Weak stabilization of solutions to PDEs with hysteresis in thermovisco-elastoplasticity

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Abstract. We present a thermodynamically consistent description of the uniaxial behavior of thermovisco-elastoplastic materials for which the total stress σ contains, in addition to elastic, viscous and thermic contributions, a plastic component $\sigma^p(x,t) = \mathcal{P}[\varepsilon(x,\cdot), \theta(x,t)](t)$. Here, ε and θ are the fields of strain and absolute temperature, respectively, and $\{\mathcal{P}[\cdot,\theta]\}_{\theta>0}$ denotes a family of (rate-independent) hysteresis operators of Prandtl-Ishlinskii type, parametrized by the absolute temperature. The momentum and energy balance equations governing the space-time evolution of the material form a system of two highly nonlinearly coupled partial differential equations involving partial derivatives of hysteretic nonlinearities at different places. It is shown that under no external forcing, the unique global strong solution of a corresponding initial-boundary value problem remains bounded in the energy norm and the velocity asymptotically vanishes for large times.

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1 Introduction

For many materials the stress-strain (σ - ε) relations measured in uniaxial load-deformation experiments strongly depend on the absolute (Kelvin) temperature θ and, at the same time, exhibit a strong plastic behavior witnessed by the occurrence of rate-independent hysteresis loops. Figure 1 shows a typical diagram, where the elasticity modulus and the yield limit depend on temperature.

Among the materials exhibiting temperature-dependent, but rate-independent hysteretic effects are shape memory alloys (see, for instance, Chapter 5 in [1]) and even, although to a smaller extent, quite ordinary steels.

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If the σ - ε relation exhibits a hysteresis, it can no longer be expressed in terms of simple single-valued functions since the latter are certainly not able to give a correct account of the inherent memory structures that are responsible for the complicated loopings in the interior of experimentally observed hysteresis loops.

To avoid these difficulties, a different approach to thermoelastoplastic hysteresis based on the notion of *hysteresis operators* introduced by the Russian group around M. A. Krasnoselskii in the seventies (see [5]) has been proposed by the authors in [7]. The temperature-dependent plastic stress σ^p has been assumed in the form of an *operator equation* with a temperature-dependent hysteretic constitutive operator $\mathcal{P}$ of *Prandtl-Ishlinskii type*, namely

$$\sigma^p \;=\; \mathcal{P}[\varepsilon,\theta] \;:=\; \int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r[\varepsilon]\,dr\,. \tag{1}$$

In this connection, $\mathfrak{s}_r$ denotes the so-called *stop operator* or *elastic-plastic element* with threshold $r>0$ (to be defined in the next section), and $\varphi(\cdot,\theta)\ge 0$ is a density function with respect to $r>0$, parametrized by the absolute temperature θ. The integral formula (1) corresponds to an infinite rheological combination in parallel of elements $\mathfrak{s}_r$.

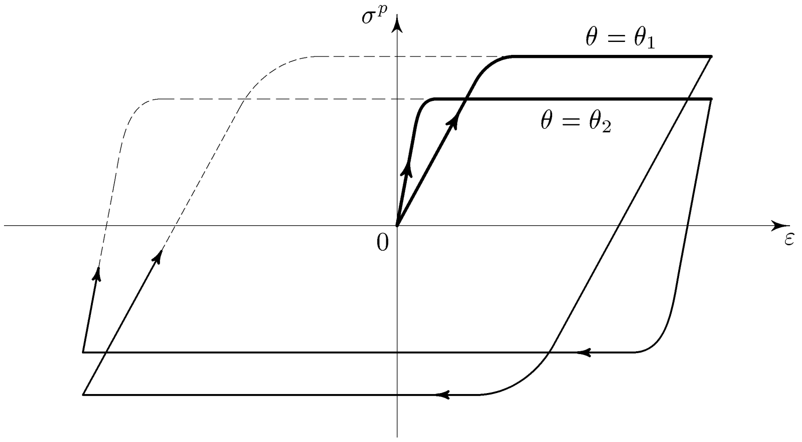

Figure 1: *Strain – plastic stress diagrams at constant temperatures* $\theta_1 \neq \theta_2$.

The advantage of this approach is that an operator equation like (1) is suited much better than a simple functional relation to keep track of *memory effects* imprinted on the material in the past history; in fact, the output at any time $t\in[0,T]$ may depend on the whole evolution of the input in the time interval $[0,t]$. Observe that the requirement of rate-independence implies that $\mathcal{P}$ cannot be expressed in terms of an integral operator of convolution type, i. e. we are not dealing with a model with fading memory.

In the isothermal case, i. e. if $\mathcal{P}$ is independent of θ, the space-time evolution is governed by the equation of motion which is of hyperbolic type, see e. g. [6]. In the temperature-dependent case, the equation of motion has to be complemented by a field equation representing the balance law of internal energy, and the second principle of thermodynamics in form of the Clausius-Duhem inequality must be obeyed. The first problem then consists in a correct definition of thermodynamic state functions like the densities of free energy, internal energy and entropy. It is natural to expect that they will be given in the form of *operators* rather than of *functions*.

A corresponding construction has been carried out in [7, 8]. It turns out, however, that we are no longer able to solve the hyperbolic case, and a further regularization is necessary. While [7] is devoted to the case when the total stress σ is composed of a plastic stress σ^p of the form (1) and a so-called *couple stress*, [7] deals with the situation when σ comprises, in addition to the plastic stress (1), (nonlinear) elastic, (linear) viscous, and (linear) thermic contributions σ^e, σ^v and σ^d, respectively; that is, we assume a constitutive law of the form

$$\sigma = \sigma^p + \sigma^e + \sigma^v + \sigma^d \,, \tag{2}$$

with σ^p given as in (1).

It should be mentioned at this place that hysteretic relations can usually not be described in an explicit form and, as a rule, enjoy only very restricted smoothness properties. Therefore, the classical techniques of one-dimensional thermovisco-elasticity developed for cases in which the stress-strain relation is given through a simple (possibly nonconvex, but differentiable) function (we only refer to the fundamental papers [2, 3]) apply only partially, and new techniques tailored to deal with the specific behavior of hysteretic nonlinearities need to be employed.

The paper is organized as follows. In Section 2, the field equations governing the space-time evolution in thermovisco-elastoplastic materials with the constitutive law (2) are derived. We obtain a system of nonlinearly coupled partial differential equations involving partial derivatives of hysteretic nonlinearities at different places, even in derivatives of highest order. Section 3 contains a summary of results of [8] on existence, uniqueness and thermodynamic consistency of solutions and their continuous dependence on given data. In Section 4, we present a new result on weak asymptotic stabilization. Section 5 is an appendix, where we derive an auxiliary convergence theorem.

2 Thermoelastoplastic constitutive laws

The stop operator $\mathfrak{s}_r : W^{1,1}(0,T) \to W^{1,1}(0,T)$ in the equation (1) is defined as the solution operator $\sigma_r = \mathfrak{s}_r[\varepsilon]$ of the variational inequality

$$|\sigma_r(t)| \le r, \quad (\dot\varepsilon(t) - \dot\sigma_r(t))(\sigma_r(t) - \tilde\sigma) \ge 0 \quad \text{for a.e. } t \in \,]0,T[\,, \quad \forall \tilde\sigma \in [-r,r], \tag{3}$$

with initial condition

$$\sigma_r(0) = \operatorname{sign}(\varepsilon(0)) \min\{r, |\varepsilon(0)|\} \tag{4}$$

which describes the strain-stress law of Prandtl's model for elastic-perfectly plastic materials with a unit elasticity modulus and yield point r, see Figure 2.

The density function φ in (1) is assumed to be given. It can be identified from the isothermal initial loading curves $\sigma = \Phi(\varepsilon, \theta)$ obtained experimentally by letting ε monotonically increase for each fixed temperature θ starting from the origin. The corresponding formula reads (see [6])

$$\Phi(\varepsilon, \theta) = \int_0^\varepsilon \int_s^\infty \varphi(r, \theta)\, dr\, ds. \tag{5}$$

We consider here only the case when φ is nonnegative, i.e. the initial loading curves at each constant temperature are concave and nondecreasing as on Figure 1.

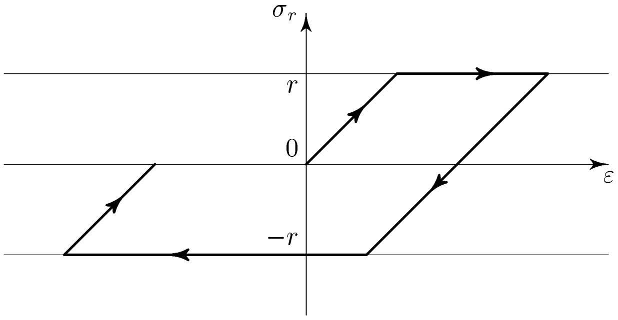

Figure 2: *Prandtl's normalized elastic-perfectly plastic element*

The operator $\mathfrak{s}_r$ has following properties (for a proof, see [1], [6]).

Proposition 1. *Let $r > 0$ be given. Then it holds:*

(i) *For every $\varepsilon \in W^{1,1}(0,T)$, we have*

$$\left(\frac{d}{dt}\mathfrak{s}_r[\varepsilon]\right)^2 = \dot\varepsilon \frac{d}{dt}\mathfrak{s}_r[\varepsilon] \quad a.e.\ in\ ]0,T[. \tag{6}$$

(ii) *For every $\varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, we have*

$$\frac{1}{2}\frac{d}{dt}\left(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2]\right)^2 \le (\dot\varepsilon_1 - \dot\varepsilon_2)(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2]) \quad a.e.\ in\ ]0,T[, \tag{7}$$

$$\int_0^T \left|\frac{d}{dt}(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])\right|(t)\, dt \le |\varepsilon_1(0) - \varepsilon_2(0)| + 2\int_0^T |\dot\varepsilon_1 - \dot\varepsilon_2|(t)\, dt, \tag{8}$$

$$|(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])(t)| \le 2 \max_{0\le\tau\le t} |\varepsilon_1(\tau) - \varepsilon_2(\tau)| \quad \forall t \in [0,T]. \tag{9}$$

(iii) *For every $r, q > 0$ and $\varepsilon \in W^{1,1}(0,T)$, we have*

$$|(\mathfrak{s}_r[\varepsilon] - \mathfrak{s}_q[\varepsilon])(t)| \le |r - q| \quad \forall t \in [0,T]. \tag{10}$$

The inequalities (8), (9) entail that the stop operator $\mathfrak{s}_r$ is Lipschitz continuous in $W^{1,1}(0,T)$ and admits a Lipschitz continuous extension onto $C([0,T])$. Moreover, we immediately see by definition that $\mathfrak{s}_r$ is a *causal* operator, that is, we have the implication

$$\varepsilon_1(\tau) = \varepsilon_2(\tau) \quad \forall \tau \in [0,t] \qquad \Rightarrow \qquad \mathfrak{s}_r[\varepsilon_1](t) = \mathfrak{s}_r[\varepsilon_2](t) \tag{11}$$

for every $t \in [0,T]$, which means that the output values at time t depend only on past values of the input. This enables us to consider $\mathfrak{s}_r$ as a family of operators acting in the spaces $C([0,t])$ for all $t \in]0,T]$.

From inequality (7) it immediately follows:

Corollary 2. *For all $\varepsilon, \varepsilon_1, \varepsilon_2 \in W^{1,1}(0,T)$, we have*

$$\mathfrak{s}_r[\varepsilon]\left(\dot{\varepsilon} - \frac{d}{dt}\mathfrak{s}_r[\varepsilon]\right) \geq 0 \quad a.e. \ \ in \ \ ]0,T[\quad (energy\ inequality)\ , \tag{12}$$

$$|(\mathfrak{s}_r[\varepsilon_1] - \mathfrak{s}_r[\varepsilon_2])(t)| \leq |\varepsilon_1(0) - \varepsilon_2(0)| + \int_0^t |\dot{\varepsilon}_1 - \dot{\varepsilon}_2|(\tau)\, d\tau \quad \forall t \in [0,T]. \tag{13}$$

In this paper we consider the one-dimensional equation of motion

$$\rho\, u_{tt} = \sigma_x + f, \tag{14}$$

where $\rho > 0$ is a constant referential density, u is the displacement, σ is the total unaxial stress and f is the volume force density.

We assume that σ can be decomposed into the sum

$$\sigma = \sigma^p + \sigma^e + \sigma^v + \sigma^d, \tag{15}$$

where

$$\sigma^e = \gamma(\varepsilon), \tag{16}$$

with a given nondecreasing Lipschitz continuous function $\gamma : \mathbb{R}^1 \to \mathbb{R}^1$, $\gamma(0) = 0$, is the (nonlinear) kinematic hardening component,

$$\sigma^v = \mu\dot{\varepsilon} \tag{17}$$

with a constant $\mu > 0$ is the viscous component,

$$\sigma^d = -\beta\theta \tag{18}$$

with a constant $\beta \in \mathbb{R}^1$ is the thermic dilation component and σ^p is the thermoplastic component given by (1). Equation (15) can be interpreted rheologically as a combination in parallel of the above components (see [9]). The stop operator $\mathfrak{s}_r$ is assumed to act on functions of x and t according to the formula

$$\mathfrak{s}_r[\varepsilon](x,t) := \mathfrak{s}_r[\varepsilon(x,\cdot)](t), \tag{19}$$

i.e. x plays the role of a parameter. The equation of motion (14) has to be coupled with the energy balance equation

$$U_t = \sigma \varepsilon_t - q_x + g, \tag{20}$$

where U is the total internal energy, q is the heat flux and g is the heat source density. The model is thermodynamically consistent provided the temperature θ and the entropy S satisfy the inequalities

$$\theta > 0, \tag{21}$$

$$S_t \ge \frac{g}{\theta} - \left(\frac{q}{\theta}\right)_x \qquad \text{(Clausius-Duhem inequality)}, \tag{22}$$

in an appropriate sense.

In [7] we derived the following expressions for thermoplastic parts of internal energy U^p and entropy S^p in operator form corresponding to the constitutive law (1),

$$U^p = \mathcal{V}[\varepsilon, \theta] := \frac{1}{2}\int_0^\infty (\varphi(r,\theta) - \theta\varphi_\theta(r,\theta))\, \mathfrak{s}_r^2[\varepsilon]\, dr, \tag{23}$$

$$S^p = \mathcal{S}[\varepsilon, \theta] := -\frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\, \mathfrak{s}_r^2[\varepsilon]\, dr. \tag{24}$$

In accordance with (15), (23), (24), we put

$$U := C_V\theta + \mathcal{V}[\varepsilon,\theta] + \Gamma(\varepsilon) + V_0, \tag{25}$$

$$S := C_V \log\theta + \mathcal{S}[\varepsilon,\theta] + \beta\varepsilon, \tag{26}$$

where $C_V > 0$, the purely caloric part of the *specific heat*, is a constant, $V_0 > 0$ is a constant which is chosen in order to ensure that $U \ge 0$ according to Hypothesis (H2) below, and $\Gamma(\varepsilon) := \int_0^\varepsilon \gamma(s) ds$. For the heat flux we assume Fourier's law

$$q = -\kappa\theta_x \tag{27}$$

with a constant heat conduction coefficient $\kappa > 0$. We complete the system (14), (20) with the small deformation hypothesis

$$\varepsilon = u_x \tag{28}$$

and rewrite it in the form

$$\rho u_{tt} - (\gamma(u_x) + \mathcal{P}[u_x,\theta] + \mu u_{xt} - \beta\theta)_x = f, \tag{29}$$

$$(C_V\theta + \mathcal{V}[u_x,\theta])_t - \kappa\theta_{xx} = (\mathcal{P}[u_x,\theta] + \mu u_{xt} - \beta\theta)\, u_{xt} + g. \tag{30}$$

3 Existence, uniqueness and thermodynamic consistency

We consider a model problem for a system of the form (29), (30), namely

$$u_{tt} - \big(\gamma(u_x)\big)_x - (\mathcal{P}[u_x,\theta])_x - \mu u_{xxt} + \beta\theta_x = f(\theta,x,t), \tag{31}$$

$$(C_V\theta + \mathcal{V}[u_x,\theta])_t - \theta_{xx} = \mathcal{P}[u_x,\theta]u_{xt} + \mu u_{xt}^2 - \beta\theta u_{xt} + g(\theta,x,t), \tag{32}$$

for $x \in]0,1[$, $t \in [0,T]$, where $T > 0$, $\mu > 0$, $C_V > 0$, $\beta \in \mathbb{R}^1$ are fixed constants, $\gamma\colon \mathbb{R}^1 \to \mathbb{R}^1$, $f,g\colon]0,\infty[\times]0,1[\times[0,T] \to \mathbb{R}^1$ are given functions, and $\mathcal{P}, \mathcal{V}$ are the operators defined by (1), (23) with a given distribution function $\varphi\colon (]0,\infty[)^2 \to [0,\infty[$ satisfying Hypothesis (H2) below.

In other words, we assume in (29), (30) that the volume force and heat source densities are given functions of x and t which may also depend on the instantaneous value of θ, and we rescale the units in such a way that $\rho \equiv \kappa \equiv 1$. The system (31), (32) is coupled with boundary and initial conditions which are chosen in the following simple form.

$$u(0,t) = u(1,t) = \theta_x(0,t) = \theta_x(1,t) = 0, \tag{33}$$

$$u(x,0) = u^0(x),\ u_t(x,0) = u^1(x),\ \theta(x,0) = \theta^0(x). \tag{34}$$

The data are assumed to satisfy the following hypotheses.

Hypothesis (H1).

(i) $u^0, u^1 \in W^{2,2}(0,1) \cap \overset{\circ}{W}{}^{1,2}(0,1)$, $\theta^0 \in W^{1,2}(0,1)$, and there exists a constant $\delta > 0$ such that

$$\theta^0(x) \geq \delta \qquad \forall x \in [0,1]. \tag{35}$$

(ii) $\gamma\colon \mathbb{R}^1 \to \mathbb{R}^1$ is an absolutely continuous function, $\gamma(0) = 0$, and there exists a constant $\gamma_0 > 0$ such that

$$0 \leq \frac{d\gamma(\varepsilon)}{d\varepsilon} \leq \gamma_0 \qquad \text{a.e. in } \mathbb{R}^1. \tag{36}$$

(iii) The functions f, g are measurable, $f(\cdot,x,t)$, $g(\cdot,x,t)$ are absolutely continuous in $[0,\infty[$ for a.e. $(x,t) \in]0,1[\times]0,T[$. Moreover, there exist a constant $K > 0$ and functions $f_0, g_0 \in L^2(]0,1[\times]0,T[)$ such that

$$g(0,x,t) = g_0(x,t) \geq 0 \qquad a.e., \tag{37}$$

$$|f(\theta,x,t)| + |f_t(\theta,x,t)| \leq f_0(x,t) \quad a.e., \tag{38}$$

$$|f_\theta(\theta,x,t)| + |g_\theta(\theta,x,t)| \leq K \quad a.e.\ . \tag{39}$$

Hypothesis (H2).

The function φ: $(]0,\infty[)^2 \to [0,\infty[$ is measurable, $\varphi(r,\cdot)$, $\varphi_\theta(r,\cdot)$ are absolutely continuous for a.e. $r > 0$, and there exist constants $L > 0$, $V_0 > 0$ such that for a.e. $\theta > 0$ the following inequalities hold.

$$\int_0^\infty \varphi(r,\theta)\,dr \le L, \tag{40}$$

$$\int_0^\infty |\varphi_\theta(r,\theta)|\,r\,dr \le L, \tag{41}$$

$$\int_0^\infty \theta\,|\varphi_{\theta\theta}(r,\theta)|\,r^2\,dr \le C_V, \tag{42}$$

where C_V is the constant introduced in (25),

$$\frac{1}{2}\int_0^\infty |\varphi(r,\theta) - \theta\varphi_\theta(r,\theta)|\,(1+r^2)\,dr \ \le\ V_0. \tag{43}$$

Example 3. A typical function φ satisfying Hypothesis (H2) can be chosen as

$$\varphi(r,\theta) = \bar{E}(\theta)\ c(r - \bar{r}(\theta)), \tag{44}$$

where $c \in \mathcal{D}(]-m,m[)$ is a mollifier such that

$$\int_{-m}^{m} c(s)\,ds = 1, \qquad c \ge 0, \tag{45}$$

with a (small) constant $m > 0$, and $\bar{E}, \bar{r}$ are given functions such that $\bar{E}(\theta) \le L$, $m \le \bar{r}(\theta) \le R$, for some constant $R \ge m$, with $(1+\theta)\left(|\bar{E}'(\theta)| + |\bar{r}'(\theta)|\right)$ bounded and $\theta\left(|\bar{E}''(\theta)| + |\bar{r}''(\theta)| + \bar{E}^{'2}(\theta) + \bar{r}^{'2}(\theta)\right)$ small, uniformly with respect to θ.

The existence result in [8] is stated as follows.

Theorem 4. *Let Hypotheses (H1), (H2) hold. Then there exists a unique solution (u,θ) to the problem (31)–(34) such that*

$$u_{tt}, u_{xx}, u_{xxt}, \theta_x \in L^\infty(0,T;\,L^2(0,1)), \tag{46}$$

$$u_{xtt}, \theta_t, \theta_{xx} \in L^2(]0,1[\times]0,T[), \tag{47}$$

$$\theta, u, u_x, u_t, u_{xt} \in C([0,1]\times[0,T]). \tag{48}$$

In addition, there exists a constant $c_0 > 0$ depending only on the given data such that for all $t \in [0,T]$ and $x \in [0,1]$ we have

$$\theta(x,t) \ge \delta e^{-c_0 t} > 0, \tag{49}$$

and (31)–(34) are satisfied almost everywhere.

We first check that the model is thermodynamically consistent according to (21), (22).

Corollary 5. *The solution from Theorem 4 satisfies the Clausius-Duhem inequality (22) with S defined by (26), (24) almost everywhere in* $]0,1[\times]0,T[$.

Proof of Corollary 5. For a.e. x and t we have

$$\begin{aligned} &\theta S_t + \theta \left(\frac{q}{\theta}\right)_x - g \qquad (50)\\ &= C_V \theta_t + \theta \left(\mathcal{S}[u_x,\theta]\right)_t + \beta\theta u_{xt} - \theta_{xx} - g + \frac{1}{\theta}\theta_x^2 \\ &= -\left(\mathcal{V}[u_x,\theta]\right)_t + \theta\left(\mathcal{S}[u_x,\theta]\right)_t + \mathcal{P}[u_x,\theta]\,u_{xt} + \mu u_{xt}^2 + \frac{1}{\theta}\theta_x^2 \\ &= \int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r[u_x]\,(u_x - \mathfrak{s}_r[u_x])_t\,dr + \mu u_{xt}^2 + \frac{1}{\theta}\theta_x^2, \end{aligned}$$

and the assertion follows from (12). □

The solutions to (31)–(34) are unique and depend continuously on the data in the following way, see [8].

Theorem 6. *Let Hypotheses (H1)(ii), (H2) hold, let* (u^0,u^1,θ^0,f,g), $(u'^0,u'^1,\theta'^0,f',g')$ *be two sets of given functions satisfying Hypothesis (H1), and let* (u,θ), (u',θ') *be solutions of (31) – (34) corresponding to these data, respectively, which satisfy (46) – (49). Assume moreover that there exist a constant* $\tilde{K} > 0$ *and functions* $d_f, d_g \in L^2(]0,1[\times]0,T[)$ *such that*

$$|f(\theta_1,x,t) - f'(\theta_2,x,t)| \le \tilde{K}|\theta_1-\theta_2| + d_f(x,t), \qquad (51)$$

$$|g(\theta_1,x,t) - g'(\theta_2,x,t)| \le \tilde{K}|\theta_1-\theta_2| + d_g(x,t), \qquad (52)$$

holds for all $\theta_1,\theta_2 \in \mathbb{R}^+$ *and a.e.* $(x,t) \in]0,1[\times]0,T[$.

Then there exists a constant C depending only on the size of the data in their respective spaces such that for all $t\in[0,T]$ *the differences* $\bar{u} = u-u'$, $\bar{\theta} = \theta - \theta'$, *satisfy*

$$\begin{aligned} &\|\bar{u}_t(t)\|^2 + \int_0^t \left(\|\bar{\theta}\|^2 + \|\bar{u}_{xt}\|^2\right)(\tau)\,d\tau \qquad (53)\\ &\le C\left(\|\bar{u}_t(0)\|^2 + \|\bar{u}_x(0)\|^2 + \|\bar{\theta}(0)\|^2 + \int_0^t\int_0^1 (d_f^2 + d_g^2)\,dx\,dt\right). \end{aligned}$$

The proofs of the above theorems depend substantially on the following properties of the hysteresis operators $\mathcal{P}$ and $\mathcal{V}$.

Proposition 7. *Let Hypothesis (H2) hold. Then the operators $\mathcal{P},\mathcal{V}$ are causal and have the following properties.*

(i) *For every* $\varepsilon, \theta \in W^{1,1}(0,T)$, $\theta > 0$, *we have*

$$|\mathcal{P}[\varepsilon,\theta](t)| \le V_0, \quad |\mathcal{V}[\varepsilon,\theta](t)| \ \le \ V_0, \tag{54}$$

$$\left|\frac{d}{dt}\mathcal{P}[\varepsilon,\theta](t)\right| \le L\left(|\dot\varepsilon(t)| + |\dot\theta(t)|\right), \qquad a.e. \ \ in \ ]0,T[. \tag{55}$$

(ii) *For every* $\varepsilon, \varepsilon_2, \theta_1, \theta_2 \in W^{1,1}(0,T)$, $\theta_1 > 0$, $\theta_2 > 0$ *and for every* $t \in [0,T]$, *we have*

$$|\mathcal{P}[\varepsilon_1,\theta_1] - \mathcal{P}[\varepsilon_2,\theta_2]|\,(t) \le L\left(|\theta_1 - \theta_2|(t) + 2\max_{0\le\tau\le t}|\varepsilon_1 - \varepsilon_2|(\tau)\right), \tag{56}$$

$$|\mathcal{V}[\varepsilon_1,\theta_1] - \mathcal{V}[\varepsilon_2,\theta_2]|\,(t) \le \frac{C_V}{2}|\theta_1 - \theta_2|(t) + 2V_0\max_{0\le\tau\le t}|\varepsilon_1 - \varepsilon_2|(\tau). \tag{57}$$

4 Asymptotic behavior

The system (31)–(34) exhibits multiple sources of energy dissipation (plasticity, viscosity, heat conduction) and it is quite justified to expect that under vanishing external forcing, that is $f \equiv g \equiv 0$, the velocity and the temperature gradient should asymptotically vanish as $t \to \infty$. This will certainly not be true for the strain because of the existence of remanent plastic deformations, cf. Section III.2 of [6] for the temperature-independent case. It turns out however that, here again, the problem is more difficult than in the case without hysteresis, due to the lack of smoothness of hysteresis operators. Below we prove that in fact, the velocity tends to 0 in L^2 as $t \to \infty$, but no asymptotics is known for the velocity gradient and for the temperature. The exact result reads as follows.

Theorem 8. *Let the hypotheses of Theorem 4 be satisfied. Assume moreover that* $\gamma(\varepsilon) = \gamma_0\,\varepsilon$ *for some* $\gamma_0 \ge 0$ *and that* $f(\theta,x,t) = g(\theta,x,t) = 0$ *for all* $\theta > 0$ *and (a.e.)* $x \in]0,1[$, $t > 0$. *Then the solution* (u,θ) *of (31)–(34) satisfies*

$$\lim_{t\to\infty}\int_0^1 u_t^2(x,t)\,dx = 0. \tag{58}$$

Proof. In the sequel, we denote by C_1, C_2, ... constants depending only on the initial conditions.

Step 1. Multiply (31) by u_t, add the result to (32) and integrate with respect to x over $]0,1[$. This yields the global energy balance identity

$$\frac{d}{dt}\int_0^1 \big(\frac{1}{2}u_t^2 + \frac{\gamma_0}{2}u_x^2 + C_V\,\theta + \mathcal{V}[u_x,\theta]\big)(x,t)\,dx \ = \ 0. \tag{59}$$

Step 2. Let us multiply (32) by $-1/\theta$. We rewrite the result in the form

$$\left(-C_V\,\log\,\theta + \frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\,\mathfrak{s}_r^2[u_x]\,dr\right)_t + \frac{1}{\theta}\big(\theta_{xx} + \mu\,u_{xt}^2\big)$$
$$+\frac{1}{\theta}\int_0^\infty \varphi(r,\theta)\,\mathfrak{s}_r[u_x]\big(u_x - \mathfrak{s}_r[u_x]\big)_t\,dr + \beta\,u_{xt} \;=0, \tag{60}$$

and integrating with respect to x and t we obtain from (12) that

$$\int_0^1 \left(-C_V\,\log\,\theta + \frac{1}{2}\int_0^\infty \varphi_\theta(r,\theta)\,\mathfrak{s}_r^2[u_x]\,dr\right)(x,t)\,dx$$
$$+\int_0^t\int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \mu\frac{u_{xt}^2}{\theta}\right)dx\,dt \;\le\; C_1. \tag{61}$$

The left-hand side can be estimated using the relations

$$\int_0^\infty \varphi_\theta(r,\theta)\,\mathfrak{s}_r^2[u_x]\,dr \;=\; \int_0^\infty \Big[\big(\varphi_\theta(r,\theta) - \varphi_\theta(r,1)\big)$$
$$+\;\big(\varphi_\theta(r,1) - \varphi(r,1)\big) + \varphi(r,1)\Big]\,\mathfrak{s}_r^2[u_x]\,dr$$
$$\ge\; -\int_0^\infty \Big[\big|\varphi_\theta(r,\theta) - \varphi_\theta(r,1)\big| + \big|\varphi_\theta(r,1) - \varphi(r,1)\big|\Big]\,r^2\,dr. \tag{62}$$

From Hypothesis (H2) it follows that

$$\int_0^\infty \big|\varphi_\theta(r,1) - \varphi(r,1)\big|\,r^2\,dr \;\le\; 2V_0, \tag{63}$$

$$\int_0^\infty \big|\varphi_\theta(r,\theta) - \varphi_\theta(r,1)\big|\,r^2\,dr \;\le\; \left|\int_1^\theta\int_0^\infty |\varphi_{\theta\theta}(r,\theta')|\,dr\,d\theta'\right| \;\le\; C_V\,|\log\,\theta|. \tag{64}$$

Using the trivial inequality

$$|\log\,\theta| \;\le\; \max\{\theta, -\log\,\theta\}, \tag{65}$$

and the estimate

$$\int_0^1 \theta(x,t)\,dx \;\le\; C_2, \tag{66}$$

which follows from (59), we conclude that

$$\int_0^1 \big|\log\,\theta(x,t)\big|\,dx \;+\; \int_0^t\int_0^1 \left(\frac{\theta_x^2}{\theta^2} + \frac{u_{xt}^2}{\theta}\right)dx\,dt \;\le\; C_3. \tag{67}$$

Step 3. For every x and t we have

$$|u_t(x,t)| \;\le\; \int_0^1 |u_{xt}(\xi,t)|\,d\xi \;\le\; \int_0^1 \frac{|u_{xt}|}{\sqrt{\theta}}\sqrt{\theta}\,d\xi \;\le\; \sqrt{C_2}\left(\int_0^1 \frac{u_{xt}^2}{\theta}\,d\xi\right)^{1/2}, \tag{68}$$

hence

$$\int_0^t \max_x |u_t(x,\tau)|^2 \, d\tau \;\leq\; C_4 . \tag{69}$$

Step 4. Analogously, for every x, y and t we have

$$\begin{aligned} \sqrt{\theta(x,t)} \;&\leq\; \sqrt{\theta(y,t)} \;+\; \frac{1}{2} \int_0^1 \frac{|\theta_x|}{\sqrt{\theta}}(\xi,t)\, d\xi \\ &\leq\; \sqrt{\theta(y,t)} \;+\; \frac{1}{2} \left(C_2 \int_0^1 \frac{\theta_x^2}{\theta^2}(\xi,t)\, d\xi \right)^{1/2} , \end{aligned} \tag{70}$$

hence

$$\max_x \theta(x,t) \;\leq\; C_5 \left(1 \;+\; \int_0^1 \frac{\theta_x^2}{\theta^2}(\xi,t)\, d\xi \right) . \tag{71}$$

Step 5. Multiply (31) by u_t and integrate over x. We obtain from (54)

$$\frac{1}{2} \frac{d}{dt} \int_0^1 u_t^2(x,t)\, dx + \mu \int_0^1 u_{xt}^2(x,t)\, dx \;\leq\; \int_0^1 \big(V_0 + |\beta|\,\theta + \gamma_0\,|u_x|\big)\, |u_{xt}|(x,t)\, dx, \tag{72}$$

and Hölder's inequality together with (59), (71) leads to the estimate

$$\begin{aligned} \frac{d}{dt} \int_0^1 u_t^2(x,t)\, dx + \int_0^1 u_{xt}^2(x,t)\, dx \;&\leq\; C_6 \left(1 \;+\; \int_0^1 \theta^2(x,t)\, dx \right) \\ &\leq\; C_7 \left(1 \;+\; \max_x \theta(x,t) \right) \\ &\leq\; C_8 \left(1 \;+\; \int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\, dx \right) . \end{aligned} \tag{73}$$

Step 6. For $t > 0$ put

$$y(t) := \int_0^1 u_t^2(x,t)\, dx, \quad h(t) := C_8 \int_0^1 \frac{\theta_x^2}{\theta^2}(x,t)\, dx. \tag{74}$$

By (69), (67) we have

$$\int_0^t y(\tau)\, d\tau \;\leq\; C_4, \quad \int_0^t h(\tau)\, d\tau \;\leq\; C_8\, C_3, \tag{75}$$

and (73) can be rewritten in the form

$$\dot{y}(t) \;+\; y(t) \;\leq\; C_8 + h(t) \quad \text{a.e.} \tag{76}$$

To complete the proof of Theorem 8, it suffices to use Theorem 9 below. □

5 Appendix: A differential inequality

We prove here the following general convergence result for differential inequalities which extends Lemma 3.1 of [12].

Theorem 9. *Let a nondecreasing function* $f : [0,\infty[\to]0,\infty[$*, an absolutely continuous function* $y : [0,\infty[\to [0,\infty[$ *and a function* $h \in L^1(0,\infty)$*,* $h \geq 0$ *a.e. be given. Assume that*

$$\int_0^\infty y(t)\,dt = Y < \infty, \qquad \int_0^\infty h(t)\,dt = H < \infty, \tag{77}$$

$$\dot{y}(t) \leq f(y(t)) + h(t) \quad \text{a.e.}, \tag{78}$$

where the dot denotes derivative with respect to t*. Then* $\lim_{t\to\infty} y(t) = 0$*.*

If moreover there exist constants $A, B \geq 0$ *such that* $f(y) \leq Ay^2 + B$ *for every* $y \geq 0$*, then*

$$y(t) \leq \begin{cases} e^{AY}(y(0) + H + B) & \text{for } \ t < 1, \\ e^{AY}(Y + H + B/2) & \text{for } \ t \geq 1. \end{cases} \tag{79}$$

The following example shows that we cannot expect any a priori pointwise bound for $y(t)$ if $f(y)$ grows faster than y^2.

Example 10. Let $\varepsilon \in]0,1[$ be given. For $n > 1$ put

$$y_n(t) = \begin{cases} |t-1|^{\varepsilon-1} & \text{for } \ t \in [0,2] \setminus\ ]1-1/n, 1+1/n[, \\ n^{1-\varepsilon} & \text{for } \ t \in [1-1/n, 1+1/n], \\ e^{2-t} & \text{for } \ t > 2. \end{cases} \tag{80}$$

Then y_n are absolutely continuous, $\int_0^\infty y_n(t)\,dt \leq 1 + 2/\varepsilon$, $y_n(0) = 1$, $\dot{y}_n(t) \leq (1-\varepsilon) y_n^{2+\varepsilon/(1-\varepsilon)}(t)$ a.e., and the sequence $\{y_n(1)\}$ is unbounded.

Proof of Theorem 9. Assume that there exists $\alpha > 0$ and a sequence $t_n \uparrow \infty$ such that

$$y(t_n) \geq 2\alpha \quad \forall n \in \mathbb{N}. \tag{81}$$

We may assume (selecting a subsequence, if necessary) that the inequality

$$t_{n+1} - t_n > \frac{2Y}{\alpha} + \beta, \tag{82}$$

holds for every $n \in \mathbb{N}$, where

$$\beta := \frac{\alpha}{2f(2\alpha)}, \quad t_1 > \frac{Y}{\alpha}. \tag{83}$$

By (77), the sets

$$A_n := \{t \in\ ]t_n - \frac{Y}{\alpha}, t_n[\ : y(t) < \alpha\} \tag{84}$$

are nonempty and we may put for all $n \in \mathbb{N}$

$$a_n := \sup A_n, \tag{85}$$

and similarly

$$b_n := \inf\{t \in \,]t_n, t_n + \frac{Y}{\alpha}[\, : y(t) < \alpha\}, \tag{86}$$

$$s_n := \min\{t \in [a_n, b_n] : y(t) \geq 2\alpha\}. \tag{87}$$

By construction we have for all $n \in \mathbb{N}$

$$a_n < s_n \leq t_n < b_n < a_{n+1}, \tag{88}$$

$$a_{n+1} - b_n > \beta, \tag{89}$$

$$y(a_n) = y(b_n) = \alpha, \quad y(s_n) = 2\alpha, \quad y(t) \geq \alpha \;\; \forall t \in [a_n, b_n]. \tag{90}$$

We now define an auxiliary function z by the formula

$$z(t) := \begin{cases} y(t) - \alpha & \text{for } \;t \in \bigcup_{n=1}^{\infty} [a_n, b_n], \\ 0 & \text{otherwise.} \end{cases} \tag{91}$$

Then z is nonnegative, absolutely continuous, and for a.e. $t > 0$ we have

$$\dot{z}(t) \leq f(z(t) + \alpha) + h(t), \quad z(t) \leq y(t). \tag{92}$$

Moreover, for $t \in [s_n - \beta, s_n]$ we have

$$z(t) \leq \alpha, \tag{93}$$

and integrating (92) from t to s_n we obtain

$$\begin{aligned} \alpha - z(t) &\leq \int_t^{s_n} \big(f(z(\tau) + \alpha) + h(\tau)\big) d\tau \\ &\leq \beta f(2\alpha) + \int_{s_n - \beta}^{s_n} h(\tau)\, d\tau. \end{aligned} \tag{94}$$

For all $t \in [s_n - \beta, s_n]$ we therefore have

$$\frac{\alpha}{2} \leq z(t) + \int_{s_n - \beta}^{s_n} h(\tau)\, d\tau, \tag{95}$$

and integrating once more we conclude that

$$\frac{1}{2}\alpha\beta \leq \int_{s_n - \beta}^{s_n} \big(z(\tau) + \beta h(\tau)\big)\, d\tau \quad \forall n \in \mathbb{N}, \tag{96}$$

which is a contradiction, since both z and h are integrable and the intervals $]s_n - \beta, s_n[$ are pairwise disjoint.

To prove (79), it suffices to rewrite (78) in the form

$$\frac{d}{dt}\left(y(t)e^{-A\int_0^t y(\tau)\,d\tau}\right) \le \big(B+h(t)\big)e^{-A\int_0^t y(\tau)\,d\tau}, \tag{97}$$

hence for every $0 \le s < t$ we have

$$\begin{aligned} y(t) &\le y(s)e^{A\int_s^t y(\tau)\,d\tau} + \int_s^t \big(B+h(\tau)\big)e^{A\int_\tau^t y(\sigma)\,d\sigma}d\tau \\ &\le e^{AY}\big(y(s)+H+B(t-s)\big). \end{aligned} \tag{98}$$

For $t \le 1$ we simply put $s = 0$, for $t \ge 1$ we integrate (98) with respect to s from $t-1$ to t. □

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