# Image Representations via a Finite Radon Transform

František Matúš and Jan Flusser

*Abstract*—**This paper presents a model of finite Radon transforms composed of Radon projections. The model generalizes to finite group projections in the classical Radon transform theory. The Radon projector averages a function on a group over cosets of a subgroup. Reconstruction formulae that were formally similar to the convolved backprojection ones are derived, and an iterative reconstruction technique is found to converge after a finite number of steps. Applying these results to the group $Z_p^2$, new computationally favorable image representations have been obtained. A numerical study of the transform coding aspects is attached.**

*Index Terms*—**Finite Radon transform, Fourier transform, image compression, image representations, iterative reconstruction, Radon projections, transform coding.**

## I. Introduction

DEVELOPMENT of various representations of image data continues to be an area of active research. It has had substantial impact on many image processing and image analysis problems including manipulation, compression, pattern recognition, coding, computer vision, etc. The most common manner of image representation is to use an energy-preserving transform. Classical and thoroughly investigated examples include Fourier, cosine, sine, Hadamard, Haar, and other unitary transforms [15], [5].

The Radon transform entered the center of interest with its first radioastronomic and tomographic applications [8] and spread quickly into many fields. Recall that the Radon transform (this notion is from integral geometry, cf. [7], [11], [18]) of a real function $f$ defined on a Euclidean plane is the function $\hat{f}$, which is defined on the family of all lines in the plane having the value $\hat{f}(p)$ equal to the line integral along the line $p$. It is often advantageous to view the Radon transform as the family $(\hat{f}_q)$ of projections where the projection $\hat{f}_q$ is the restriction of $\hat{f}$ on the set of all lines parallel to the line $q$ containing the origin.

Numerous discretizations of the Radon transforms employed in practice have been devised. For example, tomographic data can also be arranged into a finite subfamily of sampled projections. The utilization of the discretized Radon transforms penetrates the range of image analysis and processing techniques (for an account, see [17]). Projection space representations and manipulations of digital images have given rise to new algorithms and have opened new possibilities. A substantial role in this progress has been played by the parallel pipeline projection engine ($P^3E$) computer architecture. This is even sometimes compared with the role of the FFT algorithm supporting the Fourier transform.

During the last decade, elements of a finite Radon transform theory have appeared mainly in the field of combinatorics [2], [4], [12], [18]. The fundamental concept due to Bolker is as follows. The finite Radon transform of a real function $f$ defined on a finite set $S$ (with respect to a collection $\mathcal{T}$ of its subsets: blocks) is the function $\hat{f}$ on $\mathcal{T}$, the values of which are obtained by summing (we prefer averaging) $f$ over the blocks. It seems quite impossible to say anything about this transform without additional conditions on the collection $\mathcal{T}$; however, when considered on special combinatorial or algebraical structures like designs, matroids, lattices, groups, etc., some results have been attained (for a review, see [12]). There seem to be very few resemblences to the classical Radon transform theory.

The main goal of our paper is to introduce and investigate a scheme of finite Radon transforms on groups, to present new results on finite Radon transforms on finite Euclidean planes, and to discuss corresponding projection representations of images and their applications. Hence, we are not concerned with the *discretized* Radon transforms but with the *discrete, finite* ones.

Accordingly, the paper is divided into three parts.

First, we investigate Radon transforms viewed as families of Radon projections on finite groups: In Bolker's setting, $S$ is taken to be a group and $\mathcal{T}$ a quotient group of $S$ getting the Radon projection $f \rightarrow \hat{f}$. The reconstruction from projections is the main theme explored here. Analogies with the classical, analytical situation appear to be much deeper than one can expect. Even an inversion of this Radon transform can be written in the form of filtered backprojection (cf. Theorem 1). Moreover, an iterative reconstruction technique (cf. Section II-E.) akin to Kacmarz's [17] converges after a finite number of steps (which is equal to the number of the Radon projections; see Theorem 3).

Second, we translate these results into the additive groups $Z_p^2$ of Euclidean planes. A discrete version of projection slice theorems connecting the finite Radon and finite Fourier transforms is presented. We also examine Radon transforms composed of partial projections, i.e., of the projections in the above sense admitting that $\mathcal{T}$ is an arbitrary subset of a quotient group.

Third, image representations via aforementioned finite Radon transforms are suggested, and their numerical aspects are discussed. Similarly, as with other transforms, maximum information is packed into a small number of samples. In this way, image compression is achieved by the storage of the most informative samples only (this technique, which is

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The authors are with the Institute of Information Theory and Automation, Academy of Sciences of Czech Republic, Prague, Czech Republic.

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called *transform coding*, is described for other transforms in [1], [5], [10], and [16]). In Section IV, compression results are visualized, the efficiency of the compression algorithms is discussed, and the feasibility of real-time implementation through the P$^3$E architecture is approved.

## II. Finite Radon Transforms on Groups

Let $G$ be a finite group and $\mathbf{A}(G)$ be the linear algebra of all complex functions on $G$, i.e., $\mathbf{A}(G)$ is a $|G|$-dimensional complex linear space with the convolution of two functions $f_1,\ f_2 \in \mathbf{A}(G)$ (see [3])

$$f_1 * f_2(x) = |G|^{-1} \sum_{y\in G} f_1(y) f_2(y^{-1}x)$$

as the product. We start with simple notations and technicalities.

### A. Radon Projections

The Radon projection of a function $f \in \mathbf{A}(G)$ along a normal subgroup $H$ of $G$ is defined in [14] to be the function $\Lambda_H f$ on the quotient group $G/H$ given by

$$\Lambda_H f(xH) = |H|^{-1} \sum_{y\in xH} f(y), \quad xH \in G/H.$$

The values of the Radon projection are thus equal to the averages of the function $f$ over the cosets $xH = \{xy;\ y \in H\}$.

We shall frequently use an analogy of the Euclidean plane backprojection. The backprojection of a function $g \in \mathbf{A}(G/H)$ along $H$ is the function $V_H g \in \mathbf{A}(G)$ that is constant and equal to $g(xH)$ on every coset $xH$.

Considering this scalar product on $\mathbf{A}(G)$

$$\langle f_1, f_2\rangle = |G|^{-1} \sum_{x\in G} f_1(x)\overline{f_2(x)}, \quad f_1, f_2 \in \mathbf{A}(G)$$

one can easily verify the duality of the operators $\Lambda_H$ and $V_H$

$$\langle \Lambda_H f, g\rangle = \langle f, V_H g\rangle, \quad f \in \mathbf{A}(G), g \in \mathbf{A}(G/H).$$

We shall write out a collection of simple assertions on the Radon projections and backprojections below. Proofs are omitted because of their triviality. The $|G/H|$ multiple of the indicator function of $H$ is denoted by $\lambda_H$.

**Lemma 1:**

1. $\Lambda_H V_H g = g, \quad g \in \mathbf{A}(G/H),$
2. $V_H \Lambda_H f = f * \lambda_H, \quad f \in \mathbf{A}(G),$
3. $\Lambda_H(f_1 * f_2) = \Lambda_H f_1 * \Lambda_H f_2, \quad f_1, f_2 \in \mathbf{A}(G),$
4. $V_H(g_1 * g_2) = V_H g_1 * V_H g_1, \quad g_1, g_2 \in \mathbf{A}(G/H),$
5. $V_H g * f = V_H(g * \Lambda_H f), \quad f \in \mathbf{A}(G), g \in \mathbf{A}(G/H).$

### B. Reconstruction from Projections

Let $\mathcal{H} = (H_i)_{i\in N}$ be a nonvoid and finite family of normal subgroups of $G$ (possibly $H_i = H_j$ for $i \neq j,\ i,j \in N$). Instead of $\Lambda_{H_i}, V_{H_i}$, and $\lambda_{H_i}$, we prefer to write $\Lambda_i$, $V_i$, and $\lambda_i$, respectively. Now, our finite Radon transform on finite groups will be defined in accordance with the classical theory as a family of Radon projections.

**Definition:** The operator $\Lambda : \mathbf{A}(G) \to \mathbf{A}(\mathcal{H})$ given by $\Lambda f = (\Lambda_i f)_{i\in N}$ is called the *finite Radon transform* on $G$ with respect to $\mathcal{H}$.

We remark that the symbol $\mathbf{A}(\mathcal{H})$ denotes here the product of the family $(\mathbf{A}(G/H_i))_{i\in N}$ of linear algebras (operations $+$ and $*$ are performed in $\mathbf{A}(\mathcal{H})$ coordinatewise). All the preliminaries for the precise formulation of our basic problem are now available.

**Reconstruction Problem:** Find the necessary and sufficient conditions for solvability of the equation $\Lambda f = g$ with the given right-hand side $g \in \mathbf{A}(\mathcal{H})$ and find all solutions.

In other words, we should like to solve the system of equations

$$\Lambda_i f = g_i, \quad i \in N$$

where $g_i \in \mathbf{A}(G|H_i),\ i \in N$, and $g = (g_i)_{i\in N}$. This is, however, equivalent to

$$f * \lambda_i = V_i g_i, \quad i \in N.$$

In fact, if $f$ satisfies the first set of equations, then applying $V_i$ and claim 2 of Lemma 1, we see that it satisfies the second one. Conversely, for $f$ holding the convolution equations, the observation

$$g_i = \Lambda_i V_i g_i = \Lambda_i(f * \lambda_i) = \Lambda_i V_i \Lambda_i f = \Lambda_i f$$

(we used consecutively claims 1, 2, and 1) yields that $f$ has the prescribed family of projections.

When dealing with convolution equations in algebras like $\mathbf{A}(G)$, the representation theory of groups and algebras is usually the most convenient tool. We shall present here, however, a shorter and simpler algebraic approach without introducing additional, sophisticated notions and facts.

### C. Solution

We shall denote by $H_I = \Pi_{i\in I} H_i;\ I \subset N$, which are the products of subfamilies of $\mathcal{H}$. That means that $H_\emptyset = \{e\}$, where $e$ is the identity element of $G$, $H_{\{i\}} = H_i$ for $i \in N$, $H_{\{i,j\}} = H_i H_j = \{xy \in G;\ x \in H_i,\ y \in H_j\} = H_j H_i$ for $i, j \in N$, etc. It is an elementary piece of group theory that the order of multiplicators in the formula of $H_I$ is not relevant and that every $H_I,\ I \subset N$, is a normal subgroup of $G$. To avoid double indexing, we write $\lambda_I$ instead of $\lambda_{H_I}$.

The following two expressions (we started to use them in [14] for the first time)

$$\varepsilon = \sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\lambda_I,$$

$$\omega = \sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\lambda_I \sum_{s=1}^{|I|} \frac{1}{s}$$

define two functions of $\mathbf{A}(G)$ that play the key role in our problem. The later one can be called a *convolving function*.

**Lemma 2:**

6. $\lambda_I * \lambda_J = \lambda_{I\cup J}, \quad I, J \subset N,$
7. $\varepsilon * \lambda_I = \lambda_I, \quad \emptyset \neq I \subset N,$
8. $\varepsilon * \varepsilon = \varepsilon,$
9. $\sum_{i\in N} \lambda_i * \omega = \varepsilon.$

For the proof of this lemma and the following three theorems, see Appendix A. Before stating the main results of this section, we shall find an explicit form of the dual operator of $\Lambda$. If the scalar product on $\mathbf{A}(\mathcal{H})$ is considered to be the sum of scalar products

$$\langle g^{(1)}, g^{(2)} \rangle = \sum_{i \in N} \langle g_i^{(1)}, g_i^{(2)} \rangle,$$
$$g^{(s)} = (g_i^{(s)})_{i \in N} \in \mathbf{A}(\mathcal{H}), \quad s = 1, 2$$

then from

$$\langle \Lambda f, g \rangle = \sum_{i \in N} \langle \Lambda_i f, g_i \rangle = \sum_{i \in N} \langle f, V_i g_i \rangle, \quad f \in \mathbf{A}(G), \ g \in \mathbf{A}(\mathcal{H})$$

we immediately get the form of the dual $V : \mathbf{A}(\mathcal{H}) \to \mathbf{A}(G)$ of $\Lambda$

$$Vg = \sum_{i \in N} V_i g_i, \qquad g \in \mathbf{A}(\mathcal{H}).$$

**Theorem 1:** The equation $\Lambda f = g$, where $g \in \mathbf{A}(\mathcal{H})$ is given, has a solution if and only if $\Lambda(Vg * \omega) = g$, and in this case, $Vg*\omega$ is its solution with the smallest norm. Any solution differs from this one by a function $f$ such that $f * \varepsilon = 0$.

**Remarks:**

1) We point out at the beginning other necessary and sufficient conditions for existence of a solution that are also called the *projectivity conditions* (mainly in the frame of marginal problems; see [13] and [14])

$$V_i g_i * \lambda_j = V_j g_j * \lambda_i \quad i, j \in N.$$

We explain only the necessity here. If $\Lambda f = g$, then

$$V_i g_i * \lambda_j = V_i \Lambda_i f * \lambda_j = f * \lambda_i * \lambda_j = f * \lambda_j * \lambda_i = V_j g_j * \lambda_i \quad i, j \in N.$$

The opposite implication is a nontrivial part of the Proof of Theorem 1 in Appendix A.

2) Our solution of the reconstruction problem can be written formally in the same manner as in the classical Radon transform theory. Namely, it can be written as a convolved backprojection $Vg * \omega$ or, equivalently (see 5 in Lemma 1), a backprojected convolution $V(g * \Lambda\omega)$. Theorem 1 remains valid if we consider only real functions on $G$ as $\omega$ and $\varepsilon$ are real.

3) The function $\lambda_\emptyset$ is nonzero only at the point $e$ and $\lambda_\emptyset(e) = |G|$. It is the identity of the algebra $\mathbf{A}(\mathcal{H})$ similarly like the delta function among distributions in the Euclidean plane. If we observe its projections $g = \Lambda\lambda_\emptyset$, then our reconstruction, according to the above theorem, will be (cf. 9 in Lemma 2)

$$Vg * \omega = V\Lambda\lambda_\emptyset * \omega = \sum_{i \in N} V_i \Lambda_i \lambda_\emptyset * \omega = \sum_{i \in N} (\lambda_\emptyset * \lambda_i) * \omega = \lambda_\emptyset * \sum_{i \in N} \lambda_i * \omega = \lambda_\emptyset * \varepsilon = \varepsilon.$$

More generaly, if we observe the projections $\Lambda f$ of a function $f$, the reconstruction formula yields (similar to the above) $f * \varepsilon$ (note that the mapping assigning $f * \varepsilon$ to a function $f$ is a projector due to 8 of Lemma 2).

4) If $\lambda_\emptyset \neq \varepsilon$, then the nonzero function $\lambda_\emptyset - \varepsilon$ has zero projections (we remind that $V_i$ is injective)

$$V_i \Lambda_i (\lambda_\emptyset - \varepsilon) = (\lambda_\emptyset - \varepsilon) * \lambda_i = \lambda_i - \lambda_i = 0.$$

On the contrary, if all functions of $\mathbf{A}(G)$ are uniquely given by their projections, then our reconstruction of $\lambda_\emptyset$ must be equal to $\lambda_\emptyset$, where $\lambda_\emptyset = \varepsilon$. These considerations show that the equation $\lambda_\emptyset = \varepsilon$ is the necessary and sufficient condition for unique reconstruction in the projection scheme given by the family $\mathcal{H}$.

### D. Dual Problem

An analysis of the dual operator $V$ may be also of some interest; we shall present here the dual version of Theorem 1.

**Theorem 2:** The equation $Vg = f$, where $f \in \mathbf{A}(G)$ is given, is solvable if and only if $f * \varepsilon = f$, and in this case $\Lambda(f * \omega) = \Lambda f * \Lambda\omega$ is its solution with the smallest norm. Every solution differs from this one by a $g = (g_i)_{i \in N} \in \mathbf{A}(\mathcal{H})$ satisfying $\Lambda(Vg * \omega) = 0$.

**Remarks:**

1) Similar to our observation after Theorem 1, we can see that Theorem 2 can be easily reformulated for the algebra of real functions on $G$.
2) The backprojection $V$ is one-to-one only if $|N| = 1$. Otherwise, for $i \neq j$, $i, j \in N$, we set $g_i = 1, g_j = -1$ and $g_k = 0$, $k \in N - \{i, j\}$, where $Vg = 0$ for $g = (g_i)_{i \in N} \neq 0$.

### E. Iterative Reconstruction

The solution of the reconstruction problem from Theorem 1 can be rewritten in a simple iterative form, which is computationally very tractable.

Consider $N = \{1, 2, ..., n\}$, choose arbitrary $f_{(0)} \in \mathbf{A}(G)$, and set

$$f_{(i)} = f_{(i-1)} + V_i(g_i - \Lambda_i f_{(i-1)}), \quad i = 1, 2, \ldots, n$$

where $g_i \in \mathbf{A}(G/H_i)$, $i \in N$, are given projections.

Therefore, this is nothing but a counterpart of one of the well-known standard algebraic iterative reconstruction techniques, e.g., in computerized tomography (see [8] and [17]); the new iteration $f_{(i)}$ is obtained from the old one $f_{(i-1)}$ by adding the correction in the form of backtraced subtraction of the given and current projections. We claim that $f_{(n)}$ is yet a solution of the reconstruction problem; thus, only $n$ iterations are needed, which completely removes the main drawback of the reconstruction methods of this type, namely, the slow rate of convergence.

**Theorem 3:** Let $g = (g_i)_{i \in N}$ be a family of projections satisfying the projectivity conditions. Then, $\Lambda f_{(n)} = g$, and moreover, the choice $f_{(0)} = 0$ gives $f_{(n)} = Vg * \omega$.

## III. Finite Radon Transforms on $Z_p^2$

In this section, we describe finite Radon transforms on the group $Z_p^2$ taking arbitrary families of its nontrivial subgroups. Further, we make explicit connections with the finite Fourier transform. Finally, reconstruction from partial projections will be examined, i.e., instead of having given Radon projections $g_i$, $i \in N$, on whole quotient groups, we suppose to know only some values of them.

### A. Basic Scheme

Let us consider the group $G = Z_p^2$ to be the Cartesian product $Z_p \times Z_p$ of two exemplares of the cyclic group $Z_p = \{0, 1, \ldots, p-1\}$ with addition modulo $p$. We shall suppose that $p$ is prime. This group has $p+1$ nontrivial subgroups

$$H_i = \{(k,l) \in G;\ li = k \,(\mathrm{mod}\, p)\}, \quad 0 \le i < p,$$
$$H_p = \{(k,0) \in G;\ k \in Z_p\}.$$

We put $N = \{0, 1, \ldots, p\}$, $\mathcal{H} = (H_i)_{i \in N}$. Every group $H_i$ and every factor group $G/H_i$ is isomorphical to $Z_p$. The cosets of the factor group $G/H_i$ will be indexed by $j \in Z_p$ in this way

$$H_i^j = \{(k,l) \in G;\ li + j = k \,(\mathrm{mod}\, p)\}, \quad 0 \le i < p,$$
$$H_p^j = \{(k,j) \in G;\ k \in Z_p\}.$$

Thus, $H_i^0 = H_i$, $i \in N$. Having an element $x = (k,l)$ of $G$ and a subgroup $H_i$, $i \in N$, there is just one coset $H_i^j$ of $H_i$ containing $x$; let us write $\pi_i(k,l) = j$. In other words, $\pi_p(k,l) = l$ and $\pi_i(k,l) = k - li \,(\mathrm{mod}\, p)$, $0 \le i < p$. The function $\pi_i$ is nothing but a variant of the factor mapping of $G$ on $G/H_i$.

The Radon projection of a function $f$ on $G$ is now given by

$$\Lambda_i f(H_i^j) = \frac{1}{p} \sum_{k=0}^{p-1} \sum_{l=0}^{p-1} f(k,l)\, \delta_j(\pi_i(k,l)), \quad i \in N, j \in Z_p$$

where $\delta_j(m)$ equals one or zero, according to whether or not $m$ equals $j$. The backprojection of a function $g_i$ on $G/H_i$ has the form

$$V_i g_i(k,l) = g_i(\pi_i(k,l)), \quad i \in N,\ (k,l) \in G$$

shortening the expression $g_i(H_i^j)$ to $g_i(j)$.

### B. Reconstruction Formulae

Now, we shall rewrite Theorem 1 for the presented scheme. Let us suppose more generally that $N$ is a nonempty subset of $\{0, 1, \ldots, p\}$ with cardinality $n$. Observing that $H_I = G$ as soon as $I$ has at least two elements, we immediately get

$$\varepsilon = 1 - n + \sum_{i \in N} \lambda_i$$

and after a computation (see Appendix B)

$$\omega = \frac{1}{n} - n + \sum_{i \in N} \lambda_i\,.$$

The Radon transform $\Lambda$ with respect to $\mathcal{H} = (H_i)_{i \in N}$ is injective according to Theorem 1 and Remark 4 if and only if $\varepsilon = \lambda_\emptyset$. This means $\varepsilon(e) = 1 - n + np = \lambda_\emptyset(e) = p^2$, where $e = (0,0)$ is the identity element of $G$. We arrive at the necessary condition $n = p+1$, and in this case due to $G - e = \sum_{i \in N}(H_i - e)$, we see that $\varepsilon(x) = 1 - (p+1) + p = 0$ for $x \ne e$. The Radon transform $\Lambda$ is thus injective only if it consists of all $p+1$ Radon projections.

A family $g = (g_i)_{i \in N}$ of functions on the factor groups $G/H_i$, $i \in N$, can be viewed as the family of projections of a function $f$ if it fulfils the projectivity conditions of Remark 1. These conditions require that the expression

$$\begin{aligned} V_i g_i * \lambda_{i'} &= V_i g_i * \lambda_i * \lambda_{i'} = V_i g_i * \lambda_G \\ &= \frac{1}{p^2} \sum_{(k,l) \in G} V_i g_i(k,l) \\ &= \frac{1}{p} \sum_{j=0}^{p-1} g_i(j), \quad i, i' \in N, i \ne i' \end{aligned}$$

does not depend on $i \in N$. Naturally, all projections must have the same averages; in this case, we denote them by $a(g)$. The smallest norm function having these prescribed projections is given by

$$\begin{aligned} Vg * \omega &= \sum_{i \in N} V_i g_i * \left(\frac{1}{n} - n + \sum_{j \in N} \lambda_j\right) \\ &= n\, a(g) \frac{1 - n^2}{n} + \sum_{i \in N} V_i g_i + a(g)(n^2 - n) \\ &= a(g) + \sum_{i \in N} (V_i g_i - a(g)). \end{aligned}$$

A carefull look at Theorem 3 reveals that this formula can also be obtained easily by the iterative procedure.

For the Radon transform composed of all $p+1$ Radon projections, we get the identity

$$f = V \Lambda f * \omega = a(f) + \sum_{i=0}^{p} (V_i \Lambda_i f - a(f)), \quad f \in \mathbf{A}(G)$$

where

$$a(f) = \frac{1}{p^2} \sum_{(k,l) \in G} f(k,l)$$

is the average value of $f$. Note that if the Radon transform $\Lambda$ is considered only for the functions with zero average, then its dual $V$ is also its inverse.

There is another way of viewing the above situation, namely, the group $G$ together with all cosets $H_i^j$, $0 \le i \le p$, and $0 \le j < p$ form the affine plane $AG(2,p)$. It has $p^2$ points, $p^2 + p$ lines (cosets), every point $x \in G$ lies on $p+1$ lines (in one coset of every factorgroup $G/H_i$), every line contains $p$ points, and most noteworthy, every two distinct points lie on just one line. Under some conditions, like the very last one, more general identities were proved (in [2] and [18]) for combinatorial designs.

### C. Radon and Fourier transforms

The Fourier transform is closely related to the Radon one in the classical Radon transform theory. Roughly speaking, the 1-D Fourier transform of the projection $\hat{f}_q$ is equal to the 2-D Fourier transform of the original function $f$ restricted on the line $q^{\perp}$, which is perpendicular to $q$ and contains the origin; see [7] and [18]. This assertion is called, in applications, the *projection slice theorem* [8], [17]. We shall see here that an analogy is valid also for the group $Z_p^2$.

Let us be reminded that the finite Fourier transforms can be written as

$$\mathbf{F}_2 f(k,l) = \frac{1}{p^2}\sum_{k'=0}^{p-1}\sum_{l'=0}^{p-1} f(k',l')\exp\left[-\frac{2\pi \mathrm{i}}{p}(kk'+l\,l')\right]$$

for functions on $Z_p^2$ and as

$$\mathbf{F}_1 h(j) = \frac{1}{p}\sum_{j'=0}^{p-1} h(j')\exp\left[-\frac{2\pi \mathrm{i}}{p}jj'\right]$$

for functions on $Z_p$.

The Fourier transform of a projection $\Lambda_i f$ is thus

$$\begin{aligned}\mathbf{F}_1\Lambda_i f(j) &= \frac{1}{p}\sum_{j'=0}^{p-1}\frac{1}{p}\sum_{k=0}^{p-1}\sum_{l=0}^{p-1} f(k,l)\,\delta_{j'}(\pi_i(k,l))\exp\left[-\frac{2\pi \mathrm{i}}{p}jj'\right]\\ &= \frac{1}{p^2}\sum_{k=0}^{p-1}\sum_{l=0}^{p-1} f(k,l)\exp\left[-\frac{2\pi \mathrm{i}}{p}j\,\pi_i(k,l)\right]\end{aligned}$$

i.e., we find

$$\begin{aligned}&\mathbf{F}_1\Lambda_p f(j) = \mathbf{F}_2 f(0,j), && j\in Z_p,\\ &\mathbf{F}_1\Lambda_i f(j) = \mathbf{F}_2 f(j,-ij(\mathrm{mod}\,\mathrm{p})), && 0\le i<p,\ j\in Z_p.\end{aligned}$$

Denoting by $\rho(0)=p$, $\rho(p)=0$ and by $\rho(i)=j$, which is the unique solution of the equation $1+ij=0\,(\mathrm{mod}\,p)$, we can interpret the above equations verbaly in the following form: Knowledge of the Fourier transform $\mathbf{F}_1\Lambda_i f$ of a projection $\Lambda_i f$ is nothing but knowledge of the Fourier transform $\mathbf{F}_2 f$ restricted on the subgroup $H_{\rho(i)}$, $0\le i\le p$. This claim could be called the *discrete projection slice theorem*. We remark that similar assertions can be formulated for compact groups by means of harmonic analysis [14].

### D. Reconstruction from Partial Projections

Let us denote by $P_i$ a subset of the cyclic group $Z_p$, $0\le i\le p$ with cardinality $0\le|P_i|=p_i\le p$ and by $I$ the set of those $i$ from $N=\{0,1,\ldots,p\}$ for which $P_i$ equals $Z_p$.

Having a system $h=(h_i)_{i\in N}$ of functions ($h_i$ defined on $P_i$), we shall employ the extensions $E_t h_i$ of $h_i$ on $Z_p$ induced by a number $t$

$$\begin{aligned}&E_t h_i(j) = h_i(j), && j\in P_i,\\ &E_t h_i(j) = \tfrac{1}{p-p_i}\left[pt-\sum_{j\in P_i} h_i(j)\right], && j\in Z_p-P_i.\end{aligned}$$

Since $E_t h_i$ is constant on $Z_p-P_i$ and chosen in such a way as to ensure the average of $E_t h_i$ equal to $t$, one can speak about the *uniform extensions* induced by $t$. We shall also write $E_t h=(E_t h_i)_{i\in N}$.

**Theorem 4:** The family of equations $\Lambda_i f(H_i^j)=h_i(j)$ indexed by $i\in N$, $j\in P_i$, where $h=(h_i)_{i\in N}$ is a given system of functions, has a solution if and only if the expression $\frac{1}{p}\sum_{j\in P_i} h_i(j)$ is constant on $I$; for $I\neq\emptyset$, we denote it by $a(h)$, and for $I=\emptyset$, we take

$$a(h) = \frac{\sum_{i=0}^{p}\frac{1}{p-p_i}\sum_{j\in P_i} h_i(j)}{-p+\sum_{i=0}^{p}\frac{p}{p-p_i}}.$$

If this is fulfilled, the function $\Lambda^{-1}E_{a(h)}h$ is the smallest norm solution.

For the proof of Theorem 4, see Appendix B.

**Remarks:**

1) If we do not have more than one complete projection, the family of equations is always solvable.
2) Thus, the distinguished solution has these values:

$$a(h)+\sum_{i=0}^{p}[E_{a(h)}h_i(\pi_i(k,l))-a(h)],\quad (k,l)\in Z_p^2.$$

## IV. Numerical Experiments

We performed simple computational studies of the finite Radon transforms and their inversions oriented to digital image processing. These transforms were found to be numerically well tractable, and the corresponding image representations were found to be interesting from the transform coding standpoint. In this section, we shall demonstrate favorable behavior and properties of the related algorithms and discuss the results of our numerical experiments.

### A. Computing Finite Radon Transforms of Images

We worked with 256 gray-level digital images of size $127\times127$ pixels. Interpreting them as functions $f(k,l)$, $0\le k\le126$, $0\le l\le126$, on the group $Z_{127}^2$, the finite Radon transform according to the scheme from Section III-A consists of averaging over cosets of the 128 nontrivial subgroups. They are visualized in Fig. 1; Fig. 1(a) exhibits the function $\pi_{127}$, which is constant on cosets of $H_{127}$, and Fig. 1(b)–(d) were generated to show $\pi_{126}$, $\pi_{125}$, and $\pi_{124}$, respectively. These pictures resemble contour images of discrete representations of analog lines [17]. Fig. 1(a) even coincides with them. However, the visual forms of the functions $\pi_i$ for a considerable part of the projections look like gray unstructured images; the points of lines are, as a rule, "dissipated" all over the images.

In Fig. 2(a), we can see the digital image of a portrait we analyzed. Its finite Radon transform $g_i(j)=\Lambda_i f(H_i^j)$, $0\le i\le127$, $0\le j\le126$ was arranged into an array of 128 rows, where every one was of length 127; see Fig. 2(b). Computing the $i$th Radon projection, i.e., the $i$th row of the array, we need to pass all pixels of the original image once and employ 127 histogrammers: one for every pixel in the row; the gray level $f(k,l)$ of a pixel $(k,l)$ is added to the $\pi_i(k,l)$ histogrammer. At the end, all 127 histogrammed values are divided by 127 to get the average values, and the results are

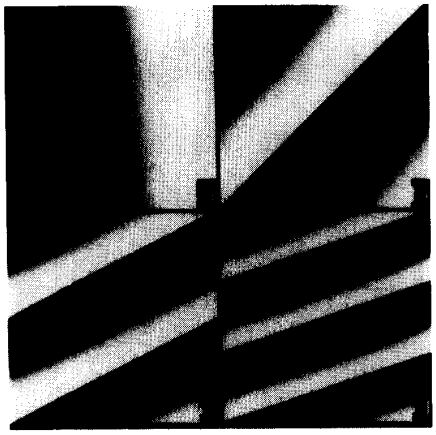

Fig. 1. Cosets $H_i^j$ of nontrivial subgroups $H_i$ (parallel lines of the geometry): (a) $i = 127$; (b) $i = 126$; (c) $i = 125$; (d) $i = 124$.

Fig. 2. (a) Original image, 127 × 127 pixels, 256 gray levels; (b) radon transform of the image (a); (c) reconstruction of the image (a) via inverse Radon transform; (d) radon transform with ordered rows according to their variances.

rounded off to integer numbers. This is the only source of noise in our model.

To illustrate this algorithm, we present the following scheme:

```
begin
  for i = 0,p − 1
    g_i = 0
    n = i
    for l = 0,p − 1
      n = n − i
      if n < 0 then n = n + p
      j = n − 1
      for k = 0,p − 1
        j = j + 1
        if j ≥ p then j = j − p
        g_i(j) = g_i(j) + f(k,l)
      endfor
    endfor
    g_i = g_i/p
  endfor
  g_p = 0
  for l = 0,p − 1
    for k = 0,p − 1
      g_p(l) = g_p(l) + f(k,l)
    endfor
  endfor
  g_p = g_p/p
end
```

Having at our disposal a pipeline of 128 stages, one pass of the image through it gives all projections, where every one is on the corresponding stage. Hence, the computing of the finite Radon transform can be pipelined and performed by $P^3E$ architecture. Another, more effective, way of computing this transform is to use the results of Section III-C. Accordingly, one performs the 2-D finite Fourier transform followed by the 128 1-D ones. This is, of course, much more noisy.

### *B. Computing the Inverse*

Our original image was adjusted to have the average value $a(f) = 128$. Its reconstruction was obtained column by column as follows. In the $l$th column $0 \le l \le 126$, we employ 127 histogrammers, where the $k$th one takes care of the pixel $(k, l)$, $0 \le k \le 126$. The initial value of all histogrammers is taken to be $g_{127}(l)$ (this is the $l$th record in the last row of our projection array). This originates from $a(f) + (g_{127}(l) - a(f))$, which is the constant plus the last term in the sum of the reconstruction formula. Now, it suffices to pass the projection array from the first row to the 126th row and to find to every datum $g_i(j) = \Lambda_i f(j)$, $0 \le i \le 126$, $0 \le j \le 126$, the uniquely given index $k$ for which $\pi_i(k, l) = j$. The correction $g_i(j) - a(f)$ is then added to the $k$th histogrammer.

More formally, the algorithm is described as follows (note that in this and foregoing schemes, we do not even use multiplications).

```
begin
  a = 0
  for j = 0,p − 1
    a = a + g_0(j)
  endfor
  a = a/p
  for l = 0,p − 1
    for k = 0,p − 1
      f(k,l) = g_p(l)
    endfor
    n = −l
    for i = 0,p − 1
      n = n + l
      if n ≥ p then n = n − p
      k = n − 1
      for j = 0,p − 1
        k = k + 1
        if k ≥ p then k = k − p
        f(k,l) = f(k,l) + g_i(j) − a
      endfor
    endfor
  endfor
end
```

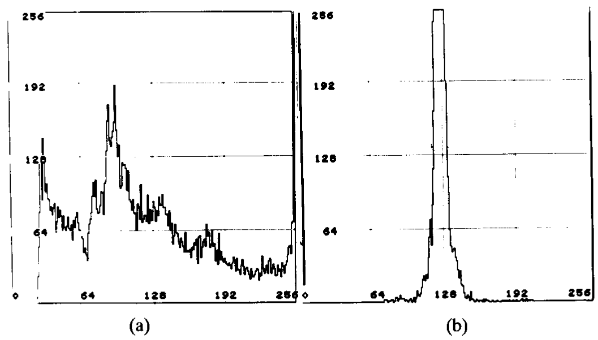


Fig. 3. (a) Histogram of the original image; (b) histogram of its Radon transform.

This algorithm can be performed again by a pipeline of 127 stages, where every one takes care of one column of the reconstructed image. Evidently, only one pass of the projection data is needed. Moreover, under the assumption that we know precisely $a(f)$ (of an integer value), this inversion procedure works without numerical errors.

The result of the reconstruction is shown in Fig. 2(c). Apparently, there is no visual difference between the original image and the reconstructed one, and the difference, which is caused exlusively by the noise accompanying the finite Radon transform, is in fact not greater than two gray levels at each pixel.

The histograms of the portrait image in Fig. 2(a) and its Radon transform array are shown in Fig. 3. Note that the original image contains all 256 gray levels, but the array only contains 96 gray levels.

### C. Compression by Rows

Typical row profiles of the finite Radon transform are shown in Fig. 4. Intuitively, profiles in Fig. 4(a) and (b) corresponding to the rows 128 and 64, respectively, retain a lot of information about the original image, whereas profiles in Fig. 4(c) and (d) (rows 116 and 52, respectively) are much less informative.

To ground this intuition, let us look more carefully at the reconstruction formula

$$a(f) + \sum_{i=0}^{127} [V_i \Lambda_i f - a(f)].$$

It consists of the constant $a(f)$ and 128 functions on $G = Z_{127}^2$ all being mutually orthogonal. In fact, since $\lambda_G = 1$, $\lambda_G * \lambda_i = \lambda_G$ and $\lambda_i * \lambda_{i'} = \lambda_G$ ($i, i' \in N$ distinct), we get

$$\begin{aligned}&\langle V_i \Lambda_i f - a(f), \lambda_G \rangle = \langle f, V_i \Lambda_i \lambda_G \rangle - a(f)\\ &= \langle f, \lambda_G * \lambda_i \rangle - a(f) = 0\\ &\langle V_i \Lambda_i f - a(f), V_{i'} \Lambda_{i'} f - a(f) \rangle\\ &= \langle f * \lambda_i * \lambda_{i'}, f \rangle\\ &\quad - \langle a(f), f * \lambda_{i'} \rangle - \langle f * \lambda_i, a(f) \rangle + |a(f)|^2 = 0.\end{aligned}$$

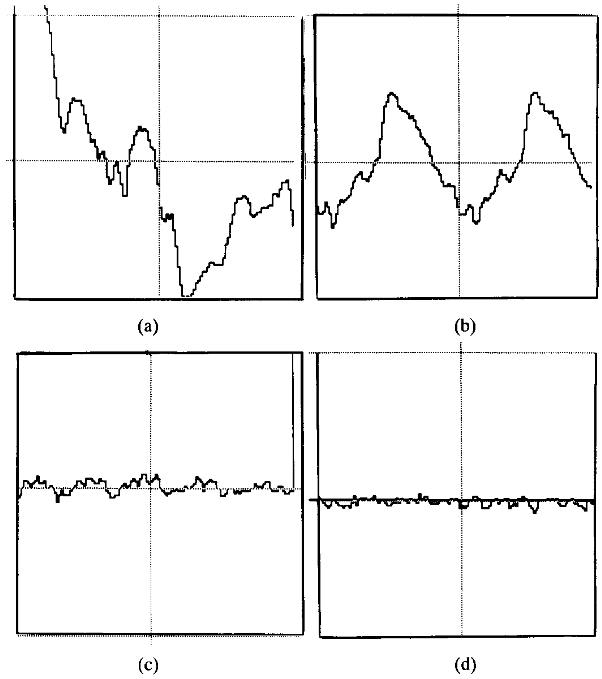


Fig. 4. Typical row profiles of the Radon transform array: (a) Row 128; (b) row 64; (c) row 116; (d) row 52.

Observing that due to

$$\begin{aligned}||V_i \Lambda_i f - a(f)||^2 &= ||V_i(\Lambda_i f - a(f))||^2\\ &= \langle \Lambda_i V_i(\Lambda_i f - a(f)), \Lambda_i f - a(f) \rangle\\ &= ||\Lambda_i f - a(f)||^2\end{aligned}$$

the norms of these functions are simply the variances of the Radon projections (rows in Fig. 2(b)), and we can conclude that projections such as those in Fig. 4(c) and (d) do not contribute substantially to the reconstruction formula, and when omitted, the resulting image differs only mildly (in the square norm) from the original one.

The rows of the Radon transform image were sorted according to their variances; the ordered finite Radon transform is shown in Fig. 2(d). Let us note that from the fourth row, the diapason of gray levels enables them to be encoded to six bits and, starting from the 40th row, to four b only. In this way, a 59% compression of the original image can be gained without any lost of information.

Further, we have used for the reconstruction only the first $n$ (more informative) rows of Fig. 2(d). The results are exhibited in Fig. 5 (from left to right and from top to bottom, we utilized 4, 7, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 110, 120, and 128 projections; the last area is occupied by the original image). The achieved compression levels are presented in Table I.

Let us remark that this way of compression is in the main equivalent to the Fourier transform coding as the omission of noninformative rows corresponds to a restriction of the Fourier transform $\boldsymbol{F}_2$ (cf., discrete projection slice theorem).

Fig. 5. Partial reconstructions from the most informative Radon transform rows (according to Table I).

TABLE I
PARTIAL RECONSTRUCTIONS FROM THE MOST INFORMATIVE ROWS (nr is the number of rows that were used; c is the compression level of the original (in percent))

| nr | c |
|---|---|
| 4 | 3 |
| 7 | 5 |
| 10 | 7 |
| 20 | 13 |
| 30 | 19 |
| 40 | 24 |
| 50 | 28 |
| 60 | 32 |
| 70 | 36 |
| 80 | 40 |
| 90 | 44 |
| 100 | 48 |
| 110 | 52 |
| 120 | 56 |
| 128 | 59 |

### D. Compression by Pixels

Another look at the reconstruction formula reveals that the pixels with the gray level $128 = a(f)$ do not contribute to the reconstructed image at all. The most informative pixels are thus the most different from the mean value. This claim is based on the identity from the beginning of the Proof of Theorem 4 (see Appendix B), which can be rewritten as

$$\|f\|^2 = \frac{1}{p^2}\sum_{k,l}|f(k,l)|^2$$
$$= |a(f)|^2 + \sum_{i=0}^{p}\frac{1}{p}\sum_{j=0}^{p-1}|\Lambda_i f(H_i^j) - a(f)|^2.$$

Of course, we have to save the knowledge about the positions of the chosen pixels, e.g., by a binary mask.

For the reconstruction of the analyzed image, seven intensity bands defining the partial projections were consecutively used. Table II reports on intensity values, numbers of pixels in each band, and compression ratios. The corresponding partial reconstructions are shown in Fig. 6. Under comparision with

TABLE II
PARTIAL RECONSTRUCTIONS FROM THE MOST INFORMATIVE PIXELS (band: the intensity band in the Radon transform array; np: the number of pixels in the band; c: Compression level of the original image (in percent))

| image | band | | np | c |
|---|---|---|---|---|
| a | 0 – 119, | 137 – 255 | 1 179 | 7 |
| b | 0 – 122, | 134 – 255 | 2 169 | 11 |
| c | 0 – 124, | 132 – 255 | 4 510 | 22 |
| d | 0 – 125, | 131 – 255 | 6 779 | 29 |
| e | 0 – 126, | 130 – 255 | 10 009 | 40 |
| f | 0 – 127, | 129 – 255 | 14 058 | 52 |
| g | 0 – 255 | | 16 256 | 59 |

Fig. 6. Partial reconstructions from the most informative pixels (according to Table II).

the above described "row compression" method, the results here are of better visual quality at the same compression level.

### E. 2-D Convolution

The Radon transform can serve as a tool for reducing the dimensionality of certain image processing techniques as well. In our setting, the convolution of an image $f$ with a kernel $\varphi$ of small size coincides with the convolution $f * \varphi$ in $Z_p^2$ up to border effects and can be computed due to Lemma 1 as well by the formula

$$f * \varphi = \Lambda^{-1}\Lambda(f * \varphi) = \Lambda^{-1}(\Lambda f * \Lambda\varphi)$$

(cf. with the convolution theorem in [17]). In other words, 2-D convolutions in image space correspond to a series of $p+1$ 1-D row convolutions in projection representation space. The Fourier methods are, of course, more effective from a computational standpoint. Nevertheless, having compressed projection arrays by rows, the above formula implies that the convolutions are to be processed only for the informative rows, which might bring additional simplification of computations.

## APPENDIX A
## PROOFS OF ASSERTIONS FROM SECTION II.

**Proof of Lemma 2:** For any $x \in G$, we have $\lambda_I * \lambda_J(x) = |H_I|^{-1}\sum_{y\in H_I}\lambda_J(y^{-1}x)$. This yields that if $x \notin H_{I\cup J}$, then for $y \in H_I$, the element $y^{-1}x$ does not belong to $y^{-1}H_{I\cup J} = H_{I\cup J} = H_I H_J \supset H_J$, i.e., $\lambda_I * \lambda_J(x) = 0$. In the opposite case $x \in H_{I\cup J}$, we observe that $y \in H_I$ and $y^{-1}x \in H_J$ if and only if $y \in (xH_J) \cap H_I$. The last set is nonempty for having $x$ in the form $x = z_1 z_2$, $z_1 \in H_I$, $z_2 \in H_J$; the element $z_1$ belongs to $(xH_J) \cap H_I = (z_1 H_J) \cap H_I = z_1(H_I \cap H_J)$.

Now, it is easily seen that

$$\lambda_I * \lambda_J(x) = \frac{|H_I \cap H_J||G/H_J|}{|H_I|} = \frac{|H_I \cap H_J||G|}{|H_I||H_J|} = \frac{|G|}{|H_{I\cup J}|} = |G/H_{I\cup J}|, \quad x \in H_{I\cup J}$$

takes place, employing standard identities of group theory.

Having proved 6, we apply it immediately. If $i \in N$, then

$$\varepsilon * \lambda_i = \left[\lambda_\emptyset + \sum_{i\in I\subset N} (-1)^{|I|-1}(\lambda_I - \lambda_{I-\{i\}})\right] * \lambda_i = \lambda_i$$

and straighforward 7 follows for nonempty $I$, namely, if one chooses any $i \in I$, then

$$\varepsilon * \lambda_I = \varepsilon * (\lambda_i * \lambda_I) = (\varepsilon * \lambda_i) * \lambda_I = \lambda_i * \lambda_I = \lambda_I.$$

Now, the statement 8 is a trivial consequence of 7

$$\varepsilon * \varepsilon = \sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\varepsilon * \lambda_I = \varepsilon\left[1 - \sum_{I\subset N}(-1)^{|I|}\right] = \varepsilon$$

and this computation shows the validity of 9 (we use the notation $c_t = \sum_{s=1}^{t} \frac{1}{s},\ t \geq 0$)

$$\begin{aligned}\sum_{i\in N}\lambda_i * \omega &= \sum_{i\in N}\sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\lambda_{I\cup\{i\}}\, c_{|I|} \\ &= \sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\left[|I|\,\lambda_I\, c_{|I|} + \sum_{i\in N-I} \lambda_{I\cup\{i\}} c_{|I|}\right] = \\ &= \sum_{\emptyset\neq I\subset N} |I|\,\lambda_I\left[(-1)^{|I|-1}c_{|I|} + (-1)^{|I|-2}c_{|I|-1}\right] \\ &= \varepsilon.\end{aligned}$$

□

**Proof of Theorem 1:** The condition $\Lambda(Vg * \omega) = g$ is trivially sufficient for the existence of a solution. If $\Lambda f = g$, then the projectivity conditions hold as we have seen in Remark 1. From them, we derive (see Lemmas 1 and 2)

$$\begin{aligned}\Lambda_i(Vg * \omega) &= \Lambda_i V_i \Lambda_i(Vg * \omega) \\ &= \Lambda_i(Vg * \omega * \lambda_i) = \Lambda_i(\sum_{j\in N} V_j g_j * \lambda_i * \omega) = \\ &= \Lambda_i(V_i g_i * \sum_{j\in N}\lambda_j * \omega) \\ &= \Lambda_i(V_i g_i * \varepsilon) = \Lambda_i(V_i g_i * \varepsilon * \lambda_i) = g_i, \quad i \in N.\end{aligned}$$

This crucial computation closes the first part of the proof.

The difference of two solutions of the equation belongs to the kernel of the operator $\Lambda$, and this can also be characterized as $\mathcal{N}_\Lambda = \{f; \Lambda f = 0\} = \{f; f * \varepsilon = 0\}$. In fact, if $\Lambda_i f = 0$, then $V_i\Lambda_i f = f * \lambda_i = 0$ and for $\emptyset \neq I \subset N$ and $i \in I$; moreover, $f * \lambda_I = f * \lambda_i * \lambda_I = 0$; finally, $f * \varepsilon = 0$. Conversely, $f * \varepsilon = 0$ implies $f * \varepsilon * \lambda_i = f * \lambda_i = V_i\Lambda_i f = 0$ and, due to injectivity of $V_i$, $\Lambda_i f = 0$, $i \in N$ as well.

The algebra $\mathbf{A}(G)$ can be, as a complex Hilbert space, orthogonally decomposed into $\mathcal{N}_\Lambda$ and $\mathcal{R}_V$, where $\mathcal{R}_V$ is the range of the backprojection $V$. However, the solution $Vg*\omega = V(g*\Lambda\omega)$ belongs to $\mathcal{R}_V$ and, thus, has the smallest norm among all solutions of the equation. □

**Proof of Theorem 2:** The condition $f * \varepsilon = f$ is sufficient for if it is satisfied then by 9 of Lemma 2:

$$V(\Lambda f * \omega) = \sum_{i\in N} V_i\Lambda_i(f * \omega) = f * \omega * \sum_{i\in N}\lambda_i = f * \varepsilon = f$$

i.e., $g = \Lambda(f * \omega)$ solves the equation $Vg = f$. In addition, this reasoning

$$\begin{aligned}Vg * \varepsilon &= \sum_{i\in N} V_i\Lambda_i V_i g_i * \varepsilon \\ &= \sum_{i\in N} V_i g_i * \lambda_i * \varepsilon \\ &= \sum_{i\in N} V_i g_i * \lambda_i = Vg, \quad g \in \mathbf{A}(\mathcal{H})\end{aligned}$$

shows that $\mathcal{R}_V = \{f; f * \varepsilon = f\}$. The mapping $f \rightarrow f * \varepsilon$ is the orthogonal projector on $\mathcal{R}_V$ (cf., Remark 3).

Note that $Vg = 0$ if and only if $\Lambda(Vg * \omega) = 0$. In fact, to prove the nontrivial implication, we apply the operator $V$

$$0 = V\Lambda(Vg * \omega) = Vg * \omega * \sum_{i\in N}\lambda_i = Vg * \varepsilon = Vg.$$

The algebra $\mathbf{A}(\mathcal{H})$ can be orthogonally decomposed into the kernel $\mathcal{N}_V$ of the backprojection and the range $\mathcal{R}_\Lambda$ of the Radon projection; we remark that $g \rightarrow \Lambda(Vg * \omega)$ is the orthogonal projector on $\mathcal{R}_\Lambda$. As $\Lambda(f * \omega)$ belongs to $\mathcal{R}_\Lambda$, it has the smallest norm among all solutions of the equation. □

**Proof of Theorem 3:** Let us prove $f_{(i+j)} * \lambda_i = V_i g_i$ for $i \in N$ and $j = 0, 1, \ldots, n-i$ by induction on $j = 0, 1, \ldots, n-1$. For $j = 0$, we have

$$f_{(i)} * \lambda_i = f_{(i-1)} * \lambda_i + V_i g_i * \lambda_i - V_i\Lambda_i f_{(i-1)} * \lambda_i = V_i g_i, \ i \in N$$

and the induction argument has the form $(0 \leq j < n-1)$

$$\begin{aligned}f_{(i+j+1)} * \lambda_i &= f_{(i+j)} * \lambda_i \ + \ V_{i+j+1}\, g_{i+j+1} * \lambda_i \\ &\quad - f_{(i+j)} * \lambda_{i+j+1} * \lambda_i = \\ &= V_i g_i \ + \ V_{i+j+1}\, g_{i+j+1} * \lambda_i \\ &\quad - V_i g_i * \lambda_{i+j+1} = V_i g_i, \ \ 1 \leq i < n-j\end{aligned}$$

employing the projectivity conditions.

Thus, $\Lambda f_{(n)} = g$. The choice $f_{(0)} \in \mathcal{R}_V$ gives

$$f_{(n)} = f_{(0)} + \sum_{i\in N} V_i(g_i - \Lambda_i f_{(i-1)}) \ \in \mathcal{R}_V.$$

However, from the Proof of Theorem 1, we know that the equation $\Lambda f = g$ has only one solution in $\mathcal{R}_V$, which yields $f_{(n)} = Vg * \omega$ whichever $f_{(0)} \in \mathcal{R}_V$ was chosen. □

## Appendix B
## Proofs of Assertions from Section III

**Derivation of the Form of $\omega$:** We have realized that for $I$ having more than one element $\lambda_I = \lambda_G = 1$ and then

$$\begin{aligned}\omega &= \sum_{\emptyset\neq I\subset N} (-1)^{|I|-1}\lambda_I \sum_{s=1}^{|I|}\frac{1}{s} \\ &= \sum_{i\in N}\lambda_i - n + \sum_{I\subset N}(-1)^{|I|-1}\sum_{s=1}^{|I|}\frac{1}{s}.\end{aligned}$$

Rewriting the sum of the fractions into

$$\sum_{s=1}^{r}\frac{1}{s}=\int_0^1\sum_{s=1}^{r}z^{s-1}dz=\int_0^1\frac{1-z^r}{1-z}dz$$

we obtain

$$\begin{aligned}\sum_{I\subset N}(-1)^{|I|-1}\sum_{s=1}^{|I|}\frac{1}{s}&=\int_0^1\frac{1}{1-z}\sum_{I\subset N}(-1)^{|I|-1}(1-z^{|I|})dz\\&=\int_0^1\frac{1}{1-z}\sum_{I\subset N}(-1)^{|I|}z^{|I|}dz\\&=\int_0^1(1-z)^{n-1}dz=\int_0^1 z^{n-1}dz=\frac{1}{n}.\end{aligned}$$

The convolving function $\omega$ can also be found by using the finite Fourier transform and Lemma 2; we omit details. □

**Proof of Theorem 4:** The solvability condition is evidently necessary. If it is valid, then $E_{a(h)}h$ satisfies the projectivity conditions, and hence, $\Lambda^{-1}E_{a(h)}h$ is a solution.

To prove the last assertion of the theorem, we will need this identity

$$\|f\|^2=-p|a(f)|^2+\|\Lambda f\|^2,\ \ f\in\mathbf{A}(G).$$

Note that the finite Radon transform conserves the norm of functions with zero average. The supporting computations follow:

$$\begin{aligned}\|\Lambda f\|^2&=\sum_{i=0}^{p}\|\Lambda_i f\|^2=\frac{1}{p}\sum_{i=0}^{p}\sum_{j=0}^{p-1}|\Lambda_i f(H_i^j)|^2\\&=\frac{1}{p^3}\sum_{i=0}^{p}\sum_{j=0}^{p-1}|\sum_{k,l}f(k,l)\delta_j(\pi_i(k,l))|^2=\\&=\frac{1}{p^3}\sum_{k,l}\sum_{k',l'}f(k,l)\overline{f}(k',l')\sum_{i=0}^{p}\sum_{j=0}^{p-1}\\&\quad\delta_j(\pi_i(k,l))\delta_j(\pi_i(k',l')).\end{aligned}$$

The double inner sum is equal to 1 for $(k,l)\neq(k',l')$ and to $p+1$ otherwise, which yields

$$\begin{aligned}\|\Lambda f\|^2&=\frac{1}{p^3}\sum_{k,l}\sum_{k',l'}f(k,l)\overline{f}(k',l')+\frac{1}{p^2}\sum_{k,l}|f(k,l)|^2\\&=p|a(f)|^2+\|f\|^2.\end{aligned}$$

This identity also follows from the discrete projection slice theorem.

Let us suppose now that $f$ is a solution of the family of equations. Then

$$\begin{aligned}\|f\|^2=&-p|a(f)|^2+\sum_{i\in I}\|h_i\|^2+\frac{1}{p}\sum_{i\in N-I}\sum_{j\in P_i}|h_i(j)|^2\\&+\frac{1}{p}\sum_{i\in N-I}\sum_{j\in Z_p-P_i}|\Lambda_i f(H_i^j)|^2.\end{aligned}$$

Having a complete projection ($I\neq\emptyset$), we estimate by Cauchy inequality

$$\begin{aligned}\sum_{j\in Z_p-P_i}|\Lambda_i f(H_i^j)|^2&\geq\frac{1}{p-p_i}\left|\sum_{j\in Z_p-P_i}\Lambda_i f(H_i^j)\right|^2\\&=\frac{1}{p-p_i}\left|p\,a(f)-\sum_{j\in P_i}h_i(j)\right|^2,\ \ i\in N-I\end{aligned}$$

and from $a(f)=a(h)=a(f^*)$, where $f^*=\Lambda^{-1}E_{a(h)}h$, we conclude

$$\begin{aligned}\|f\|^2\geq&-p|a(f^*)|+\sum_{i\in I}\|\Lambda_i f^*\|^2\\&+\frac{1}{p}\sum_{i\in N-I}\sum_{j=0}^{p-1}|E_{a(h)}h_i(j)|^2=\|f^*\|^2\end{aligned}$$

where the equality takes place only if $\Lambda f=\Lambda f^*$, i.e., $f=f^*$.

Having no complete projection, we similarly estimate

$$\begin{aligned}\|f\|^2\geq&-p|a(f)|^2+\frac{1}{p}\sum_{i\in N}\sum_{j\in P_i}|h_i(j)|^2\\&+p\sum_{i\in N}\frac{1}{p-p_i}|a(f)-\frac{1}{p}\sum_{j\in P_i}h_i(j)|^2.\end{aligned}$$

An elementary computation shows that this function

$$-|t^2|+\sum_{i\in N}\frac{|t-t_i|^2}{p-p_i}$$

of the complex variable $t$ takes its strong global minimum at the point

$$t^*=\frac{\sum_{i\in N}\frac{t_i}{p-p_i}}{-1+\sum_{i\in N}\frac{1}{p-p_i}}.$$

Thus

$$\|f\|^2\geq-p|a(h)|^2+\sum_{i\in N}\|E_{a(h)}h_i\|^2=\|f^*\|^2$$

with equality just for $f=f^*$. □

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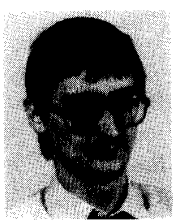
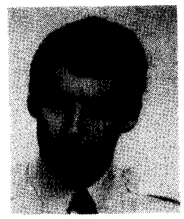


**František Matúš** was born in Poprad, East Slovakia, Czechoslovakia, in 1961. He received the M.Sc. degree in mathematical engineering from the Czech Technical University, Faculty of Nuclear Sciences and Physical Engineering, in 1984 and the CSc. degree (Ph.D. level) in theoretical cybernetics from the Czechoslovak Academy of Sciences in 1989.

From 1985, he has been with the Institute of Information Theory and Automation of the Academy of Sciences of Czech Republic, Prague. His research interests include Radon transform theory, probabilistic and combinatorial methods of artificial intelligence, random processes, coding, and information theory.



**Jan Flusser** was born in Prague, Czechoslovakia, on April 30, 1962. He received the M.Sc. degree in mathematical engineering from Czech Technical University, Prague, in 1985 and the Ph.D. degree in computer science from the Czechoslovak Academy of Sciences, Prague, in 1990.

He is employed by the Institute of Information Theory and Automation, Prague. His research interests include digital image processing, pattern recognition, and remote sensing.