k-1},\tau_k]}(t)\,, \tag{4.32}$$

$$\boldsymbol{c}(t) \ = \ \boldsymbol{c}^0\, \chi_{\{0\}}(t) + \sum_{k=1}^{m} \boldsymbol{c}^k\, \chi_{]\tau_{k-1},\tau_k]}(t)\,, \tag{4.33}$$

where χ_A is the characteristic function of a set A, $0 = \tau_0 < \tau_1 < \ldots < \tau_m = T$ is a partition of the interval $[0, T]$, $u_0, u_1, \ldots, u_m$ are vectors from X, and $\boldsymbol{c}^0, \ldots, \boldsymbol{c}^m$ are elements from $\mathbb{R}^p_+$. The solution ξ of Problem **PSP** will also be sought in the form

$$\xi(t) \ = \ \xi_0\, \chi_{\{0\}}(t) + \sum_{k=1}^{m} \xi_k\, \chi_{]\tau_{k-1},\tau_k]}(t)\,. \tag{4.34}$$

Proposition 4.1 *Let Hypothesis 1.2 hold, and let u, $\boldsymbol{c}$, be of the form (4.32), (4.33). Let $x_0 \in Z_0(\boldsymbol{c})$ be given, and let x_k for $k = 1, \ldots, m$ be defined recursively by the formula*

$$x_k = \Pi_{\tau_k}(\boldsymbol{c})(x_{k-1} + u_k - u_{k-1})\,, \tag{4.35}$$

where $\Pi_{\tau_k}(\boldsymbol{c}) : X \to Z_{\tau_k}(\boldsymbol{c})$ is the projection from Theorem 3.8. Let $\xi_k := u_k - x_k$ for $k = 0, \ldots, m$. Then the function ξ defined by Eq. (4.34) is a solution to Problem **PSP**.

Proof. By definition, we have $x(t) := u(t) - \xi(t) \in Z_t(\boldsymbol{c})$ for every $t \in [0,T]$. Moreover, we have for all k that

$$\begin{aligned} \xi_k - \xi_{k-1} &= (x_{k-1} + u_k - u_{k-1}) - \Pi_{\tau_k}(\boldsymbol{c})(x_{k-1} + u_k - u_{k-1}) \\ &\in \mathcal{R}_{\tau_k}(\boldsymbol{c}, x) \ = \ \mathcal{R}_{\tau_{k-1}+}(\boldsymbol{c}, x)\,. \end{aligned} \tag{4.36}$$

Condition (iii) of Problem **PSP** is then automatically fulfilled. To check that (iv) holds, we fix $t_1 < t_2$ and find $0 \le i \le j \le m$ such that $\tau_{i-1} < t_1 \le \tau_i$, $\tau_{j-1} < t_2 \le \tau_j$, with the convention $\tau_{-1} = -\infty$. From Eq. (4.36) we obtain that

$$\begin{aligned} \xi(t_2) - \xi(t_1) &= \sum_{k=i+1}^{j} (\xi_k - \xi_{k-1}) \\ &\in \mathcal{R}_{\tau_{i+1}}(\boldsymbol{c}, x) + \ \ldots \ + \mathcal{R}_{\tau_{j-1}}(\boldsymbol{c}, x) + \mathcal{R}_{\tau_{j-1}+}(\boldsymbol{c}, x) \\ &\subset \mathcal{R}_{[t_1,t_2]}(\boldsymbol{c}, x)\,, \end{aligned}$$

and the proof is complete. ■

Proposition 4.2 *Let Hypothesis 1.2 hold, let B be a symmetric $\mathcal{P}$-invariant set with nonempty interior as in Theorem 2.3, and let m_B be its Minkowski functional. Let $(\boldsymbol{c}, u, x_0), (\boldsymbol{c}', u', x'_0) \in \mathcal{D}$ be such that there exist respective solutions $\xi, \xi' \in G_L(0,T;\, X)$ to Problem* **PSP**. *For $t \in [0,T]$ put $\bar\xi(t) := \xi(t) - \xi'(t)$, $\bar u(t) := u(t) - u'(t)$, $\bar{\boldsymbol{c}}(t) := \boldsymbol{c}(t) - \boldsymbol{c}'(t)$. Then for every $t \in [0,T]$ we have*

$$m_B(\bar\xi(t)) \ \le \ \max\{m_B(\bar\xi(0)), |\bar u|_{[0,t]} + |\bar{\boldsymbol{c}}|'_{[0,t]}\}\,. \tag{4.37}$$

Proof. For $t \in [0,T]$ put $\psi(t) := m_B(\bar\xi(t))$, $x(t) := u(t) - \xi(t)$, $x'(t) := u'(t) - \xi'(t)$, $\bar x(t) := x(t) - x'(t)$, and

$$\alpha(t) := \max\{m_B(\bar\xi(0)), |\bar u|_{[0,t]} + |\bar{\boldsymbol{c}}|'_{[0,t]}\},$$

and assume that there exists $t_1 \in]0,T[$ such that $\psi(t_1) > \alpha(t_1)$. We fix some $\psi_0 \in]\alpha(t_1), \psi(t_1)[$, and put

$$t_0 := \inf\{t \in [0,t_1]\,;\, \psi(t) \ge \psi_0\}\,. \tag{4.38}$$

The function ψ is left-continuous and $\psi(0) < \psi_0$. We therefore have $0 \le t_0 < t_1$, and one of the following two cases occurs:

(i) $\Psi := \psi(t_0+) > \max\{\psi(t_0), \alpha(t_1)\}$,

(ii) $\psi(t_0+) = \psi(t_0) = \psi_0 > \psi(t)$ for every $t \in [0, t_0[$, $t_0 > 0$.

In case **(i)** we put $z := \bar\xi(t_0+)/\Psi$. Then $m_B(z) = 1$, hence $z \in \partial B$, and for every $y \in \partial m_B(z)$, where ∂m_B is the subdifferential of m_B, we have by definition that

$$\langle y, z - \bar z\rangle \ \ge \ m_B(z) - m_B(\bar z) \quad \forall\, \bar z \in X\,. \tag{4.39}$$

In particular, we have $y \in \mathcal{N}_B(z)$, and putting $\bar{z} := \bar{\xi}(t_0)/\Psi$ in (4.39), we obtain

$$\langle y, \bar{\xi}(t_0+) - \bar{\xi}(t_0)\rangle \ \geq \ \psi(t_0+) - \psi(t_0) \ > \ 0\,. \tag{4.40}$$

Condition (iii) in Problem **PSP** reads

$$\xi(t_0+) - \xi(t_0) \in \mathcal{R}_{t_0+}(\boldsymbol{c}, x)\,, \quad \xi'(t_0+) - \xi'(t_0) \in \mathcal{R}_{t_0+}(\boldsymbol{c}', x')\,.$$

We thus infer from (4.40) that there exists either some $j \in \tilde{J}_{t_0+}(\boldsymbol{c}, x)$ such that $\langle y, r_j\rangle > 0$, or some $i \in \tilde{J}_{t_0+}(\boldsymbol{c}', x')$ such that $\langle y, r_i\rangle < 0$. Both cases are symmetric to each other, let us assume therefore that $\langle y, r_j\rangle > 0$ for some $j \in \tilde{J}_{t_0+}(\boldsymbol{c}, x)$. Then Corollary 2.2 yields $\langle z, n_j\rangle \geq 0$. On the other hand, by definition of $\tilde{J}_{t_0+}(\boldsymbol{c}, x)$ we have

$$\langle \bar{x}(t_0+), n_j\rangle = c_j(t_0+) - \langle x'(t_0+), n_j\rangle \geq \bar{c}_j(t_0+)\,.$$

We conclude that

$$\begin{aligned} 0 \ &\leq \ \langle z, n_j\rangle = \frac{1}{\Psi}\left(\langle \bar{u}(t_0+), n_j\rangle - \langle \bar{x}(t_0+), n_j\rangle\right) \\ &\leq \ \frac{1}{\Psi}(\langle \bar{u}(t_0+), n_j\rangle - \bar{c}_j(t_0+)) \ \leq \ \frac{\alpha(t_1)}{\Psi} \ < \ 1\,. \end{aligned}$$

This violates the property $(\mathcal{B})$ from Theorem 2.1, which is a contradiction.

Case **(ii)** is similar. Set $z := \bar{\xi}(t_0)/\psi_0$. Using Lemma 1.1 we find $\delta > 0$ such that

$$\tilde{J}_{[t_0-\delta,t_0]}(\boldsymbol{c}, x) \ \subset \ \tilde{J}_{t_0}(\boldsymbol{c}, x)\,, \quad \tilde{J}_{[t_0-\delta,t_0]}(\boldsymbol{c}', x') \ \subset \ \tilde{J}_{t_0}(\boldsymbol{c}', x')\,. \tag{4.41}$$

Then (4.39) holds for $y \in \mathcal{N}_B(z)$, and putting $\tilde{z} := \bar{\xi}(t_0 - \delta)/\psi_0$, we obtain similarly as in (4.40) that

$$\langle y, \bar{\xi}(t_0) - \bar{\xi}(t_0 - \delta)\rangle \ > \ 0\,. \tag{4.42}$$

By Condition (iv) of Problem **PSP** and (4.41), we have

$$\xi(t_0) - \xi(t_0 - \delta) \in \mathcal{R}_{t_0}(\boldsymbol{c}, x)\,, \quad \xi'(t_0) - \xi'(t_0 - \delta) \in \mathcal{R}_{t_0}(\boldsymbol{c}', x')\,. \tag{4.43}$$

We assume again that there exists $j \in \tilde{J}_{t_0}(\boldsymbol{c}, x)$ such that $\langle y, r_j\rangle > 0$, hence $\langle z, n_j\rangle \geq 0$, $\langle \bar{x}(t_0), n_j\rangle \leq \bar{c}_j(t_0)$. Repeating the argument of **(i)** we obtain that

$$\begin{aligned} 0 \ &\leq \ \langle z, n_j\rangle = \frac{1}{\psi_0}\left(\langle \bar{u}(t_0), n_j\rangle - \langle \bar{x}(t_0), n_j\rangle\right) \\ &\leq \ \frac{1}{\psi_0}(\langle \bar{u}(t_0), n_j\rangle - \bar{c}_j(t_0)) \ \leq \ \frac{\alpha(t_1)}{\psi_0} \ < \ 1\,, \end{aligned}$$

which is again a contradiction. Proposition 4.2 is proved. ∎

Proposition 4.3 *Let* $\{(\boldsymbol{c}^{(n)}, u^{(n)}, x_0^{(n)})\,;\, n \in \mathbb{N}\}$ *be a sequence in* $\mathcal{D}$ *such that for every* $n \in \mathbb{N}$ *there exists the respective solution* $\xi^{(n)} \in G_L(0,T;X)$ *to Problem* **PSP**. *Assume that* $u^{(n)}, \boldsymbol{c}^{(n)}$ *converge uniformly to some* $u \in G_L(0,T;X)$, $\boldsymbol{c} \in G_L(0,T;\mathbb{R}^p_+)$, *respectively, and* $x_0^{(n)}$ *converge to some* x_0. *Then* $\xi^{(n)}$ *converge uniformly to* ξ, *and* ξ *is a solution to Problem* **PSP** *corresponding to* $(\boldsymbol{c}, u, x_0)$.

Proof. We obviously have $(\boldsymbol{c}, u, x_0) \in \mathcal{D}$. From Proposition 4.2 it follows that $\{\xi^{(n)}\}$ is a Cauchy sequence in $G_L(0,T;\, X)$, there exists therefore $\xi \in G_L(0,T;\, X)$ such that $|\xi^{(n)} - \xi|_{[0,T]} \to 0$ as $n \to \infty$, and $x(t) := u(t) - \xi(t) \in Z_t(\boldsymbol{c})$ for every $t \in [0,T]$. It remains to check that Conditions (iii), (iv) of Problem **PSP** are fulfilled.

Let us first fix some $t \in [0,T[$. By Lemma 1.1 (iv), we have

$$\mathcal{R}_{t+}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \mathcal{R}_{t+}(\boldsymbol{c}, x)$$

for n sufficiently large, hence $\xi^{(n)}(t+) - \xi^{(n)}(t) \in \mathcal{R}_{t+}(\boldsymbol{c}, x)$ for large n and it suffices to pass to the limit and check that Condition (iii) is verified.

Let now $[t_1, t_2] \subset [0,T]$ be an arbitrary interval. For every $t \in [t_1, t_2]$ we find $\varepsilon(t) > 0$, $\delta(t) > 0$ as in Lemma 1.1. By compactness, we may fix $t_1 \le s_1 < \ldots < s_m \le t_2$ such that

$$[t_1, t_2] \subset \bigcup_{k=1}^{m} \,]s_k - \delta(s_k), s_k + \delta(s_k)[\ ,$$

and put $\varepsilon_0 := \min\{\varepsilon(s_k)\,;\, k = 1, \ldots, m\}$. We choose n_0 sufficiently large such that

$$|x^{(n)} - x|_{[0,T]} < \varepsilon_0 \,, \qquad |\boldsymbol{c}^{(n)} - \boldsymbol{c}|'_{[0,T]} < \varepsilon_0 \quad \text{for } \ n \ge n_0 \,. \tag{4.44}$$

For an arbitrary $t \in [t_1, t_2]$ we find s_k such that $|t - s_k| < \delta(s_k)$. By Lemma 1.1, one of the following alternatives occurs for $n \ge n_0$:

(a) $t \in]s_k, s_k + \delta(s_k)[\,, \quad \tilde{J}_t(\boldsymbol{c}^{(n)}, x^{(n)}) \cup \tilde{J}_{t+}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \tilde{J}_{s_k+}(\boldsymbol{c}, x)$,

(b) $t \in]s_k - \delta(s_k), s_k[\,, \quad \tilde{J}_t(\boldsymbol{c}^{(n)}, x^{(n)}) \cup \tilde{J}_{t+}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \tilde{J}_{s_k}(\boldsymbol{c}, x)$,

(c) $t = s_k \,, \quad \tilde{J}_t(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \tilde{J}_{s_k}(\boldsymbol{c}, x), \ \tilde{J}_{t+}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \tilde{J}_{s_k+}(\boldsymbol{c}, x)$.

We therefore conclude that $\tilde{J}_{[t_1,t_2]}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \tilde{J}_{[t_1,t_2]}(\boldsymbol{c}, x)$ for $n \ge n_0$, and passing to the limit in the inclusion

$$\xi^{(n)}(t_2) - \xi^{(n)}(t_1) \in \mathcal{R}_{[t_1,t_2]}(\boldsymbol{c}^{(n)}, x^{(n)}) \subset \mathcal{R}_{[t_1,t_2]}(\boldsymbol{c}, x)$$

we complete the proof. ∎

We are now ready to prove Theorems 1.3 and 1.4.

Proof of Theorem 1.3. For an arbitrary $(\boldsymbol{c}, u, x_0) \in \mathcal{D}$ there exists a sequence of step functions of the form (4.32), (4.33) which converge uniformly to u and $\boldsymbol{c}$, respectively, and such that $u_0 = u(0)$, $\boldsymbol{c}^0 = \boldsymbol{c}(0)$. The piecewise constant solutions from Proposition 4.1 of the approximate problems converge to a solution $\xi \in G_L(0,T;\, X)$ of Problem **PSP** according to Proposition 4.3. The uniqueness and the Lipschitz estimate then follows from Proposition 4.2, as the Minkowski functional m_B is an equivalent norm in X. ∎

Proof of Theorem 1.4. For an arbitrary $u \in U$ we define the singular set

$$S \ = \ \{t \in]0,T]\,;\, |u(t+) - u(t)| > \varrho\} \,. \tag{4.45}$$

Then S is finite, $S = \{s_i\}_{i=1}^{n}$, with

$$0 < s_1 < \ldots < s_n < T \,, \quad 0 \le n \le N_1 \tag{4.46}$$

with N_1 independent of $u \in U$ by virtue of the compactness of U. We further put $s_0 = 0$, $s_{n+1} = T$, and for $i = 1, \ldots, n+1$ define recursively the sequences

$$
\begin{aligned}
t_0^i &= s_{i-1}\,, & (4.47)\\
t_\ell^i &= \sup\,\{t \in]t_{\ell-1}^i, T]\,;\, |u(\tau) - u(t_{\ell-1}^i+)| \le \varrho/2 \quad \forall \tau \in]t_{\ell-1}^i, t[\,\} & (4.48)
\end{aligned}
$$

as long as $t_{\ell-1}^i < s_i$. It follows from (4.48) that $t_\ell^i \le s_i$. Indeed, assuming $t_\ell^i > s_i$ would imply $|u(s_i) - u(t_{\ell-1}^i+)| \le \varrho/2$, $|u(s_i+) - u(t_{\ell-1}^i+)| \le \varrho/2$, hence $|u(s_i+) - u(s_i)| \le \varrho$ in contradiction with (4.45).

Let us consider now a fixed $i \in \{1, \ldots, n+1\}$, and assume that for some $q \in \mathbb{N}$ we have $s_{i-1} = t_0^i < t_1^i < \ldots < t_q^i < s_i$. For each $\ell = 1, \ldots, q$ we have by (4.48) that $|u(t_\ell^i+) - u(t_{\ell-1}^i+)| \ge \varrho/2$. We thus can find points $a_\ell^i < t_\ell^i \le b_\ell^i$, $a_\ell^i \ge b_{\ell-1}^i$, such that

$$|u(b_\ell^i) - u(a_\ell^i)| \ge \varrho/4 \qquad \text{for} \quad \ell = 1, \ldots, q\,. \tag{4.49}$$

From the compactness of U we infer again that q cannot be arbitrarily large. In other words, there exists $N_2 > 0$ independent of $u \in U$ such that for each $i = 1, \ldots, n+1$ we have

$$s_{i-1} = t_0^i < t_1^i < \ldots < t_{q_i}^i = s_i\,, \qquad q_i \le N_2\,. \tag{4.50}$$

We now estimate the total variation of u over a fixed interval $[t_{\ell-1}^i, t_\ell^i]$, $i \in \{1, \ldots, n+1\}$, $\ell \in \{1, \ldots, q_i\}$. To this end, we consider step functions $\hat{u}$, $\hat{\boldsymbol{c}}$ of the form (4.32), (4.33), that is

$$
\begin{aligned}
\hat{u}(t) &= \hat{u}_0\, \chi_{\{0\}}(t) + \sum_{k=1}^m \hat{u}_k\, \chi_{]\tau_{k-1}, \tau_k]}(t)\,, & (4.51)\\
\hat{\boldsymbol{c}}(t) &= \hat{\boldsymbol{c}}^0\, \chi_{\{0\}}(t) + \sum_{k=1}^m \hat{\boldsymbol{c}}^k\, \chi_{]\tau_{k-1}, \tau_k]}(t)\,, & (4.52)
\end{aligned}
$$

corresponding to a partition $0 = \tau_0 < \tau_1 < \ldots < \tau_m = T$ with $\hat{u}_0 = u(0)$, $\hat{\boldsymbol{c}}^0 = \boldsymbol{c}(0)$, $\tau_r = t_{\ell-1}^i$, $\tau_s = t_\ell^i$ for some $r, s \in \mathbb{N}$, $0 \le r < s \le m$. By Proposition 4.1, the solution $\hat{\xi}$ of Problem **PSP** associated with $(\hat{\boldsymbol{c}}, \hat{u}, x_0) \in \mathcal{D}$ has the form

$$\hat{\xi}(t) \;=\; \hat{\xi}_0\;\chi_{\{0\}}(t) + \sum_{k=1}^m \hat{\xi}_k\;\chi_{]\tau_{k-1}, \tau_k]}(t) \tag{4.53}$$

with $\hat{\xi}_0 = \hat{u}_0 - x_0$, and putting $\hat{x} := \hat{u} - \hat{\xi}$ we have

$$\hat{\xi}_k - \hat{\xi}_{k-1} \;=\; (\hat{u}_k - \hat{\xi}_{k-1}) - \Pi_{\tau_k}(\hat{\boldsymbol{c}})(\hat{u}_k - \hat{\xi}_{k-1}) \quad \in \; \mathcal{R}_{\tau_k}(\hat{\boldsymbol{c}}, \hat{x}) \tag{4.54}$$

for $k = 1, \ldots, m$. We choose the functions $\hat{\boldsymbol{c}}$, $\hat{u}$ in such a way that

$$|u - \hat{u}|_{[0,T]} \le \varrho/8\,, \quad |\boldsymbol{c} - \hat{\boldsymbol{c}}|' \le \varrho/4\,, \tag{4.55}$$

and for $k = r+1, \ldots, s$ we put

$$y_k = v(\tau_k) + \hat{u}_k - \hat{u}_r\,. \tag{4.56}$$

By (4.48) and (4.55) we have $|\hat{u}_k - \hat{u}_r| \le 3\varrho/4$ for all $k = r+1, \dots, s$, and from (1.11) it follows for each $k = r+1, \dots, s$ and $j \in J$ that

$$\langle y_k, n_j \rangle \ \le \ c_j(\tau_k) - \varrho/4 \ \le \ \hat{c}_j(\tau_k) \,, \tag{4.57}$$

that is,

$$y_k \in Z_{\tau_k}(\hat{\boldsymbol{c}}) \qquad \forall k \in \{r+1, \dots, s\} \,. \tag{4.58}$$

Theorem 3.8 with $x = \hat{u}_k - \hat{\xi}_{k-1}$, $y = y_k$, $Z = Z_{\tau_k}(\hat{\boldsymbol{c}})$ together with Eqs. (4.54), (4.56) yield

$$\begin{aligned} \gamma \|\hat{\xi}_k - \hat{\xi}_{k-1}\| \ &\le \ \|\hat{u}_k - \hat{\xi}_{k-1} - y_k\| - \|\hat{u}_k - \hat{\xi}_k - y_k\| \\ &= \ \|\hat{u}_r - \hat{\xi}_{k-1} - v(\tau_k)\| - \|\hat{u}_r - \hat{\xi}_k - v(\tau_k)\| \end{aligned} \tag{4.59}$$

for all $k = r+1, \dots, s$. Summing up over k the above inequalities and using the fact that $\|\cdot\|$ is an equivalent norm in X we find a constant $C_1 > 0$ such that

$$\begin{aligned} C_1 \operatorname*{Var}_{[t^i_{\ell-1}, t^i_\ell]} \hat{\xi} \ &\le \ \gamma \sum_{k=r+1}^{s} \|\hat{\xi}_k - \hat{\xi}_{k-1}\| \\ &\le \ \|\hat{x}_r - v(\tau_{r+1})\| - \|\hat{u}_r - \hat{\xi}_s - v(\tau_s)\| \\ &\quad + \sum_{k=r+1}^{s-1} \|\hat{u}_r - \hat{\xi}_k - v(\tau_{k+1})\| - \|\hat{u}_r - \hat{\xi}_k - v(\tau_k)\| \\ &\le \ \|\hat{x}_r\| + \|v(\tau_{r+1})\| + \sum_{k=r+1}^{s-1} \|v(\tau_{k+1}) - v(\tau_k)\| \,. \end{aligned} \tag{4.60}$$

By hypothesis, the right-hand side of the above inequality is bounded above by a constant which we denote by C_2 and which is independent of the choice of $u \in U$, $\boldsymbol{c} \in V$, and of $\hat{u}$ and $\hat{\boldsymbol{c}}$ satisfying (4.55). Considering sequences of such functions which uniformly converge to u and $\boldsymbol{c}$, respectively, we obtain from Theorem 1.3 that the sequence of piecewise constant solutions $\hat{\xi}$ uniformly converges to ξ. The estimate (4.60) then implies that

$$\operatorname*{Var}_{[t^i_{\ell-1}, t^i_\ell]} \xi \ \le \ \frac{C_2}{C_1} \,, \tag{4.61}$$

and the assertion with $C = (N_1 + 1) N_2 C_2 / C_1$ follows from (4.46) and (4.50). ∎

Corollary 4.4 *Let the hypotheses of Theorem 1.3 hold, let an element* $(\boldsymbol{c}, u, x_0) \in \mathcal{D}$ *be given, and let* $\xi \in G_L(0,T; X)$ *be the solution to Problem* **PSP**. *Let* $t \in [0, T[$ *be such that* $u(t+) = u(t)$, $\boldsymbol{c}(t+) = \boldsymbol{c}(t)$. *Then* $\xi(t+) = \xi(t)$. *In particular, if both* u *and* $\boldsymbol{c}$ *are continuous in* $[0,T]$, *then* ξ *is continuous in* $[0,T]$.

Proof. We have by hypothesis $Z_{t+}(\boldsymbol{c}) = Z_t(\boldsymbol{c})$, hence we are in the situation of Lemma 3.7 with $Z = Z_{t+}(\boldsymbol{c}) = Z_t(\boldsymbol{c})$, $x_1 = x_2 = u(t+) - \xi(t) = u(t) - \xi(t)$, $z_1 = x(t+)$, $z_2 = x(t)$. We have $x_1 - z_1 = \xi(t+) - \xi(t) \in \mathcal{R}(z_1)$, $x_2 - z_2 = 0$, and Lemma 3.7 yields $m_B(\xi(t+) - \xi(t)) = 0$, hence $\xi(t+) = \xi(t)$. ∎

5 Examples

Example 5.1 We give here a geometric characterization of ℓ-paracontractivity in the case of $\dim X = 2$, $p = 2$, with the unit vectors n_1, n_2, r_1, r_2 such that $|\langle n_1, n_2\rangle| < 1$, $\langle n_j, r_j\rangle > 0$ for $j = 1,2$. Put

$$a := \frac{\langle n_2, r_1\rangle}{\langle n_1, r_1\rangle}, \quad b := \frac{\langle n_1, r_2\rangle}{\langle n_2, r_2\rangle}. \tag{5.62}$$

Clearly, the system $\{P_1, P_2\}$ given by (1.9) is transversal. We show below that the following statement holds.

Proposition 5.2 *The system $\{P_1, P_2\}$ is ℓ-paracontracting if and only if $|ab| < 1$.*

Proof. Put $n_j^\perp := R\, n_j$, $j = 1,2$, where R is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We have

$$n_2^\perp \;=\; a\, n_1^\perp + \frac{\langle n_2^\perp, n_1\rangle}{\langle n_1, r_1\rangle}\, r_1\,, \qquad n_1^\perp \;=\; b\, n_2^\perp + \frac{\langle n_1^\perp, n_2\rangle}{\langle n_2, r_2\rangle}\, r_2\,,$$

hence $P_1\, n_2^\perp = a\, n_1^\perp$, $P_2\, n_1^\perp = b\, n_2^\perp$, $P_j\, n_j^\perp = n_j^\perp$ for $j = 1,2$.
Assume first that the system is ℓ-paracontracting. Then

$$\begin{aligned} |ab|\, \|n_1^\perp\| \;&=\; \|P_1 P_2\, n_1^\perp\| \;\le\; \|P_2 n_1^\perp\| - \gamma\, \|P_2\, n_1^\perp - P_1 P_2\, n_1^\perp\| \\ &\le\; \|n_1^\perp\| - \gamma\, \|n_1^\perp - P_2\, n_1^\perp\| - \gamma\, \|P_2\, n_1^\perp - P_1 P_2\, n_1^\perp\|\,, \end{aligned}$$

hence $|ab| < 1$.
Conversely, if $|ab| < 1$, we fix some c such that $|a| < c < 1/|b|$, with the convention $1/0 = \infty$. Any element $x \in X$ can be written the form

$$x = \alpha\, n_1^\perp + \beta\, n_2^\perp \tag{5.63}$$

for some $\alpha, \beta \in \mathbb{R}$, and we have $P_1 x = (\alpha + \beta a)\, n_1^\perp$, $P_2 x = (\alpha b + \beta)\, n_2^\perp$. Putting $\|x\| := |\alpha| + c\,|\beta|$ for $x \in X$ of the form (5.63), we obtain $\|P_1 x\| = |\alpha + \beta a| \le |\alpha| + |\beta|\,|a|$, $\|P_2 x\| = c\,|\alpha b + \beta| \le c\,(|\alpha|\,|b| + |\beta|$, $\|x - P_1 x\| = |\beta|\,(|a| + c)$, $\|x - P_2 x\| = |\alpha|\,(1 + c|b|)$, and Eq. (1.10) holds with $\gamma = \min\{(c - |a|)/(c + |a|), (1 - |b|c)/(1 + |b|c)\}$. ■

Example 5.3 In the situation of Example 5.1, consider

$$n_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; n_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \; r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; r_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

Then $a = b = -1$, hence the ℓ-paracontractivity condition is not fulfilled. The assertion of Theorem 1.3 for $Z = \{x \in \mathbb{R}^2\,;\, \langle x, n_j\rangle \le 0 \text{ for } j = 1,2\}$ nevertheless still holds provided we add extra vectors $n_3 = r_3 = R\, r_1$, where R is the matrix from the proof of

Proposition 5.2, which do not change the shape of Z, but fill the reflection cone at the vertex $x=0$, see the example on Figure 2.1 of [4]. This is due to the fact that the set

$$B \;=\; \{x \in \mathbb{R}^2\,;\, |x_1| \le 1+\sqrt{2}\,,\, |x_2| \le 1\}$$

is a bounded symmetric $\mathcal{P}$-invariant set with non-empty interior, and the oblique projection onto Z is explicitly given by the formula

$$\Pi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \;=\; \begin{cases} \begin{pmatrix} \max\{-x_2, \min\{x_2, x_1\}\} \\ x_2 \end{pmatrix} & \text{if } \; x_2 > 0\,, \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{if } \; x_2 \le 0\,, \end{cases}$$

hence the argument of Propositions 4.1 – 4.3 still works.

Example 5.4 As another illustration to Theorem 3.8, consider the 'dual' situation to Example 5.3 (more information about this type of duality can be found in [13]), where we interchange the roles of normal and reflection vectors, that is,

$$r_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; r_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ -1 \end{pmatrix}, \; n_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; n_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix},$$

and put $Z = \{(0,w)\,;\, w \in \mathbb{R}\}$. Then the ℓ-paracontractivity condition holds for the norm $\|(x,y)\| = c|x|+|y|$ with any $c>1$ and $\gamma = (c-1)/(c+1)$, but the transversality condition is violated. The projection Π is not uniquely determined, since $(0,x)-(0,z) \in \mathcal{R}(0,z)$ for every $z \ge x$.

References

[1] Aumann, G.: 1954, *Reelle Funktionen.* Berlin: Springer-Verlag, (German).

[2] Desch, W. and J. Turi: 1999, 'The stop operator related to a convex polyhedron'. *J. Differ. Equations* **157**, 329–347.

[3] Drábek, P., P. Krejčí, and P. Takáč: 1999, *Nonlinear differential equations*, Vol. 404 of *Research Notes in Mathematics.* London: Chapman & Hall/CRC.

[4] Dupuis, P. and H. Ishii: 1991, 'On Lipschitz continuity of the solution mapping to the Skorokhod problem with applications'. *Stoch. and Stoch. Rep.* **35**, 31–62.

[5] Dupuis, P. and K. Ramanan: 1999, 'Convex duality and the Skorokhod problem, I, II'. *Probab. Theory Relat. Fields* **115**, 153–195, 197–236.

[6] Elsner, L., I. Koltracht, and M. Neumann: 1990, 'On the convergence of asynchronous paracontractions with applications to tomographic reconstructions from incomplete data'. *Linear Algebra Appl.* **130**, 65–82.

[7] Fraňková, D.: 1991, 'Regulated functions'. *Math. Bohem.* **119**, 20–59.

[8] Krasnosel'skii, M. A. and A. V. Pokrovskii: 1988, *Systems with hysteresis*. Berlin: Springer-Verlag.

[9] Krejčí, P. and A. Vladimirov: 2001, 'Lipschitz continuity of polyhedral Skorokhod maps'. *J. Analysis Appl. (ZAA)* **20**, 817–844.

[10] Krejčí, P. and P. Laurençot: 2002, 'Generalized variational inequalities'. *J. Convex Anal.* **9**, 159–183.

[11] Monteiro Marques, M. D. P.: 1993, *Differential inclusions in nonsmooth mechanical problems—shocks and dry friction*. Basel: Birkhäuser.

[12] Moreau, J. J.: 1977, 'Evolution problem associated with a moving convex set in a Hilbert space'. *J. Differential Equations* **20**, 347–374.

[13] Vladimirov, A. A.: 2000, 'Duality principle for discrete linear inclusions'. Preprint no. 555, Weierstrass Institute for Applied Analysis and Stochastics, Berlin.

[14] Vladimirov, A. A.: 1990, 'Averaging properties of multidimensional hysteresis operators'. *Soviet Math. Dokl.* **40**(3), 623–626.

[15] Vladimirov, A. A., L. Elsner, and W.-J. Beyn: 2000, 'Stability and paracontractivity of discrete linear inclusions'. *Linear Algebra Appl.* **312**, 125–134.