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Asymptotic Stability of Periodic Solutions to the Wave Equation with Hysteresis

Abstract

The wave equation $F_\lambda(u_t)_t - u_{xx} = g(x,t)$ with an Ishlinskii operator F_λ and a given ω-periodic right-hand side g is considered here with suitable boundary conditions. Sufficient conditions are given for the existence, uniqueness and global asymptotic stability of a periodic solution. The proof is based on the strict monotonicity of a "strictly convex" Ishlinskii operator.

Introduction

Global solvability and asymptotics for the wave equation $F_\lambda(u_t)_t - u_{xx} = g(x,t)$ with an Ishlinskii operator F_λ have been established in [7] for a given bounded right-hand side g and simple boundary and initial conditions. In fact, the paper [7] deals with a more general case, where a Preisach operator W is considered instead of F_λ. We suppose here moreover that the function g is ω-periodic with respect to t.

The existence and uniqueness of periodic solutions of the wave equation with hysteresis has been already proved earlier (cf. e.g. [4]) by the Fourier method. We use here the Ficken-Fleishman method (cf. [9] for further references) which enables us to construct the periodic solution and to prove at the same time its global asymptotic stability.

It turns out that the Ficken-Fleishman method requires to consider more general initial configurations λ of the Ishlinskii operator F_λ. For this reason the description of the structure of memory given in [6] is no longer applicable and we have to derive new formulas here. The basic properties of an Ishlinskii operator with an arbitrary admissible initial configuration are summarized in §§1-3. In §§4,5 we study the monotonicity of F_λ. The main auxiliary result is Proposition (4.3) which characterizes the strict monotonicity of the Ishlinskii operator. In §6 we mention parameter-dependent Ishlinskii operators and §7 contains the main results and proofs.

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1. Elementary hysteresis operators

Let $u \in W^{1,1}(0,T)$ be a given function and $h > 0$, $x_h^o \in [-h,h]$ given numbers. The problem of finding a function $x_h \in W^{1,1}(0,T)$ such that

$$
(1.1) \qquad \begin{aligned}
&(i) && x_h(t) \in [-h,h] \qquad \forall t \in [0,T],\\
&(ii) && (x_h'(t) - u'(t))(x_h(t) - \phi) \le 0 \quad \text{a.e.} \quad \forall \phi \in [-h,h],\\
&(iii) && x_h(0) = x_h^o
\end{aligned}
$$

has a unique solution (cf. e.g. [5]). We mention here for the sake of completeness the following simple regularity result. We denote by u'_+ (u'_-) the right derivative (left derivative, respectively) of u and similarly for x_h.

(1.2) Lemma. *Let u, h, x_h, x_h^o be as above. If $u'_+(t)$ exists for some $t \in [0,T]$, then $x'_{h+}(t)$ exists and $x'_{h+}(t)(x'_{h+}(t) - u'_+(t)) = 0$. Analogously, if $u'_-(t)$ exists for some $t \in [0,T]$, then $x'_{h-}(t)$ exists and $x'_{h-}(t)(x'_{h-}(t) - u'_-(t)) = 0$.*

The proof of this lemma is elementary and we omit it here. It follows easily from the implication

$$[\forall \sigma \in (a,b); x_h(\sigma) \in (-h,h)] \Rightarrow [\forall t,s \in [a,b]; x_h(t) - x_h(s) = u(t) - u(s)].$$

The existence and uniqueness result for (1.1) enables us to define an operator $f_h(\cdot, x_h^o) : W^{1,1}(0,T) \to W^{1,1}(0,T)$ for every $h > 0$ and $|x_h^o| \le h$ by the formula

$$
(1.3) \qquad f_h(u, x_h^o)(t) := x_h(t), \qquad t \in [0,T],
$$

where $x_h(t)$ is the solution of (1.1). This operator is usually called *stop.* Its properties have been extensively studied (cf. e.g. the monograph [2]). The operator $f_h(\cdot, x_h^o)$ is Lipschitz continuous in $W^{1,1}(0,T)$ and continuous in $W^{1,p}(0,T)$ for $1 < p < \infty$ (cf. [1], [5]). We need here another typical property of the stop (cf. [8]).

(1.4) Lemma (Semigroup property). *For every $u \in W^{1,1}(0,T)$, $t_1, t_2 \ge 0$, $t_1 + t_2 \le T$, $x_h^o \in [-h,h]$ we have*

$$f_h(u(\cdot + t_1), f_h(u, x_h^o)(t_1))(t_2) = f_h(u, x_h^o)(t_1 + t_2).$$

Proof. Put $x(t) := f_h(u, x_h^o)(t + t_1)$, $y(t) := f_h(u(\cdot + t_1), f_h(u, x_h^o)(t_1))(t)$ for $t \in [0, t_2]$. We have $x(0) = y(0)$, $(x'(t) - u'(t + t_1))(x(t) - \phi) \le 0$, $(y'(t) - u'(t + t_1)(y(t) - \phi) \le 0$ a.e. for all $\phi \in [-h,h]$, hence $x(t) \equiv y(t)$ in $[0, t_2]$.

■

We now introduce the *configuration space*

$$\Lambda := \left\{ \lambda \in W^{1,\infty}(0,\infty);\ |\lambda'(h)| \le 1\, a.e. \right\}.$$

For $\overline{h} > 0$ we denote

$$\Lambda(\overline{h}) := \left\{\lambda \in \Lambda;\ \lambda(h) = 0 \quad \text{for } h \geq \overline{h}\right\}.$$

For a given $\lambda \in \Lambda$ and $h > 0$ we define the operator $\ell_h(\cdot, \lambda(h)) : W^{1,1}(0,T) \to W^{1,1}(0,T)$ by the formula

$$\ell_h(u, \lambda(h))(t) = u(t) - f_h(u, x_h^o)(t) \tag{1.5}$$

for every $u \in W^{1,1}(0,T)$ and $t \in [0,T]$, where x_h^o is given by the relation

$$x_h^o := \operatorname{sign}(u(0) - \lambda(h)) \min\{h, |u(0) - \lambda(h)|\}. \tag{1.6}$$

The operator ℓ_h is called *play.* It can be shown (cf. e.g. [5]) that for every $u, v \in W^{1,1}(0,T)$, $\lambda, \mu \in \Lambda$, $h > 0$ and $t \in [0,T]$ we have

$$|\ell_h(u, \lambda(h))(t) - \ell_h(v, \mu(h))(t)| \leq \max\left\{|\lambda(h) - \mu(h)|, ||u - v||_{[0,t]}\right\} \tag{1.7}$$

where we denote $||w||_{[0,t]} = \max\{|w(s)|; 0 \leq s \leq t\}$. Consequently, $\ell_h(\cdot, \lambda(h))$ can be considered as a Lipschitz continuous operator in $C([0,T])$ for every fixed $\lambda \in \Lambda$.

The operators ℓ_h, f_h are *hysteresis operators* in the sense of Visintin's definition (they are causal and rate independent, cf. e.g. [8]), hence it is meaningful to investigate their structure of memory. Let us note that the case $\lambda \equiv 0$ (so-called *reference* or *virgin state*) has been studied in [6] in detail.

The following result is an immediate consequence of (1.1) (we make use of Lemma (1.2), (1.7) and the density of $W^{1,1}(0,T)$ in $C([0,T])$.

(1.8) Lemma. *Let $u \in C([0,T])$ be monotone in $[t_1, t_2] \subset [0,T]$. Then for $t \in [t_1, t_2]$ we have*

$$\ell_h(u, \lambda(h))(t) = \begin{cases} \max\{\ell_h(u, \lambda(h))(t_1), u(t) - h\}, & \text{if } u \text{ is nondecreasing in } [t_1, t_2] \\ \min\{\ell_h(u, \lambda(h))(t_1), u(t) + h\}, & \text{if } u \text{ is nonincreasing in } [t_1, t_2]. \end{cases}$$

This formula can be used as a definition of the play (cf [2]).

2. Structure of memory

Intuitively, we call *memory* of a system of evolution at the time t the set of those values of the system in the past ($\tau \leq t$) which determine its present value. We will see that in our situation the memory is typically a finite or countable set.

Let us suppose now that $u \in C([0,T])$, $\overline{h} > 0$ and $\lambda \in \Lambda(\overline{h})$ are given. For $t \in [0,T]$ we denote

$$\begin{cases} r_\lambda(u)(t) := \min\{h \geq 0;\ \lambda(h) \pm h = u(t)\}, \\ R_\lambda(u)(t) := \max\{r_\lambda(u)(\tau);\ 0 \leq \tau \leq t\}. \end{cases} \tag{2.1}$$

The case $R_\lambda(u)(t) = 0$ is trivial (we have in this case $u(\tau) = u(0) = \lambda(0)$ for all $\tau \in [0,t]$, hence we always assume $R_\lambda(u)(t) > 0$.

(2.2) Lemma. *Let $t \in [0,T]$ be given and let us assume $R_\lambda(u)(t) = r_\lambda(u)(t)$. Then for every $\tau \in [0,t]$ we have*

$$\ell_h(u,\lambda(h)(\tau) = \lambda(h) \quad \textit{for} \quad h \geq r_\lambda(u)(t), \ \textit{and}$$
$$\ell_h(u,\lambda(h)(t) = \begin{cases} u(t) - h & \textit{for} \quad h < r_\lambda(u)(t), \ \textit{if } u(t) = \lambda(h) + h, \\ u(t) + h & \textit{for} \quad h < r_\lambda(u)(t), \ \textit{if } u(t) = \lambda(h) - h. \end{cases}$$

Proof. We can assume that u is piecewise monotone (otherwise we approximate u uniformly by piecewise monotone functions and use the continuity of ℓ_h). Let $0 = s_o < s_1 < \cdots < s_n = t$ be the sequence of local extrema of u.

For every $h \geq r_\lambda(u)(t)$ and every $\tau \in [0,t]$ we have $\lambda(h) - h \leq u(\tau) \leq \lambda(h) + h$. We obtain from (1.6), (1.8) by induction over i for every $i = 0, 1 \ldots, n$, putting $\lambda_i(h) := \ell_h(u,\lambda))(s_i)$

a) $\lambda_i \in \Lambda, \qquad \lambda_i(0) = u(s_i)$,

b) $\lambda_i(h) = \lambda(h) \quad$ for $\quad h \geq r_\lambda(u)(t)$.

Let us suppose for instance $\lambda(r_\lambda(u)(t)) + r_\lambda(u)(t) = u(t)$ (the other case is analogous). For $h < r_\lambda(u)(t)$ (1.8) yields

$$\ell_h(u,\lambda(h))(t) = \max\{\lambda_{n-1}(h), u(t) - h\}, \text{ where}$$

$$\begin{aligned} \lambda_{n-1}(h) &= \lambda_{n-1}(r_\lambda(u)(t)) - \int_h^{r_\lambda(u)(t)} \lambda'_{n-1}(a)da \\ &\leq \lambda(r_\lambda(u)(t)) + r_\lambda(u)(t) - h = u(t) - h, \end{aligned}$$

hence Lemma (2.2) follows easily.

■

We now introduce the concept of *memory sequence.* It will enable us to derive an explicit formula for $\ell_h(u,\lambda(h))(t)$.

We still assume that $u \in C([0,T])$ and $\lambda \in \Lambda(\overline{h})$ are given. Let $t \in [0,T]$ be fixed. We find

$$\overline{t} := \max\{\tau \in [0,t];\, r_\lambda(u)(\tau) = R_\lambda(u)(t)\}$$

and we put

$$\begin{aligned} h_o &:= R_\lambda(u)(t),\ t_o := \overline{t}, &\quad \text{if } \lambda(t_\lambda(u)(\overline{t})) - r_\lambda(u)(\overline{t}) = u(\overline{t}), \\ h_1 &:= R_\lambda(u)(t),\ t_1 := \overline{t}, &\quad \text{if } \lambda(t_\lambda(u)(\overline{t})) + r_\lambda(u)(\overline{t}) = u(\overline{t}). \end{aligned}$$

The *memory sequence* $MS_\lambda(u)(t) := \{(t_j,h_j)\}$ is then constructed by induction: We put

$$\text{(2.3)} \qquad \begin{cases} t_{2k} & := \max\{\tau \in [t_{2k-1},t]\ ; u(\tau) = \min\{u(\sigma); \sigma \in [t_{2k-1},t]\}\} \\ t_{2k+1} & := \max\{\tau \in [t_{2k},t]\ ; u(\tau) = \max\{u(\sigma); \sigma \in [t_{2k},t]\}\} \\ & \qquad \text{until } t_n = t, \\ h_{j+1} & := \frac{(-1)^j}{2}(u(t_{j+1}) - u(t_j))\,,\ j = (0), 1, 2, \ldots, n-1. \end{cases}$$

One of the following possibilities occurs:
a) the sequence $\{(t_j, h_j)\}$ is infinite, $\lim\limits_{j\to\infty} h_j = 0$,
b) the sequence $\{(t_j, h_j)\}$ is finite, $t = t_n$.
In the case b) we put $h_{n+1} := 0$.

(2.4) Proposition. *For every $t \in [0,T]$ and $h > 0$ we have*

$$\ell_h(u,\lambda(h))(t) = \begin{cases} \lambda(h), \;\; h \geq R_\lambda(u)(t), \\ u(t_j) + (-1)^j h, \;\; h \in [h_{j+1}, h_j), \;\; j = (0), 1, 2 \ldots \end{cases}$$

Proof. Let us assume for instance $\overline{t} = t_1$ (the other case is analogous), and for $h > 0$ put $\lambda_1(h) := \ell_h(u,\lambda(h))(t_1)$. By Lemma (2.2) we have

$$\lambda_1(h) = \begin{cases} \lambda(h), & h \geq R_\lambda(u)(t), \\ u(t_1) - h, & h < R_\lambda(u)(t), \end{cases}$$

hence $\lambda_1 \in \Lambda, \lambda_1(0) = u(t_1)$. Put $u_1(\tau) := u(\tau + t_1)$ for $\tau \in [0, t - t_1]$. Then (2.3) yields $R_{\lambda_1}(u_1)(t - t_1) = r_{\lambda_1}(u_1)(t_2 - t_1) = \frac{1}{2}(u(t_1) - u(t_2)) = h_2$, hence using Lemmas (2.2) and (1.4) we obtain

$$\ell_h(u,\lambda(h))(t_2) = \begin{cases} \lambda_1(h), & h \geq h_2, \\ u(t_2) + h, & h < h_2. \end{cases}$$

An easy induction over $u_j(\tau) := u(\tau + t_j), j = 1, 2, \ldots$ completes the proof.

■

(2.5) Corollary. *Let $\overline{h} > 0$, $\lambda \in \Lambda(\overline{h})$, $u \in C([0,T])$ be given. For $t \in [0,T]$ and $h > 0$ put*

$$\lambda_t(h) := \ell_h(u,\lambda(h))(t).$$

Then we have for every $t \in [0,T]$
(i) $\lambda_t \in \Lambda$, $\lambda_t(0) = u(t)$,
(ii) $\lambda_t(h) = \lambda(h)$ *for* $h \geq R_\lambda(u)(t)$,
(iii) $\left|\frac{d}{dh}\lambda_t(h)\right| = 1$ *for a.e.* $h < R_\lambda(u)(t)$,
(iv) $\lambda_t \in \Lambda(\hat{h})$, *where* $\hat{h} = \max\{\overline{h}, ||u||_{[0,t]}\}$.

The configuration $\lambda_t \in \Lambda$ characterizes the memory of the play-stop system at the time t.

3. Ishlinskii operator

(3.1) Definition. *Let $\phi \in L^1_{\mathrm{loc}}(0,\infty)$ be a given nonnegative function and let $\alpha > 0, \overline{h} > 0$ be given numbers. For $\lambda \in \Lambda(\overline{h}), u \in C([0,T])$ and $t \in [0,T]$ we put*

$$F_\lambda(u)(t) = \alpha u(t) + \int_0^\infty \ell_h(u,\lambda(h))(t)\,\phi(h)dh.$$

We have indeed $\ell_h(u, \lambda(h))(t) = 0$ for $h \geq \max\{\overline{h}, ||u||_{[0,T]}\}$, hence F_λ maps $C([0,T])$ into $C([0,T])$. The operator F_λ is called an *Ishlinskii operator.*

The local Lipschitz continuity of F_λ is an immediate consequence of (1.7). Moreover, (1.7) yields for $0 \leq s < t \leq T$

$$|F_\lambda(u)(t) - F_\lambda(u)(s)| \leq c||u(\cdot) - u(s)||_{[s,t]},$$

hence F_λ maps $W^{1,p}(0,T)$ into $W^{1,p}(0,T)$ for every $1 \leq p \leq \infty$. Using Lemma (1.2) and Lebesgue's Dominated Convergence Theorem we conclude that the identity

$$(F_\lambda(u))'_\pm(t) = \alpha u'_\pm(t) + \int_0^\infty (\ell_h(u, \lambda(h))'_\pm(t)\, \phi(h) dh$$

holds provided $u'_\pm(t)$ exists. The continuity of F_λ in $W^{1,p}(0,T)$ for $1 \leq p < \infty$ is an immediate consequence of the continuity of $\ell_h(\cdot, \lambda(h))$.

In the sequel we assume

$$\phi(h) > 0 \qquad \text{for a.e.} \quad h > 0 \tag{3.3}$$

It is clear that $(F_\lambda(u))'(t) = 0$ if $u'(t) = 0$. For $u'(t) \neq 0$ the following lemma holds:

(3.4) Lemma. *Let $t \in (0,T)$ be given such that $u'(t) \neq 0$ and $(F_\lambda(u))'(t)$ exists. Put $\rho(t) = \inf\{h_j;\ (t_j, h_j) \in MS_\lambda(u)(t)\}$. Then $\rho(t) > 0$ and*

$$(\ell_h(u, \lambda(h)))'(t) = \begin{cases} 0 & \text{for } h > \rho(t), \\ u'(t) & \text{for } h < \rho(t). \end{cases}$$

Proof. We can suppose $u'(t) > 0$ (the other case is analogous). The memory sequence is obviously finite (otherwise we would have $u'(t) = 0$), hence $t = t_{2k+1}$ for some $k \geq 0$ and

a) $\rho(t) = R_\lambda(u)(t) = r_\lambda(u)(t)$, if $k = 0$, or

b) $\rho(t) = h_{2k+1} = \frac{1}{2}(u(t) - u(t_{2k}))$, if $k \geq 1$.

In the case a) we have for $h > \rho(t)$ $\quad (\ell_h(u, \lambda(h)))'_-(t) = 0$, for $h < \rho(t)$ we have

$$x_h(t) := u(t) - \ell_h(u, \lambda(h))(t) = h,$$

hence Lemma (1.2) yields $x'_{h_+}(t) = 0$. Consequently,

$$\begin{aligned} 0 &= (F_\lambda(u))'_+(t) - (F_\lambda(u))'_-(t) \\ &= \int_0^{\rho(t)} [u'(t) - (\ell_h(u, \lambda(h)))'_-(t)]\, \phi(h) dh + \int_{\rho(t)}^\infty (\ell_h(u, \lambda(h)))'_+(t)\, \phi(h) dh. \end{aligned}$$

We have indeed $u'(t) \geq (\ell_h(u,\lambda(h)))'_-(t)$, $(\ell_h(u,\lambda(h)))'_+t \geq 0$ by Lemma (1.2). Applying the same argument in the case b) we obtain the assertion. ■

(3.5) Lemma. *Let $\lambda \in \Lambda$, $u \in W^{1,1}(0,T)$ and $h > 0$ be given. For $t \in [0,T]$ put* $\rho(t) = \inf\{h_j; (t_j,h_j) \in MS_\lambda(u)(t)\}$.
Then the set $M_h := \{t \in [0,T]; u'(t) \neq 0, h = \rho(t)\}$ is finite or empty.

Proof. Suppose that M_h is infinite for some $h > 0$. We may assume that:

- there exists a monotone sequence $\tau_i \to \tau_o$ such that $u'(\tau_i) > 0, \rho(\tau_i) = h$ (the case $u'(\tau_i) < 0$ is analogous),
- $R_\lambda(u)(\tau_i) > r_\lambda(u)(\tau_i)$ for all $i = 1,2,\ldots$,
- there exists a convergent sequence $\sigma_i \to \sigma_o$ such that $\sigma_i < \tau_i$, $u(\tau_i) - u(\sigma_i) = 2h$, $u(t) \in [u(\sigma_i), u(\tau_i)]$ for $t \in [\sigma_i, \tau_i]$.

Therefore, $\sigma_o < \tau_o$, $u(\tau_o) - u(\sigma_o) = 2h$, $u(t) \in [u(\sigma_o), u(\tau_o)]$ for $t \in [\sigma_o, \tau_o]$. In both cases $\tau_i \nearrow \tau_o, \tau_i \searrow \tau_o$ we obtain a contradiction with (2.3). ■

4. Monotonicity

(4.1) Proposition. *Let $\alpha > 0$, $\lambda, \mu \in \Lambda(\overline{h})$, $u, v \in W^{1,1}(0,T)$ and $\phi \in L^1_{loc}(0,\infty)$* satisfying (3.3) *be given, and let F_λ, F_μ be the Ishlinskii operators* (3.1). *Then for almost all $t \in (0,T)$ we have*

$$
\begin{aligned}
&(F_\lambda(u) - F_\mu(v))'(t)(u(t) - v(t)) \\
&\quad \geq \frac{1}{2}\frac{d}{dt}\Big[\alpha(u(t) - v(t))^2 + \int_0^\infty (\ell_h(u,\lambda(h)) - \ell_h(v,\mu(h)))^2(t)\phi(h)dh\Big].
\end{aligned}
$$

Proof. Let us choose $t \in (0,T)$ in such a way that $u'(t), v'(t), (F_\lambda(u)'(t), (F_\lambda(v))'(t)$ exist. Then $(\ell_h(u,\lambda(h)))'(t), (\ell_h(v,\mu(h)))'(t)$ exist for all values of $h > 0$ except of two at most. For all such h (1.1) yields

$$
\begin{aligned}
&(\ell_h(u,\lambda(h)) - \ell_h(v,\mu(h)))'(t)(u(t) - v(t)) \\
&\quad \leq \frac{1}{2}\frac{d}{dt}(\ell_h(u,\lambda(h)) - \ell_h(v,\mu(h)))^2(t).
\end{aligned}
\tag{4.2}
$$

Integrating with respect to h we obtain the assertion. ■

(4.3) Proposition (Strict monotonicity). *Let the assumption of Proposition* (4.1) *be statisfied and let us suppose $\lambda(0) = u(0), \mu(0) = v(0)$. Then the following 4 conditions are equivalent:*

(i) $(F_\lambda(u) - F_\mu(v))'(t)(u(t) - v(t))$

$$
= \frac{1}{2}\frac{d}{dt}\Big[\alpha(u(t) - v(t))^2 + \int_0^\infty (\ell_h(u,\lambda(h)) - \ell_h(v,\mu(h)))^2(t)\,\phi(h)dh\Big]
$$

for a.e. $t \in (0,T)$*;*

(ii) *For every* $h > 0$ *we have*

$$(\ell_h(u,\lambda(h)))'(t)(x_h(t) - y_h(t)) \;=\; (\ell_h(v,\mu(h)))'(t)(x_h(t) - y_h(t)) = 0$$

for a.e. $t \in (0,T)$*, where we denote*

$$x_h(t) := u(t) - \ell_h(u,\lambda(h))(t),\; y_h(t) := v(t) - \ell_h(v,\mu(h))(t);$$

(iii) *For every* $h > 0$*,* $t \in [0,T]$*,* $\delta \in [0,1]$ *we have*

$$\begin{aligned}&\ell_h(\delta u + (1-\delta)v, \delta\lambda(h) + (1-\delta)\mu(h))(t)\\ &= \delta\ell_h(u,\lambda(h))(t) + (1-\delta)\ell_h(v,\mu(h))(t);\end{aligned}$$

(iv) *For* $t \in [0,T]$ *put* $R(t) := \max\{R_\lambda(u)(t), R_\mu(v)(t)\}$*. Then for every* $t \in [0,T]$ *and* $h > 0$ *we have*

$$\ell_h(u,\lambda(h))(t) - \ell_h(v,\mu(h))(t) = \begin{cases} \lambda(h) - \mu(h) & \text{for } h \geq R(t),\\ \lambda(R(t)) - \mu(R(t)) & \text{for } h < R(t).\end{cases}$$

Remark. We see immediately using (2.5)(i) that in (4.3)(iv) we have $\lambda(R(t)) - \mu(R(t)) = u(t) - v(t)$.

Proof of (4.3). Let us denote $\xi_h(t) = \ell_h(u,\lambda(h))(t), \eta_h(t) = \ell_h(v,\mu(h))(t)$.

(i)⇒(ii): Let us denote by M the set of all $t \in (0,T)$ such that $u'(t), v'(t)$, $(F_\lambda(u))'(t), (F_\mu(v))'(t)$ exist and (i) holds. Let $\rho(t)$ be as in Lemma (3.5) and put analogously $\sigma(t) := \inf\{h_j; (t_j,h_j) \in MS_\mu(v)(t)\}$ for $t \in M$.

The expression $(\xi_h'(t) - \eta_h'(t))(x_h(t) - y_h(t))$ is nonnegative by (1.1) (or (4.2)) and continuous with respect to h in $(0,\infty)\backslash\{\rho(t),\sigma(t)\}$ for every $t \in M$. Thus (i) implies

$$(\xi_h'(t) - \eta_h'(t))(x_h(t) - y_h(t)) = 0 \tag{4.4}$$

for every $t \in M$ and $h \in (0,\infty)\backslash\{\rho(t),\sigma(t)\}$. Let $h > 0$ be now arbitrarily chosen. By Lemma (3.5) the identity (4.4) holds for almost all $t \in (0,T)$. We have indeed by (1.1) $\xi_h'(t)(x_h(t) - y_h(t)) \geq 0$, $\eta_h'(t)(y_h(t) - x_h(t)) \geq 0$ a.e., hence (ii) follows from (4.4).

(ii)⇒(iii): Let $h > 0$ be fixed. We have $\xi_h'(x_h - \phi) \geq 0$, $\eta_h'(y_h - \phi) \geq 0$, a.e. for every $\phi \in [-h,h]$, hence (ii) yields $\xi_h'(y_h - \phi) \geq 0$, $\eta_h'(x_h - \phi) \geq 0$ a.e. for every $\phi \in [-h,h]$. Consequently, for each $\delta \in [0,1]$ we have

$$\xi_h'(\delta x_h + (1-\delta)y_h - \phi) \geq 0, \quad \eta_h'(\delta x_h + (1-\delta)y_h - \phi) \geq 0$$

a.e. for all $\phi \in [-h,h]$. Therefore,

$$(\delta\xi_h' + (1-\delta)\eta_h')(\delta x_h + (1-\delta)y_h - \phi) \geq 0 \text{ a.e. for all } \phi \in [-h,h].$$

We have $\xi_h(0) = \lambda(h), \eta_h(0) = \mu(h)$, hence (1.1), (1.5), (1.6) imply (iii).

(iii)⇒(iv): For $h \geq R(t)$ we have by (2.4) $\xi_h(t) = \lambda(h), \eta_h(t) = \mu(h)$. Let us choose $h < R(t)$ and let us suppose

$$\frac{d}{dh}(\xi_h(t) - \eta_h(t)) \neq 0.$$

Then (iii) yields for every $\delta \in (0,1)$

$$|\frac{d}{dh}\ell_h(\delta_u + (1-\delta)v, \delta\lambda(h) + 1 - \delta)\mu(h))(t)| < 1.$$

Then by Corollary (2.5)(iii) $h > R_{\delta\lambda+(1-\delta)\mu}(\delta u + (1-\delta)v)(t)$. Necessarily we must have for every $\tau \in [0,t]$ by (2.1)

$$\delta\lambda(h) + (1-\delta)\mu(h) - h < \delta u(\tau) + (1-\delta)v(\tau) < \delta\lambda(h) + (1-\delta)\mu(h) + h.$$

On the other hand, we have either $h < R_\lambda(u)(t)$ or $h < R_\mu(v)(t)$. This means that there exists $t_o \in [0,t]$ such that one of the four inequalities holds: $u(t_o) > \lambda(h) + h$ or $u(t_o) < \lambda(h) - h$ or $v(t_o) > \mu(h) + h$ or $v(t_o) < \mu(h) - h$. Choosing δ or $(1-\delta)$ sufficiently small we obtain a contradiction.

(iv)⇒(i): For $h > R(t)$ we have $\xi_h'(t) = \eta_h'(t) = 0$, for $h < R(t)$ we have by (iv) $x_h(t) = y_h(t)$ and (i) follows easily.

■

Remark. The assumption $\lambda(0) = u(0), \mu(0) = v(0)$ is not restrictive. Indeed, if we replace $\lambda(h)$ by $\lambda_o(h) := \ell_h(u, \lambda(h))(0)$, then for every $t \geq 0$ we have by (2.4) $\ell_h(u, \lambda(h))(t) = \ell_h(u, \lambda_o(h))(t)$, hence the values of $\lambda(h)$ for $h < r_\lambda(u)(0)$ are irrevelant.

5. Periodic inputs

Let $\omega > 0$ be given. We denote by C_ω ($W^{1,1}_\omega$) the space of continuous (absolutely continuous, respectively) ω-periodic functions. It follows immediately from (2.2) and (2.4) that $\ell_h(u, \lambda(h))$ is ω-periodic for every $u \in C_\omega$, $\lambda \in \Lambda$ and $h > 0$ for $t \geq \min\{\tau \in [0,\omega]; r_\lambda(u)(\tau) = R_\lambda(u)(\omega)\}$. In particular, $\ell_h(u, \lambda(h))$ and $F_\lambda(u)$ are ω-periodic for $t \geq \omega$.

The following result is an easy consequence of Proposition (4.3).

(5.1) Proposition. *Let the assumptions of* (4.3) *be satisfied for some* $u, v \in W^{1,1}_\omega$ *and let* (4.3) (i) *hold for a.e.* $t > 0$. *Then*

$$\frac{d}{dt}(\ell_h(u, \lambda(h))(t) - \ell_h(v, \mu(h))(t)) = 0$$

for all $h > 0$ *and a.e.* $t > \omega$.

We note only that the function $R(t)$ in (4.3)(iv) is constant in this case for $t \geq \omega$.

Propositions (4.3) and (5.1) are generalizations of (1.6) (ix) of [3]. The proof we present here is considerably simpler.

6. Dependence on parameters

In the sequel we deal with functions which depend also on spatial variables. We consider the spatial variable as a parameter. More precisely, if

$$u:\ [0,1]\times[0,T]\to R^1 \qquad \text{and} \qquad \lambda:\ [0,1]\times(0,\infty)\to R^1$$

are given functions such that for some $\overline{h}>0$ we have

$$\text{(6.1)} \qquad \begin{aligned} &(i) && u(x,\cdot)\in C([0,T]) && \text{for every} \quad x\in[0,1],\\ &(ii) && u(x,\cdot)\in\lambda(\overline{h}) && \text{for every} \quad x\in[0,1], \end{aligned}$$

then we define for a given $\alpha>0$ and $\phi\in L^1_{\rm loc}(0,\infty)$ the value of the Ishlinskii operator

$$\text{(6.2)} \qquad F_\lambda(u)(x,t) := F_{\lambda(x,\cdot)}(u(x,\cdot))(t),$$

where $F_{\lambda(x,\cdot)}$ is the operator (3.1). We use the same notation, since no confusion is possible. We write similarly

$$\ell_h(u,\lambda(x,h))(x,t) := \ell_h(u(x,\cdot),\lambda(x,h))(t).$$

7. The wave equation

(7.1) Assumptions

(i) $\overline{h}>0$, $\lambda\in C([0,1];\Lambda(\overline{h}))$ are given;

(ii) $\alpha>0$, $\phi\in L^1_{\rm loc}(0,\infty)$ are given such that (3.3) holds. We put for $r>0$

$$\xi(r)=\alpha+\int_0^r\phi(h)dh, \quad \gamma(r)=\operatorname{infess}\{\phi(h);0<h\le r\}$$

and we assume $\lim_{r\to\infty}\frac{\xi(r)}{r^2\gamma(r)}=0$, $\quad\lim_{r\to\infty}\frac{\xi(r)}{\gamma(r)}=+\infty$;

(iii) $u^o\in W^{2,2}(0,1)$, $u^1\in W^{1,2}(0,1)$ are given functions satisfying $u^o(0)=u^1(0)=u^{o'}(1)=0$;

(iv) $g\in L^\infty(0,\infty;L^2(0,1))$ is given function such that $g_t\in L^\infty(0,\infty;L^2(0,1))$ and $g(x,t+\omega)=g(x,t)$ for every $t\ge0$;

(v) We introduce the space

$$\begin{aligned} H^{\infty,2}_o := \big\{&u\in L^\infty(0,\infty;L^2(0,1));\ u_{tt},u_{xx},u_{xt}\in L^\infty(0,\infty;L^2(0,1)),\\ &u(0,t)=u_x(1,t)=0 \text{ for all } t\ge0\big\}. \end{aligned}$$

Assuming (7.1) we consider the problem

$$(7.2)\qquad \begin{aligned} &(i) && F_\lambda(u_t)_t - u_{xx} = g(x,t),\ x\in(0,1), t>0,\\ &(ii) && u(0,t)=u_x(1,t)=0,\ t\ge 0,\\ &(iii) && u(x,0)=u^o(x),\ u_t(x,0)=u^1(x),\ x\in[0,1], \end{aligned}$$

where F_λ is the operator (6.2).

The following theorem is one of the main results of [7] (Theorems (5.7), (5.8) and Remark (5.5)):

(7.3) Theorem. *Let* (7.1) *hold. Then the problem* (7.2) *has a unique solution* $u \in H_o^{\infty,2}$ *such that* (7.2)(i) *holds almost everywhere in* $(0,1)\times(0,\infty)$.

The main result of the present paper reads as follows:

(7.4) Theorem. *Let* (7.1) *hold and let* $u \in H_o^{\infty,2}$ *be the solution of* (7.2). *Then there exists a function* $u^* \in H_o^{\infty,2}$ *such that* $u^*(x,t+\omega)=u^*(x,t)$ *for all* $(x,t)\in[0,1]\times[0,\infty)$ *and*

$$\lim_{t\to\infty}\max_{x\in[0,1]}\left(|u_t(x,t)-u_t^*(x,t)|+|u_x(x,t)-u_x^*(x,t)|\right)=0.$$

(7.5) Theorem. *Let the assumptions of* (7.4) *hold. Then* u^* *satisfies* (7.2)(i) *for almost all* $(x,t)\in(0,1)\times(0,\infty)$ *and* u^* *is the unique* ω*-periodic solution of* (7.2)(i),(ii) *for* $t\ge\omega$.

Proof of (7.4). The sequences $\{u_t(x,n\omega)\},\{u_x(x,n\omega)\}, n=0,1,2,\dots$ are equibounded and equicontinuous in $C([0,1])$. Putting $\lambda_n(x,h) := \ell_h(u_t,\lambda(x,h))(x,n\omega)$, $\hat h := \max\{\overline{h},V\}, V := \sup\{|u_t(x,t)|;\ x\in[0,1], t\ge 0\}$ we see by (1.7) that the sequence $\{\lambda_n\}$ is equibounded and equicontinuous in $C([0,1];\Lambda(\hat h))$ (note that $\Lambda(\hat h)$ is compact in the sup-norm).

By Arzelà-Ascoli theorem there exist $v_o, w_o \in C([0,1])$ and $\lambda^*\in C([0,1];\Lambda(\hat h))$ and a subsequence $\{n_i\}$ of $\{n\}$ such that

$$u_t(\cdot,n_i\omega)\to v_o,\ u_x(\cdot,n_i\omega)\to w_o \text{ uniformly },\lambda_{n_i}\to\lambda^* \text{ uniformly }.$$

Let us denote $u^n(x,t) := u(x,t+n\omega)$. The semigroup property (1.4) yields

$$(7.6)\qquad F_{\lambda_n}(u_t^n)_t - u_{xx}^n = g(x,t) \text{ a.e. for } n=0,1,2,\dots.$$

We further define the expression $D(v,w,\mu,\nu)(t)$ for $v,w\in H_o^{\infty,2}, \mu,\nu\in C([0,1]);\Lambda(\hat h))$ and $t\ge 0$ by the formula

$$(7.7)\qquad D(v,w,\mu,\nu)(t) := \int_0^1 [\alpha(v_t-w_t)^2(x,t)+(v_x-w_x)^2(x,t)+ \\ +\int_0^\infty (\ell_h(v,\mu(x,h))(x,t)-\ell_h(w,\nu(x,h))(x,t))^2\phi(h)dh]dx.$$

Indeed, if $F_\mu(v_t)_t - v_{xx} = g(x,t), F_\nu(w_t)_t - w_{xx} = g(x,t)$ a.e., then (4.1) implies

$$\frac{d}{dt}D(v,w,\mu,\nu)(t) \le 0 \text{ a.e.} \tag{7.8}$$

This yields in particular

$$D(u^{n_i}, u^{n_j}, \lambda_{n_i}, \lambda_{n_j})(t) \le D(u^{n_i}, u^{n_j}, \lambda_{n_i}, \lambda_{n_j})(0).$$

By hypothesis, the right-hand side of the last inequality tends to 0 for $i,j \to \infty$. Consequently, $\{u_t^{n_i}\}, \{u_x^{n_i}\}$, are Cauchy sequences in $L^\infty(0,\infty; L^2(0,1))$ and there exists a continuous function u^* such that $u_t^*, u_x^* \in L^\infty(0,\infty; L^2(0,1))$ and $u_t^{n_i} \to u_t^*, u_x^{n_i} \to u_x^*$ in $L^\infty(0,\infty; L^2(0,1))$ strong.

On the other hand, the sequence $\{u^n\}$ is bounded in $H_o^{\infty,2}$, hence $u^* \in H_o^{\infty,2}$ and $u^{n_i} \to u^*$ in $H_o^{\infty,2}$ weakly-star, $u_t^{n_i} \to u_t^*$, $u_x^{n_i} \to u_x^*$ locally uniformly. This implies $u_t^*(x,0) = v_o(x)$, $u_x^*(x,0) = w_o(x)$ and

$$F_{\lambda^*}(u_t^*)_t - u_{xx}^* = g(x,t) \qquad \text{a.e.} \tag{7.9}$$

Put $u^{**}(x,t) := u^*(x,t+\omega)$, $\lambda^{**}(x,h) := \ell_h(u_t^*, \lambda^*(x,h))(x,\omega)$ for $x \in [0,1], t \ge 0$, $h > 0$.

Our next goal is to prove $u^{**} = u^*, \lambda^{**} = \lambda^*$.

Put $n = 1$ and $r := \lim_{t\to\infty} D(u^1, u, \lambda_1, \lambda)(t)$. We have for every $t \ge 0$

$$D(u^{**}, u^*, \lambda^{**}, \lambda^*)(t) = \lim_{i\to\infty} D(u^1, u, \lambda_1, \lambda)(t + n_i\omega),$$

hence

$$D(u^{**}, u^*, \lambda^{**}, \lambda^*)(t) \equiv r = \text{ const. for all } t \ge 0. \tag{7.10}$$

Differentiating (7.10) with respect to t and using Proposition (4.1) we conclude that the condition (4.3)(i) is satisfied for almost all $x \in (0,1)$ for $u := u_t^{**}(x,\cdot)$, $v := u_t^*(x,\cdot)$, $\lambda := \lambda^{**}(x,\cdot)$, $\mu := \lambda^*(x,\cdot)$.

The function $R(t)$ in (4.3)(iv) is nondecreasing, hence by Proposition (4.3) for almost every $x \in (0,1)$ and every $h > 0$ there exist the limits

$$\begin{aligned} &\lim_{t\to\infty} u_t^{**}(x,t) - u_t^*(x,t) =: U(x), \\ &\lim_{t\to\infty} (\ell_h(u_t^{**}, \lambda^{**}(x,h))(x,t) - \ell_h(u_t^*, \lambda^*(x,h))(x,t)) =: L(x,h). \end{aligned}$$

The function u_t^* is bounded in $[0,1] \times [0,\infty)$, hence necessarily $U(x) = L(x,h) \equiv 0$.

Let now $\varepsilon > 0$ be given. We denote by $||\cdot||$ the norm in $L^2(0,1)$. There exists $T_o > 0$ such that for $t > T_o$ we have $||u_t^*(\cdot, t+\omega) - u_t^*(\cdot,t)|| < \varepsilon$, and $J > 0$ such that for every $t \ge 0$ and $j \ge J$ we have

$$||u_t^*(\cdot,t) - u_t(\cdot, t + n_j\omega)|| < \varepsilon.$$

Put $T_1 = T_o + n_J\omega$. For $t > T_1$ we have $t - n_J\omega > T_o$, hence

$$||u_t(\cdot, t+\omega) - u_t(\cdot, t)|| < 3\varepsilon.$$

Let now $t \geq 0$ be arbitrary. We find $K \geq J$ such that $t + n_K\omega > T_1$. We have

$$\begin{aligned}&||u_t^*(\cdot, t+\omega) - u_t^*(\cdot, t)|| \leq ||u_t^*(\cdot, t+\omega) - u_t(\cdot, t+\omega+n_K\omega)||+\\&+ ||u_t(\cdot, t+\omega+n_K\omega) - u_t(\cdot, t+n_K\omega)|| + ||u_t(\cdot, t+n_K\omega) - u_t^*(\cdot, t)|| < 5\varepsilon\end{aligned}$$

consequently $u_t^{**} = u_t^*$ a.e. The boundedness of u^* implies immediately $u^{**} = u^*$.

We now pass to the limit in (7.10) for $t \to \infty$. This yields $r = 0$, hence $\lambda^{**} = \lambda^*$.

Using (1.4), (7.6), (7.8) and (7.9) we obtain for each n

$$D(u_t^{n+1}, u_t^*, \lambda_{n+1}, \lambda^*)(0) = D(u_t^n, u_t^*, \lambda_n, \lambda^*)(\omega) \leq D(u_t^n, u_t^*, \lambda_n, \lambda^*)(0),$$

hence the sequence $\{d_n\}$, $d_n := D(u_t^n, u_t^*, \lambda_n, \lambda^*)(0)$ is nonincreasing.

We have indeed $\lim\limits_{i\to\infty} d_{n_i} = 0$, hence $d_n \to 0$ as $n \to \infty$.

In the same way as above we conclude that $u_t^n \to u_t^*, u_x^n \to u_x^*$ locally uniformly in $[0,1] \times [0,\infty)$.

Let now $\varepsilon > 0$ be fixed. We find n_o such that for $n \geq n_o$ and $(x,t) \in [0,1] \times [0,\omega]$ we have

$$|u_t^n(x,t) - u_t^*(x,t)| + |u_x^n(x,t) - u_x^*(x,t)| < \varepsilon.$$

For each $t \geq n_o\omega$ we find $n \geq n_o$ such that $t \in [n\omega, (n+1)\omega)$ and we obtain

$$\begin{aligned}&|u_t(x,t) - u_t^*(x,t)| + |u_x(x,t) - u_x^*(x,t)|\\&= |u_t^n(x,t-n\omega) - u_t^*(x,t-n\omega)| + |u_x^n(x,t-n\omega) - u_x^*(x,t-n\omega)| < \varepsilon.\end{aligned}$$

Theorem (7.4) is proved.

■

Theorem (7.5) is now an obvious consequence of (7.9), (4.1) and (5.1).

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