A REVIEW OF AGGREGATION FUNCTIONS

R. MESIAR, A. KOLESÁROVÁ, T. CALVO, AND M. KOMORNÍKOVÁ

Abstract. Several local and global properties of (extended) aggregation functions are discussed and their relationships are examined. Some special classes of averaging, conjunctive and disjunctive aggregation functions are reviewed. A special attention is paid to the weighted aggregation functions, including some construction methods.

1. Introduction

Aggregation of a finite number of observed values from a scale I into a single output value from the same scale is an indispensable tool in each discipline based on data processing. The variability of spheres dealing with aggregation (fusion) techniques is so rich that we frequently meet the same results and techniques under different names. Nowadays, when aggregation theory becomes a well established field of mathematics, it is the time to unify the notations and terminology. This is one of the aims of the monograph on aggregation [25] which is currently under preparation by Grabisch, Marichal, Mesiar and Pap. Therefore in this chapter we will use the notations and terminology from [25]. The main aim of this chapter is to bring a review of some recent results in aggregation. To achieve the readability, we will also recall some older results whenever necessary. Note that comprehensive state-of-art overviews on aggregation can be found in [20] (dated to 1985) and in [6] (dated to 2002). The chapter is organized as follows. In the next section, basic notions, notations and properties are given, including the classification of aggregation functions. In Section 3, averaging aggregation functions are discussed. Section 4 is devoted to conjunctive aggregation functions, and by duality, also to disjunctive aggregation functions. Weighted aggregation functions are discussed in Section 5. Finally, some conclusions are given.

Note that though aggregation can be discussed on an arbitrary scale I (equipped with linear order), we restrict our considerations to the real intervals. Moreover, a major attention will be put to the case $I = [0, 1]$.

2. Basic notions, notations and properties

Unless otherwise stated, the letter I will denote a subinterval of the extended real line, $I = [a, b] \subseteq [-\infty, \infty]$. Aggregation on I for a fixed number n of inputs always means a processing of input data by a special n–ary function defined on I^n. Similarly, aggregation on I for an arbitrary (but fixed) finite number of inputs can be seen as a data processing by a system of such functions. One of the crucial problems in that case is the relationship of the functions from the system differing in the number of inputs.

Note that to shorten some expressions, we will write $\mathbf{x}$ instead of $(x_1, \dots, x_n)$.

Definition 1.

(i) An *n–ary aggregation function* is a function $A^{(n)} : I^n \to I$ that is non–decreasing in each place and fulfills the following boundary conditions

$$\inf_{\mathbf{x}\in I^n} A^{(n)}(\mathbf{x}) = \inf I \quad \text{and} \quad \sup_{\mathbf{x}\in I^n} A^{(n)}(\mathbf{x}) = \sup I.$$

(ii) An *extended aggregation function* is a function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ such that for all $n > 1$, $A^{(n)} = A_{|I^n}$ is an n–ary aggregation function and $A^{(1)}$ is the identity on I.

We first recall several examples of extended aggregation functions on I:

- The *sum* Σ,
$$\Sigma(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i,$$
in the case of an interval I with the left–end point $-\infty$ or 0, the right–end point 0 or ∞, and with the convention $(-\infty) + \infty = -\infty$ if necessary.
- The *product* Π,
$$\Pi(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i,$$
if I is an interval with the left–end point 0 or 1, the right–end point 1 or ∞ and with the convention $0 \,.\, \infty = 0$ if necessary.
- The *arithmetic mean* M,
$$M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i,$$
on an arbitrary interval I, and if $I = [-\infty, \infty]$, the convention $(-\infty)+\infty = -\infty$ is adopted.
- The *geometric mean* G,
$$G(x_1, \ldots, x_n) = \left(\prod_{i=1}^{n} x_i\right)^{1/n},$$
where $I \subseteq [0, \infty]$, and $0 \,.\, \infty = 0$ by convention.
- The *minimum* Min,
$$Min(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\} = \bigwedge_{i=1}^{n} x_i.$$
- The *maximum* Max,
$$Max(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\} = \bigvee_{i=1}^{n} x_i.$$

In all above mentioned extended aggregation functions there is some relationship between aggregation functions $A^{(n)}$ and $A^{(m)}$ for all $n, m \in \mathbb{N}$. This is not guaranteed by Definition 1, in general. Before discussing this problem in more details, we recall some basic properties of (n–ary/extended) aggregation functions. Unless otherwise specified, a property of a discussed extended aggregation function $A : \bigcup_{n\in\mathbb{N}} I^n \to I$ means that each n–ary aggregation function $A^{(n)} : I^n \to I$

possesses the mentioned property. Therefore we will define the next properties for n–ary aggregation functions only.

Definition 2. For a fixed $n \in \mathbb{N} \setminus \{1\}$, let $A^{(n)} : I^n \to I$ be an n–ary aggregation function on I. Then $A^{(n)}$ is called:

(i) *symmetric* (anonymous) if for each permutation $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ and each $\mathbf{x} \in I^n$

$$A^{(n)}(\mathbf{x}) = A^{(n)}(x_{\sigma(1)}, \dots, x_{\sigma(n)})\,;$$

(ii) *idempotent* (unanymous) if for each $c \in I$

$$A^{(n)}(c, \dots, c) = c\,;$$

(iii) *strictly monotone* if for all $x_i,\, y_i \in I$, $i \in \{1, \dots, n\}$ such that $x_i \le y_i$ and $(x_1, \dots, x_n) \ne (y_1, \dots, y_n)$ it follows that

$$A^{(n)}(x_1, \dots, x_n) < A^{(n)}(y_1, \dots, y_n)\,;$$

(iv) *continuous* if for each $\mathbf{x_0} \in I^n$,

$$\lim_{\mathbf{x} \to \mathbf{x_0}} A^{(n)}(\mathbf{x}) = A^{(n)}(\mathbf{x_0}),$$

i.e., if $A^{(n)}$ is a continuous function of n variables in the usual sense;

(v) 1–*Lipschitz*, if for all $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in I^n$,

$$|A^{(n)}(x_1, \dots, x_n) - A^{(n)}(y_1, \dots, y_n)| \le \sum_{i=1}^{n} |x_i - y_i|;$$

(vi) *bisymmetric* if for all $n \times n$ matrices $X = (x_{ij})$, with entries $x_{ij} \in I$ for all $i,\, j \in \{1, \dots, n\}$,

$$\begin{aligned} & A^{(n)}\left(A^{(n)}(x_{11}, \dots, x_{1n}), \dots, A^{(n)}(x_{n1}, \dots, x_{nn})\right) \\ = \;& A^{(n)}\left(A^{(n)}(x_{11}, \dots, x_{n1}), \dots, A^{(n)}(x_{1n}, \dots, x_{nn})\right). \end{aligned}$$

We can equivalently say that, for example, an n–ary aggregation function $A^{(n)}$ is symmetric if and only if for all $\mathbf{x} \in I^n$ it holds

$$A^{(n)}(\mathbf{x}) = A^{(n)}(x_2, x_1, x_3, \dots, x_n) = A^{(n)}(x_2, \dots, x_n, x_1).$$

Similarly, the idempotency of $A^{(n)}$ is equivalent to the property

$$Min^{(n)} \le A^{(n)} \le Max^{(n)}.$$

Definition 3. For a fixed $n \in \mathbb{N} \setminus \{1\}$, let $A^{(n)} : I^n \to I$ be an n–ary aggregation function on I.

(i) An element $e \in I$ is called *neutral element* of $A^{(n)}$ if for each $i \in \{1, \dots, n\}$ and each $x_i \in I$ it holds that

$$A^{(n)}(e, \dots, e, x_i, e, \dots, e) = x_i\,.$$

(ii) An element $a \in I$ is called *annihilator* of $A^{(n)}$ if for all $(x_1, \dots, x_n) \in I^n$ it holds that
if $x_i = a$ for some $i \in \{1, \dots, n\}$ then $A^{(n)}(x_1, \dots, x_n) = a$.

For extended aggregation functions we can also introduce stronger versions of idempotency, neutral element and bisymmetry.

Definition 4. Let $A : \bigcup\limits_{n\in\mathbb{N}} I^n \to I$ be an extended aggregation function. Then

(i) A is *strongly idempotent* whenever

$$A(\underbrace{\mathbf{x},\dots,\mathbf{x}}_{k-times}) = A(\mathbf{x})$$

for all $k \in \mathbb{N}$ and $\mathbf{x} \in \bigcup\limits_{n\in\mathbb{N}} I^n$.

(ii) An element $e \in I$ is said to be a *strong neutral element* of A if for each $n \in \mathbb{N}$, each $\mathbf{x} \in \bigcup\limits_{n\in\mathbb{N}} I^n$ and $i \in \{1,\dots,n+1\}$ it holds

$$A(\mathbf{x}) = A(x_1,\dots,x_{i-1},e,x_i,\dots,x_n).$$

(iii) A is *strongly bisymmetric* if for any $n\times m$ matrix $X = (x_{ij})$ with all entries $x_{ij} \in I$, it holds

$$A^{(n)}(A^{(m)}(\mathbf{x}_{1.}),\dots,A^{(m)}(\mathbf{x}_{n.})) = A^{(m)}(A^{(n)}(\mathbf{x}_{.1}),\dots,A^{(n)}(\mathbf{x}_{.m})),$$

where for all $i \in \{1,\dots,n\}$, $j \in \{1,\dots,m\}$,

$$\mathbf{x}_{i.} = (x_{i1},\dots,x_{im}) \quad \text{and} \quad \mathbf{x}_{.j} = (x_{1j},\dots,x_{nj}).$$

Classical properties linking different input arities of extended aggregation functions are:

- *associativity*, that is, for each n, $m \in \mathbb{N}$, $\mathbf{x} \in I^n$, $\mathbf{y} \in I^m$
$$A^{(n+m)}(\mathbf{x},\mathbf{y}) = A^{(2)}(A^{(n)}(\mathbf{x}),A^{(m)}(\mathbf{y}))\,;$$
- *decomposability*, that is, for all integers $0 \le k \le n$, $n \in \mathbb{N}$, and all $\mathbf{x} \in I^n$

$$A^{(n)}(x_1,\dots,x_k,x_{k+1},\dots,x_n) = A^{(n)}(\underbrace{A^{(k)}(x_1,\dots,x_k)}_{k-times},\underbrace{A^{(n-k)}(x_{k+1},\dots,x_n)}_{(n-k)-times}).$$

The associativity of an extended aggregation function A is equivalent to the standard associativity of the corresponding binary aggregation function $A^{(2)}$,

$$A^{(2)}(x,A^{(2)}(y,z)) = A^{(2)}(A^{(2)}(x,y),z)$$

for all x, y, $z \in I$, and $A^{(n)}$ for $n > 2$, being the genuine n–ary extension of $A^{(2)}$ given by

$$A^{(n)}(x_1,\dots,x_n) = A^{(2)}\left(A^{(n-1)}(x_1,\dots,x_{n-1}),x_n\right)$$

defined by induction. Evidently, using this way, any binary aggregation function $A^{(2)}$ can be extended to an extended aggregation function $\overline{A^{(2)}} = A$. More generally, a huge class of extended aggregation functions can be constructed from a system $\mathcal{A} = (A_n^{(2)})_{n\in\mathbb{N}}$ of binary aggregation functions by induction. We define $A_{\mathcal{A}} = A$ as follows:

$$\begin{aligned}
&A^{(1)}(x_1) = x_1,\\
&A^{(2)}(x_1,x_2) = A_1^{(2)}(x_1,x_2),\\
&\vdots\\
&A^{(n)}(x_1,\dots,x_n) = A_{n-1}^{(2)}\left(A^{(n-1)}(x_1,\dots,x_{n-1}),x_n\right)\\
&\vdots
\end{aligned}$$

Extended aggregation functions $A_{\mathcal{A}} = A$ were called recursive by Montero, see e.g. [35], compare also [17]. Evidently, each associative extended aggregation function

is recursive but not vice–versa.
From the examples introduced above, the sum Σ is symmetric, associative and bisymmetric. If $0 \in I$, then 0 is the strong neutral element of Σ, if $-\infty \in I$ then this element is the annihilator of Σ, and if $+\infty \in I$ and $-\infty \notin I$ then $+\infty$ is the annihilator of Σ. The extended aggregation function Σ is 1–Lipschitz and strictly monotone if $I \subset \mathbb{R}$, continuous if $I \neq [-\infty, \infty]$.

The arithmetic mean M is recursive, symmetric, strongly idempotent and bisymmetric on any interval I. It is 1–Lipschitz and strictly monotone if $I \subset \mathbb{R}$ and continuous if $I \neq [-\infty, \infty]$. It has an annihilator a only if I is an unbounded interval, namely, $a = -\infty$ if $-\infty \in I$; $a = \infty$, if $\infty \in I$ and $-\infty \notin I$.

Let the extended aggregation function $A : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ be given by

$$A(x_1, \ldots, x_n) = \min\left(x_1, \prod_{i=2}^{n} x_i\right)$$

whenever $n > 1$. Evidently, $e = 1$ is the neutral element of A, but it is not a strong neutral element. Indeed, if we take $(x_1, x_2) = (0.5, 0.5)$ then, for $i = 1$ we have $A(1, x_1, x_2) = 0.25$, for $i = 2$ and $i = 3$ we have $A(x_1, 1, x_2) = A(x_1, x_2, 1) = 0.5$. Observe that A is a quasi–copula, see Section 4, i.e., A is 1-Lipschitz.
To simplify notation, if no confusion can arise, n–ary aggregation functions $A^{(n)}$ will simply be denoted by A without stressing their arity.

The basic classification of aggregation functions takes into account the main fields of applications. Following Dubois and Prade [21], we will distinguish four classes of (n–ary/extended) aggregation functions:

- *conjunctive aggregation functions*: aggregation functions $A \leq Min$;
- *averaging aggregation functions*: aggregation functions A, $Min \leq A \leq Max$, or, equivalently, idempotent aggregation functions;
- *disjunctive aggregation functions*: aggregation functions $A \geq Max$;
- *mixed aggregation functions*: aggregation functions which do not belong to any of other three classes.

Observe that the interval I may be crucial for the classification of a discussed aggregation function. For example, the product Π is a conjunctive aggregation function on $[0, 1]$, disjunctive on $[1, \infty]$ and mixed on $[0, \infty]$.

For any decreasing one–to–one mapping $\varphi : I \to I$, $A : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ is a conjunctive (disjunctive) extended aggregation function if and only if the function $A_\varphi : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ given by

$$A_\varphi(x_1, \ldots, x_n) = \varphi^{-1}\left(A(\varphi(x_1), \ldots, \varphi(x_n)\right)$$

is a disjunctive (conjunctive) extended aggregation function. This duality allows to investigate, construct and discuss conjunctive aggregation functions only, and to transfer all the results by this duality to the disjunctive aggregation functions.

3. Averaging aggregation functions

We first recall the basic averaging aggregation functions, for more details we recommend [6]:

- The *arithmetic mean* M,
$$M(x_1, \dots, x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i \,.$$
- *Quasi–arithmetic means* M_f, where $f : I \to [-\infty, \infty]$ is a continuous strictly monotone function and
$$M_f(x_1, \dots, x_n) = f^{-1}(M(f(x_1), \dots, f(x_n)),$$
as, for example, the geometric, harmonic and quadratic means.
- *Weighted arithmetic means* $M_{\mathbf{w}}$, where $\mathbf{w} = (w_1, \dots, w_n)$, $w_i \geq 0$, $\sum\limits_{i=1}^{n} w_i = 1$ and
$$M_{\mathbf{w}}(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i \, x_i,$$
see also Section 5.
- *Weighted quasi–arithmetic means* $M_{f,\mathbf{w}}$,
$$M_{f,\mathbf{w}}(x_1, \dots, x_n) = f^{-1} \left(\sum_{i=1}^{n} w_i \, f(x_i) \right).$$
- *OWA* (*ordered weighted average*) *operator* $M'_{\mathbf{w}}$,
$$M'_{\mathbf{w}}(x_1, \dots, x_n) = M_{\mathbf{w}}(x'_1, \dots, x'_n) = \sum_{i=1}^{n} w_i \, x'_i,$$
where x'_i is the i–th order statistics from the sample $(x_1, \dots, x_n)$.
- *OWQA* (*ordered weighted quasi–arithmetic*) *operator* $M'_{f,\mathbf{w}}$,
$$M'_{f,\mathbf{w}}(x_1, \dots, x_n) = M_{f,\mathbf{w}}(x'_1, \dots, x'_n) = f^{-1} \left(\sum_{i=1}^{n} w_i \, f(x'_i) \right).$$
- *Idempotent uninorms*, [14].
- *Idempotent nullnorms*, i.e., a–medians, given for a fixed $a \in I$ by
$$Med_a(x_1, \dots, x_n) = \mathrm{med}(x_1, a, x_2, a, x_3, a, \dots, a, x_n).$$
- *Fuzzy integrals*, [28, 46].

Recall that for any 2–copula $C : [0,1]^2 \to [0,1]$ (for the definition of a copula see the next section) and for any fuzzy measure $m : \mathcal{P}(\{1, \dots, n\}) \to [0,1]$, i.e., a non–decreasing set function such that $m(\emptyset) = 0$ and $m(\{1, \dots, n\}) = 1$, we can define a fuzzy integral $F_{C,m} : [0,1]^n \to [0,1]$ by
$$F_{C,m}(x_1, \dots, x_n) = \sum_{i=1}^{n} \left(C\left(x'_i, m(\{j \,|\, x_j \geq x'_i\})\right) - C\left(x'_{i-1}, m(\{j \,|\, x_j \geq x'_i\})\right) \right),$$
with the convention $x'_0 = 0$, where x'_i is the i–th order statistics from the sample $(x_1, \dots, x_n)$. Then $F_{\Pi,m}$ is the Choquet integral [11, 18, 37] and $F_{Min,m}$ is the Sugeno integral [42, 37]. Also observe that if m is additive then $F_{\Pi,m} = M_{\mathbf{w}}$ is the weighted arithmetic mean with the weights given by $w_i = m(\{i\})$. Similarly, if m is symmetric, i.e., $m(A) = h\left(\frac{cardA}{n}\right)$ for some increasing function $h : [0,1] \to [0,1]$, then $F_{\Pi,m}$ is the OWA operator $M'_{\mathbf{w}}$ with the weights $w_i = h\left(\frac{i}{n}\right) - h\left(\frac{i-1}{n}\right)$.

Note that averaging aggregation functions are closed under composition, i.e., for any averaging (extended) aggregation functions A, $A_1, \ldots, A_n$ on I, also the function $D = A(A_1, \ldots, A_n) : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$, given by $D(\mathbf{x}) = A(A_1(\mathbf{x}), \ldots, A_n(\mathbf{x}))$, is an averaging extended aggregation function.

An interesting class of averaging aggregation functions are the *internal* aggregation functions characterized by $A(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$. Continuous internal aggregation functions are exactly lattice polynomials, whose prescription formula contains inputs $x_1, \ldots, x_n$, symbols for join $\vee$ and meet $\wedge$, i.e., Max and Min in infix form, and parentheses. Independently of the interval I, they have the same formula, and on any open interval I they are the only aggregation functions invariant under any increasing $I \to I$ one–to–one transformation φ. On $[0,1]$, they are in a one–to–one correspondence with $\{0,1\}$– valued fuzzy measures (and then we can apply any fuzzy integral based on a copula C, e.g., the Choquet or Sugeno integrals). As an example we give all 18 ternary aggregation functions which are internal and continuous on any interval I:

$A^{(3)}(x_1, x_2, x_3) =$

$$
\begin{array}{lll}
x_1; & x_2; & x_3; \\
x_1 \wedge x_2; & x_1 \wedge x_3; & x_2 \wedge x_3; \\
x_1 \vee x_2; & x_1 \vee x_3; & x_2 \vee x_3; \\
x_1 \wedge (x_2 \vee x_3); & x_2 \wedge (x_1 \vee x_3); & x_x \wedge (x_1 \vee x_2); \\
x_1 \vee (x_2 \wedge x_3); & x_2 \vee (x_1 \wedge x_3); & x_3 \vee (x_1 \wedge x_2); \\
x_1 \wedge x_2 \wedge x_3 = x_1'; & (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3) = x_2'; & x_1 \vee x_2 \vee x_3 = x_3' .
\end{array}
$$

Another interesting and still not completely described family of averaging extended aggregation functions are the mixture operators $M^g : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ given by

$$
M^g(x_1, \ldots, x_n) = \frac{\sum\limits_{i=1}^{n} g(x_i)\, x_i}{\sum\limits_{i=1}^{n} g(x_i)},
$$

where $g : I \to]0, \infty[$ is a given weighting function [32, 45]. Evidently, mixture operators are idempotent and they generalize the arithmetic mean M, since $M = M^g$ for any constant weighting function g. Mixture operators are extended aggregation functions if and only if they are monotone, which is not a general case. For example, let $I = [0, b]$ and let $g : I \to]0, \infty[$ be given by $g(x) = x + 1$. Then M^g is an averaging extended aggregation function only if $b \in]0, 1]$. Till now, only some sufficient conditions ensuring the monotonicity of mixture operators M^g are known, as, for example, for a non–decreasing differentiable function g the next two conditions:

(i) $g(x) \geq g'(x)\, l(I)$ for all $x \in I$, where $l(I)$ is the length of the interval I;
(ii) $g(x) \geq g'(x)\, (x - \inf I)$ for all $x \in I$.

Also other generalizations of mixture operators are interesting, as, for example, the *quasi–mixture operators* M_f^g, defined by

$$M_f^g(x_1, \dots, x_n) = f^{-1}\left(\frac{\sum\limits_{i=1}^n g(x_i)\, f(x_i)}{\sum\limits_{i=1}^n g(x_i)}\right),$$

generalized mixture operators $M^{\mathbf{g}}$, where $\mathbf{g} = (g_1, \dots, g_n)$ is a vector of weighting functions, defined by

$$M^{\mathbf{g}}(x_1, \dots, x_n) = \frac{\sum\limits_{i=1}^n g_i(x_i)\, x_i}{\sum\limits_{i=1}^n g_i(x_i)},$$

and *ordered generalized mixture operators* $M'^{\mathbf{g}}$,

$$M'^{\mathbf{g}}(x_1, \dots, x_n) = M^{\mathbf{g}}(x'_1, \dots, x'_n).$$

These operators can be seen as generalizations of the quasi–arithmetic means, weighted arithmetic means and OWA operators, respectively. In general, the monotonicity of such operators is not still clarified.

An interesting composition method of aggregation functions was recently proposed in [10]. For any extended aggregation functions A, B and a binary aggregation function C on I, we define $D = A_{B,C} : \bigcup\limits_{n\in\mathbb{N}} I^n \to I$ by

$$D(x_1, \dots, x_n) = A(C(x_1, B(x_1, \dots, x_n)), \dots, C(x_n, B(x_1, \dots, x_n))).$$

Evidently, if all A, B, C are idempotent then D is also idempotent. As a special case of this method, consider $C = Min^{(2)}$, $A = F_{\Pi,m_1}$, i.e., the Choquet integral with respect to a fuzzy measure m_1 on $\{1, \dots, n\}$, and $B = F_{Min,m_2}$, i.e., the Sugeno integral with respect to a fuzzy measure m_2 on $\{1, \dots, n\}$. Then $D = A_{B,C}$ is the two–fold integral introduced by Narukawa and Torra in [47]. Observe that for m_1 equal to the strongest fuzzy measure m^* given by

$$m^*(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ 1 & \text{otherwise,} \end{cases}$$

we get $A_{B,C} = F_{Min,m_2}$. Similarly, if $m_2 = m^*$, then $A_{B,C} = F_{\Pi,m_1}$. Thus the two–fold integral is an averaging aggregation function generalizing both the Choquet and Sugeno integrals.

4. Conjunctive aggregation functions

In this section we restrict our considerations to the interval $I = [0,1]$ only. As the conjunctive aggregation functions are bounded from above by Min, the weakest extended aggregation function $A_w : \bigcup\limits_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given by

$$A_w(x_1, \dots, x_n) = \begin{cases} 1 & \text{if } \prod\limits_{i=1}^n x_i = 1, \\ 0 & \text{otherwise,} \end{cases}$$

is also the weakest conjunctive extended aggregation function, and, obviously, Min is the strongest one. Evidently, $a = 0$ is the annihilator of any conjunctive aggregation function A. Depending on the field of applications, often some kind of neutrality for the element $e = 1$ is required [3, 13].

Definition 5. An (n–ary) aggregation function A on $[0, 1]$ is called a (an n–) *semicopula* whenever $e = 1$ is its neutral element. An extended aggregation function A on $[0, 1]$ with the strong neutral element $e = 1$ is called a *conjunctor*.

Recall some distinguished classes of conjunctive aggregation functions.

- *Triangular norms* (t-norms for short) [40, 27] are associative symmetric conjunctors.
- *Quasi–copulas* [2, 26] are 1–Lipschitz conjunctive aggregation functions. Observe that each quasi–copula is necessarily a semicopula.
- *Copulas* [41, 36] are n–increasing semicopulas, where the n–increasing property means the non–negativity of all n-th differences. For $n = 2$ this means that $A : [0, 1]^2 \to [0, 1]$ is 2–increasing if and only if for all (x_1, x_2), $(y_1, y_2) \in [0, 1]^2$ such that $x_1 \le y_1$ and $x_2 \le y_2$, it holds
$$(A(y_1, y_2) - A(y_1, x_2)) - (A(x_1, y_2) - A(x_1, x_2)) \ge 0.$$
Each copula is 1–Lipschitz, and thus a quasi–copula.

Observe that conjunctive aggregation functions, semicopulas, conjunctors, quasi–copulas and copulas are convex classes, which is not the case of triangular norms. Because of the existence of exhaustive monographs on t-norms [27] and copulas [36] we will not discuss these classes in detail. However, there are some new interesting results worth mentioning.

Recall that each 1–Lipschitz t-norm is an associative copula (as a binary function) and vice–versa. Thus each associative copula is an ordinal sum [27] of Archimedean 1–Lipschitz t-norms. These later are characterized by the convexity of their additive generator, that is, a strictly decreasing continuous function $t : [0, 1] \to [0, \infty]$, $t(1) = 0$.
A related problem concerning k–Lipschitz Archimedean t-norms, $k > 1$, was stated as an open problem in [1]. Mesiarová has recently characterized [34] k–Lipschitz Archimedean t-norms by the k–convexity of their additive generators. The k–convexity of an additive generator t means that for all $0 < x < y < 1$ and $\epsilon \in]0, \min(1 - y, 1 - k\,x)]$ it holds
$$t(x + k\,\epsilon) - t(x) \le t(y + \epsilon) - t(y).$$
Evidently, the 1–convexity reduces to the standard convexity.
The weakest 1–Lipschitz conjunctor is the Łukasiewicz t-norm T_L, in the framework of copulas also called the lower Fréchet–Hoeffding bound, which in the binary form is given by $T_L(x, y) = \max(x + y - 1, 0)$, $(x, y) \in [0, 1]^2$.

Note that the class of all k–Lipschitz t-norms for $k > 1$ has no weakest element, though there are several minimal k–Lipschitz t-norms. The weakest k–Lipschitz conjunctor in the binary form is given by $C_k(x, y) = \max(x + k\,y - k, k\,x + y - k, 0)$.

For each ternary conjunctive aggregation function $C : [0, 1]^3 \to [0, 1]$ we can introduce three binary functions C_{12}, C_{23}, $C_{13} : [0, 1]^2 \to [0, 1]$ given by
$$C_{12}(x, y) = C(x, y, 1), \quad C_{23}(x, y) = C(1, x, y), \quad C_{13}(x, y) = C(x, 1, y).$$

All functions C_{12}, C_{23}, C_{13} are conjunctive. Evidently, if C is the ternary form of some conjunctor, then $C_{12} = C_{23} = C_{13}$. In general these equalities fail even for semicopulas (quasi–copulas, copulas). An interesting problem is also the reverse compatibility problem, namely, under which conditions binary functions A, B, D of some type are the marginal functions of a ternary conjunctive aggregation function C of the same type. In the case of t-norms it is evident that $A = B = D$ are necessarily the binary forms and C is the ternary form of the same t-norm. In the case of semicopulas (quasi–copulas), for any A, B, D there is a ternary semicopula (quasi–copula) C, not necessarily unique, such that $C_{12} = A$, $C_{23} = B$, $C_{13} = D$, for example, $C : [0,1]^3 \to [0,1]$ given by

$$C(x,y,z) = Min(A(x,y), B(y,z), D(x,z))\,. \tag{1}$$

However, for 2–copulas A, B, D the ternary operation C given by (1) need not be a copula, in general. This is, e.g., in the case $A = B = D = T_L$, when C is a 3–quasi–copula but not a 3–copula. For any 2–copulas A, B, let $A * B = D : [0,1]^2 \to [0,1]$ be given by

$$D(x,y) = \int_0^1 \frac{\partial A(x,t)}{\partial t}\,\frac{\partial B(t,y)}{\partial t}\,dt\,.$$

Then D is also a 2–copula [12], and $C : [0,1]^3 \to [0,1]$ given by

$$C(x,y,z) = \int_0^y \frac{\partial A(x,t)}{\partial t}\,\frac{\partial B(t,z)}{\partial t}\,dt$$

is a 3–copula and moreover, $C_{12} = A$, $C_{23} = B$, $C_{13} = D$, compare also [30]. For example, if $A = B = T_L$ then $A * B = Min^{(2)}$ and (T_L, T_L, D) are marginal 2–copulas of a 3–copula $C : [0,1]^2 \to [0,1]$ if and only if $D = T_L * T_L = Min^{(2)}$ and $C(x,y,z) = \max(\min(x,z) + y - 1, 0)$.

Let $\mathcal{C} = \left(C_n^{(2)}\right)_{n\in\mathbb{N}}$ be a system of binary conjunctive aggregation functions. Then the recursive extended aggregation function $C = C_{\mathcal{C}}$,

$$\begin{aligned} C^{(n)}(x_1,\dots,x_n) &= C_{n-1}^{(2)}\left(C^{(n-1)}(x_1,\dots,x_{n-1}), x_n\right) && (2)\\ &= C_{n-1}^{(2)}\left(\dots C_2^{(2)}\left(C_1^{(2)}(x_1,x_2),x_3\right),\dots,x_n\right) && (3)\end{aligned}$$

is conjunctive. If all $C_n^{(2)}$, $n \in \mathbb{N}$, are semicopulas (quasi–copulas) then C is an extended semicopula (quasi–copula). In the case of copulas, it is an open problem under which conditions $C^{(n)}$ is a copula and C is an extended copula. In the case when $C = \overline{C_1^{(2)}}$, i.e., $C_n^{(2)} = C_1^{(2)}$ for all $n \in \mathbb{N}$, and $C_1^{(2)} : [0,1]^2 \to [0,1]$ is an Archimedean 2–copula, then C is an extended copula, that is, an n–copula for each $n \in \mathbb{N}$, if and only if C is generated by a decreasing bijection $t : [0,1] \to [0,\infty]$ whose inverse $t^{-1} : [0,\infty] \to [0,1]$ is totally monotone, that is, whose all derivatives at each point from $]0,\infty[$ exist and are non–negative [36]. Each such copula is necessarily bounded by the product, $C \geq \Pi$, which is an important example of an extended copula, reflecting the independence of random variables. To see a negative example, let $C_1^{(2)} = \Pi^{(2)}$ and $C_n^{(2)} = Min^{(2)}$ for all $n > 1$. Then C is an extended quasi–copula but not an extended copula. Also note that not each extended copula is recursive. For example, the extended aggregation function $C : \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ given by

$$C(x_1,\dots,x_n) = x_1\, Min(x_2,\dots,x_n), \quad n \geq 2,$$

is an extended copula which is not recursive.

Finally, we introduce a useful proposition.

Proposition 1. Let $\mathcal{F}$ be a class of (n–ary/extended) aggregation functions on $[0,1]$ and let $\mathcal{H}_{\mathcal{F}}$ be the set of all binary aggregation functions $D : [0,1]^2 \to [0,1]$ such that for all A, $B \in \mathcal{F}$ also $C = D(A,B)$ given by $C(\mathbf{x}) = D(A(\mathbf{x}), B(\mathbf{x}))$, is an element of $\mathcal{F}$. Then

(i) For the class $\mathcal{F} = \mathcal{A}$ of all (n–ary/extended) aggregation functions on $[0,1]$ it holds $\mathcal{H}_{\mathcal{A}} = \mathcal{A}^{(2)}$, that is, D can be an arbitrary binary aggregation function.

(ii) For the class $\mathcal{F} = \mathcal{B}$ of all conjunctive (n–ary/extended) aggregation functions we have $\mathcal{H}_{\mathcal{B}} = \{D \in \mathcal{A}^{(2)} \mid D \leq Max^{(2)}\}$.

(iii) For the class $\mathcal{F} = \mathcal{S}$ of all (n–ary/extended) semicopulas we have

$$\mathcal{H}_{\mathcal{S}} = \{D \in \mathcal{A}^{(2)} \mid D \text{ is idempotent}\}.$$

(iv) For the class $\mathcal{F} = \mathcal{Q}$ of all quasi–copulas we have

$$\mathcal{H}_{\mathcal{Q}} = \{D \in \mathcal{A}^{(2)} \mid \|D\|_\infty = 1\}.$$

Note that the Chebyshev norm of a binary aggregation function D is given by

$$\|D\|_\infty = \sup\left(\frac{|D(x,y) - D(u,v)|}{\max(|x-u|, |y-v|)}\right),$$

where the supremum is taken over all (x,y), $(u,v) \in [0,1]^2$, $(x,y) \neq (u,v)$.

(v) For the class $\mathcal{F} = \mathcal{C}$ of all copulas we have

$$\mathcal{H}_{\mathcal{C}} = \{D \in \mathcal{A}^{(2)} \mid D \text{ is a weighted mean}\}.$$

(vi) For the class $\mathcal{F} = \mathcal{T}$ of all t-norms we have

$$\mathcal{H}_{\mathcal{T}} = \{P_F,\, P_L\},$$

where $P_F(x,y) = x$ and $P_L(x,y) = y$ for all $(x,y) \in [0,1]^2$.

Evidently,

$$\mathcal{H}_{\mathcal{T}} \subset \mathcal{H}_{\mathcal{C}} \subset \mathcal{H}_{\mathcal{Q}} \subset \mathcal{H}_{\mathcal{S}} \subset \mathcal{H}_{\mathcal{B}} \subset \mathcal{H}_{\mathcal{A}}.$$

By duality, similar notions can be introduced and similar results can be obtained for disjunctive aggregation functions. For an (n-ary/extended) aggregation function A on $I = [0,1]$, the standard duality, here called simply duality, is related to the order reversing bijection $n : [0,1] \to [0,1]$, $n(x) = 1 - x$, the so–called standard negation on $[0,1]$. Then an (n-ary/extended) aggregation function A^d on $[0,1]$ is called the dual of A, if for all $\mathbf{x}$ it holds $A^d(\mathbf{x}) = 1 - A(1 - \mathbf{x})$.

For example, duals of t–norms are t–conorms, that is, associative symmetric aggregation functions with 0 as the strong neutral element. For binary 1–Lipschitz aggregation functions another type of duality was introduced, see [31], compare also [44]. For an aggregation function $A : [0,1]^2 \to [0,1]$ its reverse $A^* : [0,1]^2 \to [0,1]$ is given by $A^*(x,y) = x + y - A(x,y)$. Evidently $(A^*)^* = A$. An interesting problem is under which conditions $A^* = A^d$, i.e., for which binary 1–Lipschitz aggregation functions it holds

$$A(x,y) = x + y - 1 + A(1-x, 1-y) \quad \text{for all } (x,y) \in [0,1]^2. \tag{4}$$

Restricting our considerations to the associative aggregation functions we end up with the famous Frank functional equation [24] and the only solutions to equation (4) are Frank's t–norms and the symmetric ordinal sums of Frank's t–norms, see [29].

5. Weighted Aggregation Functions

This section is devoted to a proposal how to introduce weights (importances) into aggregation. For an input vector $\mathbf{x} = (x_1, \ldots, x_n)$, the corresponding weights $w_1, \ldots, w_n$ can be understood as cardinalities of single inputs $x_1, \ldots, x_n$, respectively. We will deal with weighting vectors $\mathbf{w} = (w_1, \ldots, w_n)$, $w_i \in [0, \infty[\,, i \in \{1, \ldots, n\}$, and $\sum_{i=1}^{n} w_i > 0$. If $\sum_{i=1}^{n} w_i = 1$, $\mathbf{w}$ will be called a normal weighting vector.

For an extended aggregation function $A : \bigcup_{n \in \mathbb{N}} I^n \to I$, and a weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$ (for some $n \in \mathbb{N}$), we will discuss an n–ary aggregation function $A_{\mathbf{w}} : I^n \to I$, which will be called a *weighted aggregation function.* We expect the next quite natural properties of weighted aggregation functions, compare also [4].

(W1) If $\mathbf{w} = (1, \ldots, 1) = \mathbf{1}$ then

$$A_{\mathbf{1}}(x_1, \ldots, x_n) = A(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in I^n$.

(W2) For any $(x_1, \ldots, x_n) \in I^n$ and any $\mathbf{w} = (w_1, \ldots, w_n)$,

$$A_{\mathbf{w}}(x_1, \ldots, x_n) = A_{\mathbf{w}^*}(x_{m_1}, \ldots, x_{m_k}),$$

where $\{m_1, \ldots, m_k\} = \{i \in \{1, \ldots, n\} \mid w_i > 0\}$, $m_1 < \ldots < m_k$, $\mathbf{w}^* = (w_{m_1}, \ldots, w_{m_k})$.

(W3) If $\mathbf{w}$ is a normal weighting vector then $A_{\mathbf{w}}$ is an idempotent aggregation function.

Observe that (W1) simply embeds the aggregation function A into weighted aggregation functions. Further, due to (W2), a zero weight w_i in a weighting vector $\mathbf{w}$ means that we can omit the corresponding score x_i (and the weight $w_i = 0$) from aggregation. Finally, the property (W3) expresses the standard boundary condition for extended aggregation functions, namely, that the aggregation of a unique input x results in x, $A(x) = x$. Then $A_{\mathbf{w}}(x_1, \ldots, x_n)$ with $\sum_{i=1}^{n} w_i = 1$ can be seen as the aggregation of x with cardinality $\sum_{i=1}^{n} w_i = 1$, i.e., $A_{\mathbf{w}}(x, \ldots, x) = A(x) = x$, which is exactly the idempotency of the function $A_{\mathbf{w}}$.

The standard summation on $[0, +\infty]$ can be understood as a typical aggregation on $[0, +\infty]$. For a given weighting vector $\mathbf{w} = (w_1, \ldots, w_n)$, the weighted sum $\sum_{i=1}^{n} w_i\, x_i$ is simply the sum of inputs x_i transformed by means of weights w_i into new inputs $y_i = w_i\, x_i$. Note that the common multiplication of reals applied in the next transformation can be straightforwardly deduced from the original summation (and the standard order of real numbers), i.e., for $w \geq 0$, $x \in [0, +\infty]$

$$w \cdot x = \sup \Big(y \in [0, +\infty] \mid \exists\, i, j \in \mathbb{N},\ \tfrac{i}{j} < w \text{ and } u \in [0, +\infty] \text{ such that } \sum_{k=1}^{j} u < x \text{ and } y = \sum_{k=1}^{i} u \Big) \tag{5}$$

Recall that the weighted sum $\sum_{i=1}^{n} w_i\, x_i$ for weights w_i such that $\sum_{i=1}^{n} w_i = 1$ is just the weighted arithmetic mean. The above discussed approach can be applied to any continuous symmetric associative aggregation function defined on $I = [0, c]$ with neutral element 0, as, for example, to any continuous t–conorm S. The weighted t–conorm $S_{\mathbf{w}} : [0, 1]^n \to [0, 1]$, where $n = \dim \mathbf{w}$, is simply defined as

$$S_{\mathbf{w}}(x_1, \ldots, x_n) = S(w_1 \odot x_1, \ldots, w_n \odot x_n) \tag{6}$$

where the transformed input data $w_i \odot x_i$ are obtained from the weights w_i and the original inputs x_i by means of a binary operation $\odot : [0, +\infty[\times [0, 1] \longrightarrow [0, 1]$,

$$w \odot x = \sup \ (y \in [0, 1] \mid \exists\, i, j \in \mathbb{N},\ \tfrac{i}{j} < w \text{ and } u \in [0, 1] \text{ such that } S(\underbrace{u, \ldots, u}_{j-times}) < x \text{ and } y = S(\underbrace{u, \ldots, u}_{i-times})). \tag{7}$$

Evidently, (7) is an appropriate modification of (5). Note that $0 \odot x = 0$ and $1 \odot x = x$ for all $x \in [0, 1]$. In the case when S has unit multipliers, i.e., $S(x, y) = 1$ for some x, $y \in [0, 1[$ we should require $\sum_{i=1}^{n} w_i \geq 1$ to keep the boundary condition $S_{\mathbf{w}}(1, \ldots, 1) = 1$. Obviously, the weighted t–conorm $S_{\mathbf{w}}$ for any continuous t–conorm S fulfills axioms (W1), (W2), (W3). More details about weighted t–conorms can be found in [5], including several examples. Recall some facts:

- $Max_{\mathbf{w}}(x_1, \ldots, x_n) = \max(x_i \mid w_i > 0)$, (due to $w \odot x = x$ if $w > 0$);
- $S_{\mathbf{w}}$ is lower semi–continuous (left continuous);
- $S_{\mathbf{w}}$ (with some nontrivial $w_i \notin \{0, 1\}$) is continuous if and only if either $S = Max$ or S is a continuous Archimedean t–conorm;
- If S is continuous Archimedean t–conorm with an additive generator $g : [0, 1] \to [0, +\infty]$, and $\mathbf{w}$ is a normal weighting vector, then $S_{\mathbf{w}}(x_1, \ldots, x_n) = g^{-1}\left(\sum_{i=1}^{n} w_i g(x_i)\right)$, i.e., $S_{\mathbf{w}}$ is a weighted quasi–arithmetic mean (because $w \odot x = g^{-1}(w \cdot g(x))$ for $w \in [0, 1]$). It is either cancelative (if S is a nilpotent t–conorm; e.g., the Yager t–conorm for $p = 2$, see [27], leads to the weighted quadratic mean) or it has annihilator $a = 1$ (if S is a strict t–conorm).

Dual operators to t–conorms are t–norms [27]. Weighted t–norms can be defined in the spirit of (6) and (7), or, equivalently, by the duality, i.e.,

$$T_{\mathbf{w}}(x_1, \ldots, x_n) = 1 - S_{\mathbf{w}}(1 - x_1, \ldots, 1 - x_n), \tag{8}$$

where T is an arbitrary continuous t–norm and $S = T^d$ is the corresponding dual t–conorm. Note that axioms (W1), (W2) and (W3) are also fulfilled for weighted t–norms. Similarly as in the case of weighted t–conorms we have the following facts:

- $Min_{\mathbf{w}}(x_1, \ldots, x_n) = \min(x_i \mid w_i > 0)$;
- $T_{\mathbf{w}}$ is upper semi–continuous (right continuous);
- $T_{\mathbf{w}}$ (with some nontrivial $w_i \notin \{0, 1\}$) is continuous if and only if either $T = \min$ or T is a continuous Archimedean t–norm;
- If T is a continuous Archimedean t–norm with an additive generator $f : [0, 1] \to [0, +\infty]$, and $\mathbf{w}$ is a normal weighting vector, then

$$T_{\mathbf{w}}(x_1, \ldots, x_n) = f^{-1}\left(\sum_{i=1}^{n} w_i f(x_i)\right),$$

  i.e., $T_{\mathbf{w}}$ is a weighted quasi–arithmetic mean. It is cancelative whenever T is nilpotent and it has annihilator 0 whenever T is a strict t–norm.

For example, for the product t–norm Π, the relevant normal weighted function $\Pi_{\mathbf{w}}$ is just the weighted geometric mean.

Observe that if $\sum\limits_{i=1}^{n} w_i = n$, then for a continuous Archimedean t–norm T generated by an additive generator f the corresponding weighted operator is given by $T_{\mathbf{w}}(x_1, \ldots, x_n) = f^{(-1)}\left(\sum\limits_{i=1}^{n} w_i f(x_i)\right)$ what is just a weighted generated t–norm as proposed by Dubois and Prade in [20].

Several aggregation functions can be built by means of t–norms and t–conorms, for example, nullnorms, uninorms, Γ–operators, etc. Their weighted versions are then built from the corresponding weighted t–norms and t–conorms. For more details we recommend [**?**].

The basic idea of quantitative weights as cardinalities can be straightforwardly illustrated on the example of the weighted mean arising from the arithmetic mean. In statistics, starting with integer weights n_i, which are simply frequencies of observations x_i, the weighted mean is

$$M_{\mathbf{n}}(x_1, \ldots, x_n) = \frac{\sum\limits_{i=1}^{n} n_i\, x_i}{\sum\limits_{i=1}^{n} n_i},$$

where $\mathbf{n} = (n_1, \ldots, n_n)$. Because of the strong idempotency of the standard arithmetic mean, $M_{\mathbf{n}}$ can be easily generalized into the form

$$M_{\mathbf{w}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i\, x_i, \quad w_i \geq 0, \ \ \sum_{i=1}^{n} w_i = 1.$$

The previous property of the standard arithmetic mean we can apply on any symmetric strongly idempotent extended aggregation function A. The strong idempotency of a symmetric extended aggregation function A allows to introduce integer and rational quantitative weights – simply looking at them as cardinalities. In fact, we repeat the standard approach applied to the arithmetic mean as mentioned above. Indeed, for inputs $x_1, \ldots, x_n \in I$ and integer weights $\mathbf{w} = (w_1, \ldots, w_n) \in (\mathbb{N} \cup \{0\})^n$, we put

$$A_{\mathbf{w}}(x_1, \ldots, x_n) = \mathbf{A}(\underbrace{x_1, \ldots, x_1}_{w_1-times}, \underbrace{x_2, \ldots, x_2}_{w_2-times}, \ldots, \underbrace{x_n, \ldots, x_n}_{w_n-times}). \tag{9}$$

Obviously, if $\mathbf{k} = (k, \dots, k)$, $k \in \mathbb{N}$, is a constant weighting vector, the symmetry and the strong idempotency of A result in $A_{\mathbf{k}}(\mathbf{x}) = A(\mathbf{x})$. This fact allows to define consistently the weighted aggregation in the case of rational weights $w_i \in \mathbb{Q}^+$. In that case we find such an integer $k \in \mathbb{N}$ that $k\, w_i \in \mathbb{N} \cup \{0\}$ for all $i = 1, \dots, n$, and we put

$$A_{\mathbf{w}}(\mathbf{x}) = A_{k\mathbf{w}}(\mathbf{x}). \tag{10}$$

The resulting fused value in (10) does not depend on the actual choice of $k \in \mathbb{N}$. Further, because of (10) and (9), $A_{\mathbf{w}} = A_{p\mathbf{w}}$ for each positive rational p and each rational weighting vector $\mathbf{w} \in (\mathbb{Q}^+)^n$, $\mathbf{w} \neq (0, \dots, 0)$. Therefore we can deal with normed (rational) weighting vectors only, that is, we may suppose that $\sum\limits_i w_i = 1$. The last problem we need to solve, is the case when also irrational weights w_i are admitted.

Definition 6. Let $A : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ be a symmetric strongly idempotent extended aggregation function. For any non–zero weighting vector $\mathbf{w} = (w_1, \dots, w_n) \in [0, \infty[^{\,n}$, the corresponding n–ary weighted function $A_{\mathbf{w}} : I^n \to I$ is defined as follows:

(i) If all weights w_i are rational, we apply formulas (10) and (9).

(ii) If there is some irrational weight w_i, denote $\mathbf{w}^* = (w_1^*, \dots, w_n^*)$ the corresponding normed weighting vector, that is, $\mathbf{w} = \left(\sum\limits_i w_i\right) \mathbf{w}^*$.

For any $m \in \mathbb{N}$, $i \in \{1, \dots, n\}$, let

$$w_i^{(m)} = \min\left(\frac{j}{m\,!} \mid j \in \mathbb{N} \cup \{0\},\ \frac{j}{m\,!} \geq w_i^*\right),$$

and $\mathbf{w}^{(m)} = (w_1^{(m)}, \dots, w_n^{(m)})$.
Then $w_i^{(m)} \in \mathbb{Q}^+$ and $\sum\limits_i w_i^{(m)} \geq 1$ for all $m \in \mathbb{N}$ (and if already all weights $w_i^* \in \mathbb{Q}^+$, then also $w_i^{(m)} = w_i^*$ for all i and all sufficiently large m) and we define

$$A_{\mathbf{w}}(\mathbf{x}) = \liminf_{m \to \infty} A_{\mathbf{w}^{(m)}}(\mathbf{x}) \quad \text{for all } \mathbf{x} \in I^n. \tag{11}$$

The following result can be straightforwardly checked from Definition 6.

Proposition 2. Let $\triangle = (\mathbf{w}^{(n)})_{n=1}^{\infty}$ be a weighting triangle, i.e., for each $n \in \mathbb{N}$, let $\mathbf{w}^{(n)} = (w_{1,n}, \dots, w_{n,n})$ be a non–zero weighting vector. Under the notations and requirements in Definition 6, define the function $A_\triangle : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$, $A_\triangle(\mathbf{x}) = A_{\mathbf{w^{(n)}}}(\mathbf{x})$, whenever $\mathbf{x} \in I^n$. Then $A_\triangle$ is a well defined idempotent extended aggregation function.

Note that the approach allowing to introduce integer (rational) weights as given in formulas (9) and (10) was already applied to decomposable idempotent symmetric extended aggregation functions, see [23]. However, our results cover a wider class of symmetric strongly idempotent extended aggregation functions. For example, let $g : [0,1] \to [0,1]$ be given by $g(x) = 2\,x - x^2$. Define the function $A : \bigcup\limits_{n \in \mathbb{N}} I^n \to I$ by

$$A(x_1, \dots, x_n) = \sum_{i=1}^{n} \left(g\left(\frac{i}{n}\right) - g\left(\frac{i-1}{n}\right)\right) x_i',$$

where x'_i is the i–th order statistics from the sample $(x_1, \ldots, x_n)$. Then A, which is an extended OWA operator, is a symmetric strongly idempotent extended aggregation function which is not decomposable. Further observe that the limit in formula (11) need not exist, in general.

The idea of qualitative weights incorporation into aggregation is linked to the transformation of the inputs by means of the corresponding weights from $[0,1]$ (as parameters expressing the importance of the corresponding input coordinates/criteria),

$$A_{\mathbf{w}}(\mathbf{x}) = A(h(w_1, x_1), \ldots, h(w_n, x_n)), \tag{12}$$

where $h : [0,1] \times I \to [0,1]$ is an appropriate binary function. This idea was already applied, e.g., in expert systems, and for $I = [0,1]$ it was introduced by Yager in [50], where h is a function called a RET operator. More details about RET operators can also be found in [43].

To ensure (W1), the following property of h is required:
(RET1) $h(1, x) = x$ for all $x \in I$.
Similarly, to ensure (W2), A is supposed to have a neutral element e and then
(RET2) $h(0, x) = e$ for all $x \in I$.
Further, to ensure the monotonicity of $A_{\mathbf{w}}$, one requires
(RET3) $h(w, \cdot)$ is non-decreasing for all $w \in [0,1]$.
Finally, to ensure the boundary conditions of aggregation functions, one requires
(RET4) $h(\cdot, b)$ is non-decreasing for all $b \geq e$;
(RET5) $h(\cdot, b)$ is non-increasing for all $b \leq e$.

Proposition 3. Let $A : \bigcup_{n \in \mathbb{N}} I^n \to I$ be an extended aggregation function with neutral element e and let $h : [0,1] \times I \to I$ fulfil properties (RET1)–(RET5). For any weighting vector $\mathbf{w} \in [0,1]^n$, $\max w_i = 1$, define the function $A_{\mathbf{w}}$ by (12). Then $A_{\mathbf{w}}$ is an n-ary aggregation function satisfying axioms (W1), (W2) and (W3).

We only recall a typical example of a RET operator given by $h: \ h(w,x) = (x-e)w + e$. If $e = 0$ and $I = [0,1]$, any binary semicopula fulfills (RET1)–(RET5), while for $e = 1$, any fuzzy implication satisfying the neutrality principle, which corresponds to (RET1), see, e.g. [27], can be applied.

In some special cases, h can also be defined for weights exceeding 1, that is, h maps $[0, \infty[\times I$ into I. For example, recall the introduction of weights for continuous t–norms and t–conorms. Take, e.g., a strict t-norm T with an additive generator $f : [0,1] \to [0, \infty]$. Then $h(w, x) = f^{-1}(wf(x))$, and for an arbitrary weighting vector $\mathbf{w}$ (the only constraint is $\sum w_i > 0$) we can put $T_{\mathbf{w}}(\mathbf{x}) = f^{-1}\left(\sum w_i f(x_i)\right)$.

Recall that special classes of anonymous (i.e., symmetric) aggregation functions with neutral elements appropriate for qualitative weights incorporation are triangular norms, triangular conorms, uninorms.

Projections to a distinguished subspace of some metric space are often applied operators which are usually related to some (constraint) optimization problem. The crucial role is played here by the underlying metric, and in fact, we are always looking for the best approximation of a discussed point by some point from the considered subspace. A similar philosophy can be found in defuzzification methods [19, 49], where a fuzzy quantity is characterized by a unique real number. Based on the just mentioned ideas, we introduce a metric–like function on the space of all possible scores (finitely dimensional inputs from some real interval or ordinal

scale). Next we transform our metric–like function into a fuzzy relation. This approach is already standard in the domain of T–equivalence relations where the transformation was done, see, e.g., [16, 38, 39]. For a fixed score $(x_1, x_2, \ldots, x_n)$, we will look for an appropriate "projection" to the subspace of all unanymous scores $(r, r, \ldots, r)$, $r \in I$, applying some defuzzification method. Thus, in fact, we will define a function with inputs and outputs from some real interval I. In the special case of the MOM defuzzification method we will rediscover a generalization of the penalty method introduced by Yager and Rybalov [51], see also [8].
For a fixed real interval I and $n \in \mathbb{N}$ we introduce a dissimilarity function $D : I^n \times I^n \to [0, \infty[$ by

$$D(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} D_i(x_i, y_i), \tag{13}$$

where all $D_i : I^2 \to [0, \infty[$ are particular one-dimensional dissimilarity functions, $D_i(x, y) = K_i(f_i(x) - f_i(y))$, with $K_i :]-\infty, \infty[\to]-\infty, \infty[$ a convex function with the unique minimum $K_i(0) = 0$, and $f_i : I \to]-\infty, \infty[$, a strictly monotone continuous real function. For more details see [33]. Note that if K_i are even functions then D is a metric on I^n.

Definition 7. For a given dissimilarity D, the function $U : I^n \to [0, 1]^I$ which assigns to a score $\mathbf{x}$ the fuzzy subset $U_{\mathbf{x}}$ of I with the membership function

$$U_{\mathbf{x}}(r) = \frac{1}{1 + D(\mathbf{x}, \mathbf{r})}, \tag{14}$$

where $\mathbf{r} = (r, \ldots, r)$, will be called a D–fuzzy utility function.

Proposition 4. Each D–fuzzy utility function U assigns to each score $\mathbf{x} \in I^n$ a continuous quasi-convex fuzzy quantity $U_{\mathbf{x}}$, i.e., for all $r, s \in I$, $\lambda \in [0, 1]$,

$$U_{\mathbf{x}}(\lambda \cdot r + (1 - \lambda)s) \geq \min(U_{\mathbf{x}}(r), U_{\mathbf{x}}(s)),$$

and thus for any $\alpha \in]0, 1]$ the α–cut $U_{\mathbf{x}}^{\alpha} = \{r \in I \mid U_{\mathbf{x}}(r) \geq \alpha\}$ is a closed subinterval of I in the standard topology.

For each defuzzification method DEF acting on quasi–convex (continuous) fuzzy quantities, we can assign to each score $\mathbf{x}$ a characteristic $DEF(U_{\mathbf{x}})$. Supposing that for any fuzzy quantity Q, $DEF(Q) \in supp(Q)$, $DEF(U)$ is an $I^n \to I$ function. In general, this function must be neither idempotent nor non–decreasing. Note that in [33], the conditions on DEF ensuring the idempotency and monotonicity of the aggregation function $DEF(U)$ are discussed. Observe that the MOM defuzzification method (Mean of Maxima) satisfies these conditions and thus we will illustrate our approach on the MOM defuzzification. Note that $MOM(U)(\mathbf{x}) = \frac{1}{2}\left(\inf U_{\mathbf{x}}^{\alpha^*} + \sup U_{\mathbf{x}}^{\alpha^*}\right)$, where $\alpha^* = \sup\{\alpha \in]0, 1] \mid U_{\mathbf{x}}^{\alpha} \neq \emptyset\}$.

Definition 8. For a given dissimilarity D, the MOM-based operator $MOM(U)$ will be denoted by A_D.

As already mentioned above, for any dissimilarity D, A_D is an idempotent aggregation function.

Example 1.

(i) For $D(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} (f(x_i) - f(y_i))^2$, we have $A_D(\mathbf{x}) = f^{-1}\left(\frac{1}{n} \sum_{i=1}^{n} f(x_i)\right)$, i.e., A_D is a quasi–arithmetic mean.

(ii) For $D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} \mid x_i - y_i \mid$, we have $A_D(\mathbf{x}) = \mathrm{med}(x_1,\ldots,x_n)$, i.e., the median operator.

(iii) For $n=2$, $D(\mathbf{x},\mathbf{y}) = \mid x_1 - y_1 \mid + (x_2-y_2)^2$, we have $A_D(\mathbf{x}) = \mathrm{med}(x_1, x_2 - 1/2, x_2 + 1/2)$.

(iv) For $D(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} D_c(x_i,y_i)$, where $D_c(x,y) = \begin{cases} c(y-x), & \text{if } x \le y \\ x-y, & \text{else} \end{cases}$, A_D is the α–quantil (order statistics) with $\alpha = \frac{1}{(1+c)}$.

(v) For $D(\mathbf{x},\mathbf{y}) = \max_{i=1}^{n} \mid x_i - y_i \mid$ we have $A_D(\mathbf{x}) = \frac{\min_i x_i + \max_i x_i}{2}$, i.e., A_D is a special OWA operator.

Dissimilarity based approach to aggregation functions allows a straightforward incorporation of weights. For a weighting vector $\mathbf{w} = (w_1,\ldots,w_n)$, the weighted dissimilarity $D_{\mathbf{w}}$ will be given by $D_{\mathbf{w}}(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{n} w_i\, D_i(x_i,y_i)$ and then we will apply Definition 8 to obtain the corresponding weighted aggregation function. In the case of standard aggregation functions we have obtained in Example 1 (i) and (ii), the standard weighted quasi–arithmetic mean and the weighted median are obtained, respectively. The weighted aggregation function corresponding to Example 1 (iii) is given by $A_{D_{\mathbf{w}}}(\mathbf{x}) = \mathrm{med}(x_1, x_2 - \frac{w_1}{2w_2}, x_2 + \frac{w_1}{2w_2})$.

Finally, following the ideas of Yager [48], we propose to introduce OWAF (ordered weighted aggregation functions) as follows.

Definition 9. Let $A_{\mathbf{w}} : I^n \to I$ be a weighted aggregation function. Then the operator $A'_{\mathbf{w}} : I^n \to I$ given by $A'_{\mathbf{w}}(\mathbf{x}) = A_{\mathbf{w}}(x_{\sigma(1)},\ldots,x_{\sigma(n)})$, where $\sigma : \{1,\ldots,n\} \to \{1,\ldots,n\}$ is a permutation for which $x_{\sigma(1)} \le \ldots \le x_{\sigma(n)}$, will be called an OWAF.

Evidently, starting from a weighted arithmetic mean $M_{\mathbf{w}}$, Definition 9 yields the OWA operator $M'_{\mathbf{w}}$. Note that the ordered weighted t–norm $T'_{(0,1,1)}(x,y,z) = \beta \cdot \gamma$ and its dual ordered weighted t–conorm $S'_{(1,1,0)}(x,y,z) = \alpha + \beta - \alpha\beta$, $\alpha = \min(x,y,z)$, $\beta = \mathrm{med}(x,y,z)$, $\gamma = \max(x,y,z)$, were found to be important in the study of fuzzy preference structures [15].

6. Conclusion

We have discussed some aspects of the theory of aggregation functions, including the review of some properties and classes of aggregation functions, and some construction methods. Especially, we have splitted the properties of extended aggregation functions into local properties, i.e., the properties of relevant n–ary aggregation functions for each fixed n, and into global properties which are often called "strong". Global properties properties constraint different arities functions involved in each extended aggregation function and thus, in the next development of the theory of aggregation functions they should be investigated in more detail. We expect interesting generalizations based on modifications of these standard approaches in the near future. For example, copulas are due to their probabilistic nature strongly connected with the standard operations, especially with the sum. Switching to the possibilistic background, we end up with semicopulas. However, there are many appropriate pseudo–additions (t–conorms) varying between the sum and maximum.

Acknowledgments

This work was supported by the grants VEGA 1/3012/06, 1/2032/05 and GAČR 402/04/1026.

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Department of Mathematics, SvF, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia, and ÚTIA AV ČR, 182 08 Prague, Czech Republik
E-mail address: mesiar@math.sk

Institute IAM, Fac. of Chemical and Food Technology, Slovak University of Technology, Radlinského 9, 812 37 Bratislava, Slovakia
E-mail address: anna.kolesarova@stuba.sk

University of Alcalá, 28871 Alcalá de Henares , Spain
E-mail address: tomasa.calvo@uah.es

Department of Mathematics, SvF, Slovak University of Technology, Radlinského 11, 813 68 Bratislava, Slovakia, and ÚTIA AV ČR, 182 08 Prague, Czech Republik
E-mail address: magda@math.sk