Error estimates for the discrete inversion of hysteresis and creep operators ⋆

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Abstract

The accuracy of a numerical scheme for real-time inverse control of piezoelectric actuators taking into account both hysteresis and creep effects is analyzed with respect to the time step and the memory discretization parameter. It is shown that the error is of the first order for Lipschitz continuous inputs.

Key words: Prandtl-Ishlinskii operator, hysteresis, creep, discretization, error estimates
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Introduction

The aim of this paper is to investigate the convergence rate of the numerical scheme proposed in [5] and based on a model from [7,8] for real-time inverse control of piezoelectric actuators which exhibit non-negligible hysteresis and creep effects in the constitutive relation between the time-dependent voltage u and deformation x. We write this relation in the form

$$F[u](t) = x(t) \quad \text{for all} \quad t \in [0, T], \tag{0.1}$$

where $T > 0$ is a given final time, and F is an operator in the space $C[0, T]$ of real-valued continuous functions. The problem consists in inverting the above

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relation, that is, for a given function $x \in C[0,T]$, we look for a function $u \in C[0,T]$ such that (0.1) holds with a prescribed accuracy.

The construction of F is based on the so-called *Prandtl-Ishlinskii model* introduced originally in [9,3] as a model of one-dimensional elastoplasticity. It is defined as a (possibly infinite) composition of elementary elastoplastic one-yield cells with different yield limits and different elasticity moduli. A mathematical justification of the model by homogenization techniques was given in [2]. Mathematical properties of the model have been systematically investigated, see [1,10,4,6], including the Lipschitz continuity of the Prandtl-Ishlinskii operator and its inverse under appropriate assumptions.

The Prandtl-Ishlinskii model is *rate-independent* and cannot account for creep effects. We therefore define a 'creep component' of F by composing elementary viscoelastic cells according to the Prandtl-Ishlinskii scheme. The combined operator is investigated in [5], in particular the existence and Lipschitz continuity of the inverse operator $F^{-1} : C[0,T] \to C[0,T]$. The main result of the present paper (Theorem 2.1 below) consists in estimating the accuracy of the time-discrete and memory-discrete approximation of F^{-1}. We derive an error estimate of the order $\alpha + \boldsymbol{m}_x(h)$, where h is the time step, α is the memory discretization parameter, and $\boldsymbol{m}_x$ is the continuity modulus of the input x.

1 Hysteresis and creep operators

We first introduce the main building blocks of our construction called *play* and *creep* operators.

Definition 1.1 *Let Π denote the set of all functions $\pi \in W^{1,\infty}(0,\infty)$ with compact support such that $|\pi'(r)| \le 1$ a. e., and let $\pi^0 \in \Pi$ and $\xi^0 \in W^{1,\infty}(0,\infty)$ be given. Then*

(i) *the* play operator *$p_r[\pi^0, \cdot] : C[0,T] \to C[0,T]$ with threshold $r > 0$ is defined as the solution operator $p_r[\pi^0, u](t) := \pi_r(t)$ which with every given function $u \in C[0,T]$ associates the solution π_r of the variational inequality written in a Stieltjes integral form*

$$\begin{cases} u(t) - \pi_r(t) \in [-r, r] \qquad \forall t \in [0,T], \\ \int_0^T (u(t) - \pi_r(t) - w(t))\, d\pi_r(t) \;\ge\; 0 \quad \forall w \in C[0,T],\ ||w||_\infty \le r, \\ \pi_r(0) = \max\{u(0) - r, \min\{\pi^0(r), u(0) + r\}\}, \end{cases} \tag{1.1}$$

where $\|\cdot\|_\infty$ denotes the sup-norm;

(ii) *the* creep operator $\ell_\lambda[\xi^0, \cdot] : C[0,T] \to C^1[0,T]$ *with parameter* $\lambda > 0$ *is defined as the solution operator* $\ell_\lambda[\xi^0, u](t) := \xi_\lambda(t)$ *which with every given function* $u \in C[0,T]$ *associates the solution* ξ_λ *of the differential equation*

$$\frac{1}{\lambda}\,\dot{\xi}_\lambda(t) + \xi_\lambda(t) = u(t), \qquad \xi_\lambda(0) = \xi^0(\lambda)\,, \tag{1.2}$$

where the dot denotes derivative with respect to t.

The Stieltjes integral setting of π_r was introduced in [6]. We actually have the following explicit formulas for the operators ℓ_λ, p_r:

$$\ell_\lambda[\xi^0, u](t) = e^{-\lambda t}\,\xi^0(\lambda) + \lambda \int_0^t e^{\lambda(\tau - t)}\, u(\tau)\, d\tau\,, \tag{1.3}$$

and if the 'input' function u is monotone in an interval $[t_0, t_1]$, then for $t \in [t_0, t_1]$ we have (see also [4]) that

$$p_r[\pi^0, u](t) = \begin{cases} \max\{p_r[\pi^0, u](t_0),\ u(t) - r\} & \text{if } u \text{ increases}\,, \\ \\ \min\{p_r[\pi^0, u](t_0),\ u(t) + r\} & \text{if } u \text{ decreases}\,. \end{cases} \tag{1.4}$$

For each $u \in C[0,T]$, we define the value $F[u]$ of the *hysteresis and creep operator* $F : C[0,T] \to C[0,T]$ by the Stieltjes integral formula

$$F[u](t) := a\,u(t) + \int_0^\infty p_r[\pi^0, u](t)\ df(r) + \int_0^\infty \ell_\lambda[\xi^0, u](t)\ dg(\lambda) \tag{1.5}$$

under the following hypothesis.

Hypothesis 1.2

(i) $a > 0$ *is a given constant,* $f, g : [0,\infty[\to [0,\infty[$ *are bounded nondecreasing right-continuous functions,* $f(0) = g(0) = 0$, $f^* := f(\infty) < \infty$, $g^* := g(\infty) < \infty$, $\Gamma := \int_0^\infty (g^* - g(\lambda))\, d\lambda < \infty$.

(ii) $\xi^0 \in W^{1,\infty}(0,\infty)$ *and* $\pi^0 \in \Pi$ *are given functions,* $\pi^0(r) = 0$ *for* $r \geq R$.

The spaces Π and $W^{1,\infty}(0,\infty)$ are the *state spaces* for the operator F. This means that for every $u \in C[0,T]$ and every fixed time t, the function $r \mapsto p_r[\pi^0, u](t)$ belongs to Π, the function $\lambda \mapsto \ell_\lambda[\xi^0, u](t)$ belongs to $W^{1,\infty}(0,\infty)$ and these two functions fully determine the *state of the system* at time t. More precisely, we have the following result.

Lemma 1.3 *Let $u \in C[0,T]$ and $t \in [0,T]$ be given and let Hypothesis 1.2 (ii) hold. For $r > 0$ and $\lambda > 0$ put*

$$\pi(r) := p_r[\pi^0, u](t), \quad \xi(\lambda) := \ell_\lambda[\xi^0, u](t). \tag{1.6}$$

We then have

(i) $|\pi(r+\rho) - \pi(r)| \le \rho$ *for every* $r > 0\,,\ \rho > 0\,,$
(ii) $\pi(r) = 0$ *for* $r \ge \max\{R, \|u\|_\infty\}\,,$
(iii) $\|\xi\|_\infty \le \max\{\|\xi^0\|_\infty, \|u\|_\infty\}$,
(iv) $|\xi(\lambda+\mu) - \xi(\lambda)| \le \mu \max\{1, 2T\}\big(\|\xi^0\|_{1,\infty} + \|u\|_\infty\big) \quad \forall \lambda > 0\,,\ \mu > 0\,,$

where $\|\cdot\|_{1,\infty}$ denotes the norm in $W^{1,\infty}(0,\infty)$.

Proof. Statements (i), (ii) follow from Corollary II.2.6 of [6], and (iii) follows immediately from Eq. (1.3). To prove (iv), we first notice that

$$\begin{aligned}\xi(\lambda+\mu) - \xi(\lambda) = {} & e^{-(\lambda+\mu)t}\,\xi^0(\lambda+\mu) - e^{-\lambda t}\,\xi^0(\lambda) \\ & + \int_0^t \Big((\lambda+\mu)e^{-(\lambda+\mu)\tau} - \lambda e^{-\lambda\tau}\Big)\, u(t-\tau)\, d\tau\,,\end{aligned} \tag{1.7}$$

where

$$\begin{aligned}\Big|e^{-(\lambda+\mu)t}\,\xi^0(\lambda+\mu) - e^{-\lambda t}\,\xi^0(\lambda)\Big| \le {} & e^{-\lambda t}\Big((1 - e^{-\mu t})|\xi^0(\lambda+\mu)| \\ & + |\xi^0(\lambda+\mu) - \xi^0(\lambda)|\Big) \le \mu\Big(T\|\xi^0\|_\infty + \|d\xi^0/d\lambda\|_\infty\Big)\,.\end{aligned} \tag{1.8}$$

To estimate the integral on the right-hand side of (1.7), we introduce the number $\tau^* := \frac{1}{\mu}\log(1 + \frac{\mu}{\lambda})$. Then

$$t \le \tau^* \Rightarrow \int_0^t \Big|(\lambda+\mu)e^{-(\lambda+\mu)\tau} - \lambda e^{-\lambda\tau}\Big| d\tau = e^{-\lambda t} - e^{-(\lambda+\mu)t} \le \mu T \tag{1.9}$$

$$\begin{aligned} t > \tau^* \Rightarrow {} & \int_0^t \Big|(\lambda+\mu)e^{-(\lambda+\mu)\tau} - \lambda e^{-\lambda\tau}\Big| d\tau \\ & = 2e^{-\lambda\tau^*}(1 - e^{-\mu\tau^*}) - e^{-\lambda t}(1 - e^{-\mu t}) \le 2\mu\tau^* < 2\mu T.\end{aligned} \tag{1.10}$$

Combining (1.7) – (1.10) we obtain the assertion. ■

The following invertibility result was proved in [5].

Theorem 1.4 *Let F be given by (1.5), and let $L := (2/a) - (1/(a + f^*))$. Then for every $x \in C[0,T]$ there exists a unique $u \in C[0,T]$ such that*

Eq. (0.1) holds. Moreover, if $x, y \in C[0,T]$ are given and $F[u] = x$, $F[v] = y$, then

$$\|u - v\|_\infty \le L\, e^{\Gamma L T}\, \|x - y\|_\infty . \tag{1.11}$$

2 Discrete inversion and statement of main results

We first discretize the operator F with respect to the 'memory' variables r, λ. Given a discretization parameter $\alpha > 0$, we replace the functions f and g in Eq. (1.5) by step functions f_α, g_α of the form

$$f_\alpha(r) := \begin{cases} f(r_i) & \text{for } \; r \in [r_{i-1}, r_i[\,,\; i = 1, \dots, n, \\ f^* & \text{for } \; r \ge r_n \end{cases} \tag{2.1}$$

$$g_\alpha(\lambda) := \begin{cases} g(\lambda_i) & \text{for } \; \lambda \in [\lambda_{j-1}, \lambda_j[\,,\; j = 1, \dots, m, \\ g^* & \text{for } \; \lambda \ge \lambda_m, \end{cases} \tag{2.2}$$

where $0 = r_0 < r_1 < \dots < r_n$, $0 = \lambda_0 < \lambda_1 < \dots < \lambda_m$ are given sequences such that

$$f^* - f(r_n) \le \alpha\,, \tag{2.3}$$

$$\int_{\lambda_m}^{\infty} (g^* - g(\lambda))\, d\lambda \le \alpha\,, \tag{2.4}$$

$$\sum_{i=1}^{n} (f(r_i) - f(r_{i-1}))(r_i - r_{i-1}) \le \alpha\,, \tag{2.5}$$

$$\sum_{j=1}^{m} (g(\lambda_j) - g(\lambda_{j-1}))(\lambda_j - \lambda_{j-1}) \le \alpha\,. \tag{2.6}$$

For $i = 1, \dots, n$, $j = 1, \dots, m$ put

$$b_i := f(r_{i+1}) - f(r_i)\,, \quad c_j := g(\lambda_{j+1}) - g(\lambda_j)\,, \tag{2.7}$$

where we denote $f(r_{n+1}) := f^*$, $g(\lambda_{m+1}) := g^*$. For every continuous function $v : [0,\infty[\to \mathbb{R}$ we then have

$$\int_0^\infty v(r)\, df_\alpha(r) = \sum_{i=1}^{n} b_i\, v(r_i)\,, \quad \int_0^\infty v(\lambda)\, dg_\alpha(\lambda) = \sum_{j=1}^{m} c_j\, v(\lambda_j)\,, \tag{2.8}$$

and from Hypothesis 1.2 (i) it follows that

$$\sum_{j=1}^{m} c_j\lambda_j \;=\; \int_0^\infty (g^* - g_\alpha(\lambda))d\lambda \;\le\; \int_0^\infty (g^* - g(\lambda))\,d\lambda \;=\; \Gamma. \tag{2.9}$$

Let now $x \in C[0,T]$ be a given function and let $h > 0$ be a given time step. We define the sequences indexed by $k = 0, 1, \ldots, N := [T/h]$

$$t_k := kh, \quad x_k := x(t_k). \tag{2.10}$$

The method proposed in [5] consists in replacing Eq. (0.1) by the discrete system

$$a\,u_k + \sum_{i=1}^{n} b_i\,\pi_k^i + \sum_{j=1}^{m} c_j\,\xi_k^j = x_k, \quad k = 0, \ldots, N \tag{2.11}$$

with unknowns $u_0, \ldots, u_N$, where b_i, c_j are given by (2.7), and

$$\pi_0^i \;=\; \min\Big\{u_0 + r_i\,,\, \max\{\pi^0(r_i), u_0 - r_i\}\Big\}, \tag{2.12}$$

$$\xi_0^j \;=\; \xi^0(\lambda_j), \tag{2.13}$$

$$\pi_k^i \;=\; \min\Big\{u_k + r_i\,,\, \max\{\pi_{k-1}^i, u_k - r_i\}\Big\}, \tag{2.14}$$

$$\xi_k^j \;=\; e^{-\lambda_j h}\,\xi_{k-1}^j + \Big(1 - e^{-\lambda_j h}\Big)u_{k-1} \tag{2.15}$$

for $i = 1, \ldots, n$, $j = 1, \ldots, m$ and $k = 1, \ldots, N$. We solve Eq. (2.11) consecutively passing from $k-1$ to k. At each step k, Eq. (2.11) has the form $P_k(u_k) = y_k$, where $P_k(u_k) := au_k + \sum_{i=1}^{n} b_i\pi_k^i$ is an increasing piecewise affine function of u_k, $P_k'(v) \ge a$ for a. e. $v \in \mathbb{R}$, and $y_k := x_k - \sum_{j=1}^{m} \lambda_j\,\xi_k^j$ is known. The sequence $\{u_k\}_{k=0}^N$ is therefore uniquely determined by Eq. (2.11).

We measure the accuracy of the method by estimating the sup-norm of the difference between the exact solution $u = F^{-1}[x]$ of Eq. (0.1) and the linearly interpolated sequence $\{u_k\}$. The main result of this paper can be stated as follows.

Theorem 2.1 *Let Hypothesis 1.2 be satisfied, let $x \in C[0,T]$, and $\alpha, h > 0$ be given and let $\{u_k\}_{k=0}^N$ be the solution of Eq. (2.11). For $t \in [0,T]$ put*

$$u(t) \;:=\; F^{-1}[x](t)\,, \tag{2.16}$$

$$\hat{u}(t) \;:=\; \begin{cases} u_{k-1} + \frac{t - t_{k-1}}{h}\,(u_k - u_{k-1})\,, & t \in [t_{k-1}, t_k[\;, \quad k = 1, \ldots, N\,, \\ u_N\;, & t \in [t_N, T]\,. \end{cases} \tag{2.17}$$

Then there exists a constant $C > 0$ *depending only on* R, $\|\xi^0\|_{1,\infty}$, a, f^*, g^*, Γ, *and* T *(in particular, independent of* x, α, *and* h*) such that*

$$\|u - \hat{u}\|_\infty \le C\Big((\alpha + h)(1 + \|x\|_\infty) + \boldsymbol{m}_x(h)\Big), \tag{2.18}$$

where, for a function $w \in C[0,T]$, *the* continuity modulus $\boldsymbol{m}_w : [0,\infty[\to [0,\infty[$ *of* w *is defined by the formula*

$$\boldsymbol{m}_w(h) := \max\Big\{|w(t) - w(s)| : \ 0 \le s < t \le T, \ t - s \le h\Big\}. \tag{2.19}$$

We have indeed $\liminf_{h\to 0+} \boldsymbol{m}_w(h)/h > 0$ for every nonconstant function $w \in C[0,T]$. The optimal error estimate obtained from Theorem 2.1 for a Lipschitz continuous function x is therefore of the order $\alpha + h$.

For practical purposes, it would be more convenient to estimate the error of the *piecewise constant* approximation $\bar{u}$ of u defined as

$$\bar{u}(t) := \begin{cases} u_{k-1}\,, \ t \in [t_{k-1}, t_k[\,, & k = 1, \dots, N\,, \\ u_N\,, \quad t \in [t_N, T]\,. \end{cases} \tag{2.20}$$

We have $|\bar{u}(t) - \hat{u}(t)| \le \boldsymbol{m}_{\hat{u}}(h)$ for every $t \in [0,T]$, hence Lemma 4.2 below and (2.18) yield for $\|u - \bar{u}\|_\infty$ and $\|u - \hat{u}\|_\infty$ estimates of the same order.

The proof of Theorem 2.1 is divided into several steps. In Sect. 3 we estimate the error due to the discretization of the operator F, and in Sect. 4 we investigate the time discretization error and finish the proof of Theorem 2.1.

In what follows, we denote by $C_1, C_2, \dots$ any constants depending only on the data $R, \|\xi^0\|_{1,\infty}$, a, f^*, g^*, Γ, and T as in Theorem 2.1.

3 Approximation of the operator F

In this section we derive a uniform bound for the difference $\|F[u] - F_\alpha[u]\|_\infty$ for an arbitrary function $u \in C[0,T]$, where F_α is the operator

$$\begin{aligned} F_\alpha[u](t) &= a\,u(t) + \int_0^\infty p_r[\pi^0, u](t)\, df_\alpha(r) + \int_0^\infty \ell_\lambda[\xi^0, u])t)\, dg_\alpha(\lambda) \\ &= a\,u(t) + \sum_{i=1}^n b_i\, p_{r_i}[\pi^0, u](t) + \sum_{j=1}^m c_j\, \ell_{\lambda_j}[\xi^0, u](t)\,, \end{aligned} \tag{3.1}$$

according to the notation introduced in Sect. 2. We first recall an estimate for the Stieltjes integral.

Lemma 3.1 *Let γ be a right-continuous bounded nondecreasing function, $\gamma^* := \gamma(+\infty) < \infty$. Let $M > 0$ be a given constant, and let $v : [0,\infty[\to \mathbb{R}$ be a function such that*

$$|v(s+\sigma) - v(s)| \le M\sigma \qquad \forall s \ge 0\,,\ \forall \sigma \ge 0. \tag{3.2}$$

Let $\alpha > 0$ be given and let $0 = s_0 < s_1 < \cdots < s_q$ be a sequence such that

$$S_\gamma := \sum_{i=1}^{q} (\gamma(s_i) - \gamma(s_{i-1}))(s_i - s_{i-1}) \ \le\ \alpha\,. \tag{3.3}$$

Assume that

(i) *either $\gamma^* - \gamma(s_q) \le \alpha$ and $v(s) = 0$ for $s \ge R$,*

(ii) *or $\int_{s_q}^{\infty} (\gamma^* - \gamma(s))\, ds \le \alpha$.*

Let us define the step function

$$\gamma_\alpha(s) := \begin{cases} \gamma(s_i)\,, & s \in [s_{i-1}, s_i[\ ,\ \ i = 1,\dots,q, \\ \gamma^*\,, & s \ge s_q. \end{cases} \tag{3.4}$$

Then we have

$$\left| \int_0^\infty v(s)\, d(\gamma_\alpha - \gamma)(s) \right| \ \le\ \begin{cases} \alpha M(1+R) & \text{in case (i)}\,, \\ 2\alpha M & \text{in case (ii)}\,. \end{cases} \tag{3.5}$$

Proof. For every $z \ge s_q$ we have

$$\int_0^z v(s)\, d(\gamma_\alpha - \gamma)(s) = v(z)\,(\gamma^* - \gamma(z)) - \int_0^z v'(s)\,(\gamma_\alpha(s) - \gamma(s))\, ds. \tag{3.6}$$

In case (i) this yields for $z \ge z_q := \max\{R, s_q\}$ that

$$\begin{aligned} \left| \int_0^z v(s)\, d(\gamma_\alpha - \gamma)(s) \right| &\le M \int_0^{z_q} (\gamma_\alpha(s) - \gamma(s))\, ds \\ &\le M\left((z_q - s_q)(\gamma^* - \gamma(s_q)) + S_\gamma\right) \ \le\ \alpha M(1+R). \end{aligned} \tag{3.7}$$

In case (ii) we have $|v(z)| \leq |v(0)| + Mz$, and using for $z \to \infty$ the inequality

$$\int_{z/2}^{z} (\gamma^* - \gamma(s))\, ds \;\geq\; \frac{z}{2}(\gamma^* - \gamma(z)) \tag{3.8}$$

we obtain $\lim_{z\to\infty} v(z)\,(\gamma^* - \gamma(z)) = 0$. From Eq. (3.6) it follows that

$$\begin{aligned} \left| \int_0^\infty v(s)\, d(\gamma_\alpha - \gamma)(s) \right| \;&\leq\; M \int_0^\infty (\gamma_\alpha(s) - \gamma(s))\, ds \\ &\leq\; M \left(\int_{s_q}^\infty (\gamma^* - \gamma(s))\, ds + S_\gamma \right) \;\leq\; 2\alpha M\,, \end{aligned} \tag{3.9}$$

and Lemma 3.1 is proved. ■

Combining Lemmas 3.1 and 1.3 we immediately obtain the following result which concludes this section.

Proposition 3.2 *Let Hypothesis 1.2 be satisfied, and let $u \in C[0,T]$ be given. Then for every $\alpha > 0$ we have*

$$\|F[u] - F_\alpha[u]\|_\infty \;\leq\; \alpha\,\Phi\,, \tag{3.10}$$

where F_α is the operator defined by Eq. (3.1), and $\Phi := 1 + \max\{R, \|u\|_\infty\} + 2\max\{1, 2T\}\left(\|\xi^0\|_{1,\infty} + \|u\|_\infty\right)$.

4 Discretization error

The final goal of this section and of the whole paper is to prove Theorem 2.1. We start with an easy consequence of Eq. (2.14).

Lemma 4.1 *With the notation of (2.11) – (2.15), put*

$$w_k := a\,u_k + \sum_{i=1}^{n} b_i\,\pi_k^i \tag{4.1}$$

for $k = 0, \ldots, N$. Then the implications

$$u_k = u_{k-1} \;\;\Rightarrow\;\; w_k = w_{k-1}, \tag{4.2}$$

$$u_k \neq u_{k-1} \quad \Rightarrow \quad a \,\leq\, \frac{w_k - w_{k-1}}{u_k - u_{k-1}} \,\leq\, a + f^* \tag{4.3}$$

hold for all $k = 1, \dots, N$ *.*

In the next step, we prove that Eq. (2.11) preserves the continuity modulus.

Lemma 4.2 *Let the hypotheses of Theorem 2.1 be fulfilled. Then there exists a constant* C_1 *such that*

$$\boldsymbol{m}_{\hat{u}}(h) \;\leq\; C_1 \left(h \left(1 + |x(0)|\right) + \boldsymbol{m}_x(h)\right). \tag{4.4}$$

Proof. With the notation of Lemma 4.1, we can rewrite Eq. (2.11) in the form

$$w_k + \sum_{j=1}^{m} c_j \, \xi_k^j = x_k. \tag{4.5}$$

Putting

$$\delta_j := 1 - e^{-\lambda_j h} \qquad \text{for} \quad j = 1, \dots, m, \tag{4.6}$$

we have by (2.15) for $k = 1, \dots, N$ that

$$\xi_k^j \;=\; (1 - \delta_j)\, \xi_{k-1}^j \,+\, \delta_j \, u_{k-1}\,, \tag{4.7}$$

hence

$$u_k - \xi_k^j \;=\; u_k - u_{k-1} + (1 - \delta_j)\,(u_{k-1} - \xi_{k-1}^j). \tag{4.8}$$

From (4.5), (4.7) it follows that

$$w_k - w_{k-1} \,+\, \sum_{j=1}^{m} c_j \, \delta_j (u_{k-1} - \xi_{k-1}^j) \;=\; x_k - x_{k-1}. \tag{4.9}$$

For all admissible values of indices k, j we formally denote

$$U_k \;:=\; u_k - u_{k-1}\,, \tag{4.10}$$

$$X_k \;:=\; x_k - x_{k-1}\,, \tag{4.11}$$

$$V_k^j \;:=\; c_j \delta_j \left(u_k - \xi_k^j\right), \tag{4.12}$$

$$W_k \;:=\; \sum_{j=1}^{m} V_k^j. \tag{4.13}$$

By Lemma 4.1, for every $k = 1, \dots, N$ there exists $\sigma_k \in [a, a+f^*]$ such that

$$w_k - w_{k-1} = \sigma_k \left(u_k - u_{k-1} \right). \tag{4.14}$$

Eqs. (4.9), (4.8) then have the form

$$\sigma_k U_k + W_{k-1} = X_k \,, \tag{4.15}$$
$$V_k^j - (1-\delta_j) V_{k-1}^j = c_j \delta_j \, U_k \,, \tag{4.16}$$

hence the identity

$$V_k^j - (1-\delta_j) V_{k-1}^j + c_j \delta_j \, \frac{W_{k-1}}{\sigma_k} = c_j \delta_j \, \frac{X_k}{\sigma_k} \tag{4.17}$$

holds for all $j = 1, \dots, m$, $k = 1, \dots, N$. By induction we obtain from Eq. (4.17) that

$$V_k^j = (1-\delta_j)^k \, V_0^j + c_j \delta_j \sum_{p=1}^{k} \frac{X_p - W_{p-1}}{\sigma_p} (1-\delta_j)^{k-p} \,. \tag{4.18}$$

To estimate the right-hand side of (4.18), we first notice that

$$0 < \delta_j < \min\{1, \lambda_j h\} \quad \text{for } j = 1, \dots, m \,, \tag{4.19}$$

and from (2.9) it follows that

$$\sum_{j=1}^{m} c_j \, \delta_j \;\le\; \Gamma \, h \,. \tag{4.20}$$

This yields

$$\left| \sum_{j=1}^{m} c_j \delta_j \sum_{p=1}^{k} \frac{W_{p-1}}{\sigma_p} (1-\delta_j)^{k-p} \right| \;\le\; \frac{\Gamma h}{a} \sum_{p=0}^{k-1} |W_p| \tag{4.21}$$

and similarly

$$\begin{aligned} \left| \sum_{j=1}^{m} c_j \delta_j \sum_{p=1}^{k} \frac{X_p}{\sigma_p} (1-\delta_j)^{k-p} \right| &\le \frac{\boldsymbol{m}_x(h)}{a} \sum_{j=1}^{m} c_j \delta_j \sum_{p=0}^{k-1} (1-\delta_j)^p \\ &\le \frac{\boldsymbol{m}_x(h)}{a} \sum_{j=1}^{m} c_j \;\le\; \frac{g^*}{a} \, \boldsymbol{m}_x(h). \end{aligned} \tag{4.22}$$

To estimate the terms V_0^j, we use (2.11) for $k = 0$. For $i = 1, \dots, n$ put

$$\hat{\pi}_0^i \;:=\; \min\Big\{ r_i\,,\, \max\left\{\pi^0(r_i)\,,\, -r_i\right\}\Big\}. \tag{4.23}$$

Similarly as in Lemma 4.1 we have

$$u_0 \neq 0 \;\Rightarrow\; a \;\leq\; \frac{a u_0 \,+\, \sum_{i=1}^n b_i\,\pi_0^i \,-\, \sum_{i=1}^n b_i\,\hat{\pi}_0^i}{u_0} \;\leq\; a + f^*\,. \tag{4.24}$$

From (4.24) and (2.11) it follows that

$$|u_0| \leq \frac{1}{a}\left| x_0 - \sum_{j=1}^m c_j\,\xi_0^j - \sum_{i=1}^n b_i\,\hat{\pi}_0^i \right| \leq \frac{1}{a}\Big(|x(0)| + g^*\|\xi^0\|_\infty + f^* R\Big)\,, \tag{4.25}$$

and (4.20), (4.25) entail

$$\left|\sum_{j=1}^m (1-\delta_j)^k V_0^j\right| \leq \sum_{j=1}^m |V_0^j| \leq \frac{\Gamma h}{a}\Big(|x(0)| + (a+g^*)\|\xi^0\|_\infty + f^* R\Big)\,. \tag{4.26}$$

Summing Eq. (4.18) up over j and using (4.21), (4.22), and (4.26) we obtain for every $k = 0, \dots, N$ that

$$|W_k| \;\leq\; \frac{\Gamma h}{a}\sum_{p=0}^{k-1}|W_p| \;+\; C_2\Big(h\,(1+|x(0)|) \;+\; \boldsymbol{m}_x(h)\Big)\,, \tag{4.27}$$

with a constant C_2 independent of h, k, and x and the induction yields

$$|W_k| \;\leq\; C_2\Big(h(1+|x(0)|) + \boldsymbol{m}_x(h)\Big)\left(1 + \frac{\Gamma}{a}h\right)^k \quad \forall k = 0, \dots, N\,. \tag{4.28}$$

Recall that we have $N \leq T/h$, hence

$$\left(1 + \frac{\Gamma}{a}h\right)^k \;\leq\; e^{\Gamma T/a}\,, \tag{4.29}$$

and using Eq. (4.15) together with inequalities (4.28), (4.29) we find a constant $C_1 > 0$ such that

$$|u_k - u_{k-1}| \;\leq\; C_1\Big(h(1+|x(0)|) + \boldsymbol{m}_x(h)\Big) \quad \forall k = 1, \dots, N\,. \tag{4.30}$$

In particular, Ineq. (4.4) holds and Lemma 4.2 is proved. ■

Lemma 4.3 *Let $\hat{u}$ be the function defined in Theorem 2.1 and let F_α be the operator given by Eq. (3.1). Put $\hat{x} := F_\alpha[\hat{u}]$. Then there exists a constant $C_3 > 0$ such that*

$$\|\hat{x} - x\|_\infty \;\leq\; C_3\Big(h(1 + |x(0)|) + \boldsymbol{m}_x(h)\Big). \tag{4.31}$$

Proof. For every $k = 0, \ldots, N$ we have by (1.4) that

$$a\hat{u}(t_k) \;+\; \sum_{i=1}^{n} b_i\, p_{r_i}[\pi^0, \hat{u}](t_k) \;=\; au_k \;+\; \sum_{i=1}^{n} b_i\, \pi^i_k\,. \tag{4.32}$$

Put

$$\hat{\xi}^i_k \;:=\; \ell_{\lambda_j}[\xi^0, \hat{u}](t_k) \quad \text{ for } \; k = 0, \ldots, N\,,\; j = 1, \ldots, m\,. \tag{4.33}$$

Then $\hat{\xi}^j_0 = \xi^j_0 = \xi^0(\lambda_j)$ for all j, and for $t \in [t_{k-1}, t_k]$ we have

$$\ell_{\lambda_j}[\xi^0, \hat{u}](t) \;=\; e^{-\lambda_j(t - t_{k-1})}\, \hat{\xi}^j_{k-1} \;+\; \lambda_j \int_{t_{k-1}}^{t} e^{\lambda_j(\tau - t)}\, \hat{u}(\tau)\, d\tau\,, \tag{4.34}$$

hence

$$\hat{\xi}^j_k = e^{-\lambda_j h}\, \hat{\xi}^j_{k-1} + \left(1 - e^{-\lambda_j h}\right) u_{k-1} + \left(1 - \frac{1 - e^{-\lambda_j h}}{\lambda_j h}\right)(u_k - u_{k-1}) \tag{4.35}$$

for each $j = 1, \ldots, m\,,\; k = 1, \ldots, N$. From (4.7) we obtain that

$$|\hat{\xi}^j_k - \xi^j_k| \;\leq\; e^{-\lambda_j h}\left|\hat{\xi}^j_{k-1} - \xi^j_{k-1}\right| \;+\; \boldsymbol{m}_{\hat{u}}(h)\left(1 - \frac{1 - e^{-\lambda_j h}}{\lambda_j h}\right)\,, \tag{4.36}$$

hence

$$|\hat{\xi}^j_k - \xi^j_k| \;\leq\; \boldsymbol{m}_{\hat{u}}(h)\left(\frac{1}{1 - e^{-\lambda_j h}} - \frac{1}{\lambda_j h}\right) \;\leq\; \boldsymbol{m}_{\hat{u}}(h) \tag{4.37}$$

for every $k = 1, \ldots, N\,,\; j = 1, \ldots, m$ as a consequence of the elementary inequality $1/(1 - e^{-z}) < 1 + 1/z$ for every $z > 0$. From (4.37), (4.32), (2.7), (2.11) we obtain for all $k = 0, \ldots, N$ that

$$|\hat{x}(t_k) - x(t_k)| \;=\; \left|\sum_{j=1}^{m} c_j\left(\hat{\xi}^j_k - \xi^j_k\right)\right| \;\leq\; g^*\, \boldsymbol{m}_{\hat{u}}(h)\,. \tag{4.38}$$

Let now $t \in [t_k, t_{k-1}]$ be arbitrary for $k = 1, \dots, N+1$, where we put $t_{N+1} := T$. By (1.4) we have

$$\left| a\hat{u}(t) + \sum_{i=1}^{n} b_i \, p_{r_i}[\pi^0, \hat{u}](t) - a\hat{u}(t_{k-1}) - \sum_{i=1}^{n} b_i \, p_{r_i}[\pi^0, \hat{u}](t_{k-1}) \right| \tag{4.39}$$
$$\leq \; (a + f^*) \left| \hat{u}(t) - \hat{u}(t_{k-1}) \right| \; \leq \; (a + f^*) \, \boldsymbol{m}_{\hat{u}}(h)$$

and from (4.34) and Lemma 1.3 it follows for every j that

$$\left| \ell_{\lambda_j}[\xi^0, \hat{u}](t) - \ell_{\lambda_j}[\xi^0, \hat{u}](t_{k-1}) \right| \leq \left(1 - e^{-\lambda_j (t - t_{k-1})}\right) \left(\left| \hat{\xi}^j_{k-1} \right| + \max_k |u_k| \right) \tag{4.40}$$
$$\leq \lambda_j h \left(\|\xi^0\|_\infty \; + \; 2 \max_k |u_k| \right)$$

By (4.25) we have

$$\max_k |u_k| \; \leq \; |u_0| + \sum_{k=1}^{N} |u_k - u_{k-1}| \; \leq \; C_4 (1 + |x(0)|) + T \, \frac{\boldsymbol{m}_{\hat{u}}(h)}{h} \, , \tag{4.41}$$

hence

$$\left| \ell_{\lambda_j}[\xi^0, \hat{u}](t) - \ell_{\lambda_j}[\xi^0, \hat{u}](t_{k-1}) \right| \; \leq \; C_5 \, \lambda_j \; (h(1 + |x(0)|) + \boldsymbol{m}_{\hat{u}}(h)) \, . \tag{4.42}$$

Combining (4.39) with (4.42) yields for $t \in [t_{k-1}, t_k]$, $k = 1, \dots, N+1$ that

$$|\hat{x}(t) - \hat{x}(t_{k-1})| \; \leq \; C_6 \Big(h(1 + |x(0)|) + \boldsymbol{m}_{\hat{u}}(h) \Big) \, . \tag{4.43}$$

We therefore have

$$|\hat{x}(t) - x(t)| \; \leq \; |\hat{x}(t) - \hat{x}(t_{k-1})| + |\hat{x}(t_{k-1}) - x(t_{k-1})| + |x(t) - x(t_{k-1})| \tag{4.44}$$
$$\leq \; C_7 \Big(h(1 + |x(0)|) + \boldsymbol{m}_{\hat{u}}(h) \Big) + \boldsymbol{m}_x(h)$$

and to complete the proof of Lemma 4.3, it suffices to use Lemma 4.2. ∎

The above estimates enable us to prove Theorem 2.1.

Proof of Theorem 2.1. For every $\alpha > 0$, the operators F_α satisfy Hypothesis 1.2 with the same value of a, f^*, g^* and Γ. By Theorem 1.4 we therefore have

$$\|u - \hat{u}\|_\infty \; \leq \; L \, e^{L \Gamma T} \|F_\alpha[u] - F_\alpha[\hat{u}]\|_\infty \tag{4.45}$$
$$\leq \; L \, e^{L \Gamma T} \left(\|F_\alpha[u] - F[u]\|_\infty \; + \; \|x - \hat{x}\|_\infty \right) .$$

By Proposition 3.2 we have

$$\|F_\alpha[u] - F[u]\|_\infty \;\le\; \alpha\, C_8\, (1 + \|u\|_\infty)\,, \tag{4.46}$$

and Theorem 1.4 yields $\|u\|_\infty \le L\, e^{L\Gamma T}\|x - F[0]\|_\infty$, where

$$|F[0](t)| \;=\; \left| C_9 \,+\, \int_0^\infty e^{-\lambda t}\, \xi^0(\lambda)\, dg(\lambda) \right| \;\le\; C_{10}, \tag{4.47}$$

hence $\|u\|_\infty \le C_{11}(1 + \|x\|_\infty)$, and the assertion follows from (4.45), (4.46), and Lemma 4.3. ■

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