The Brownian web and net

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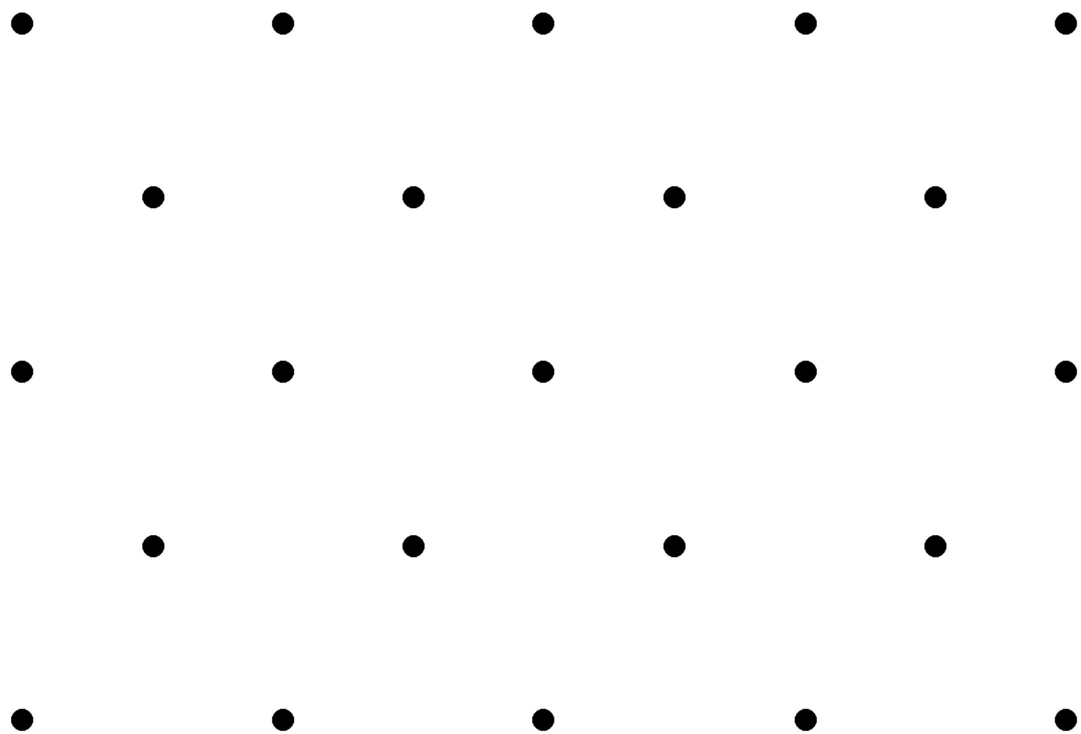
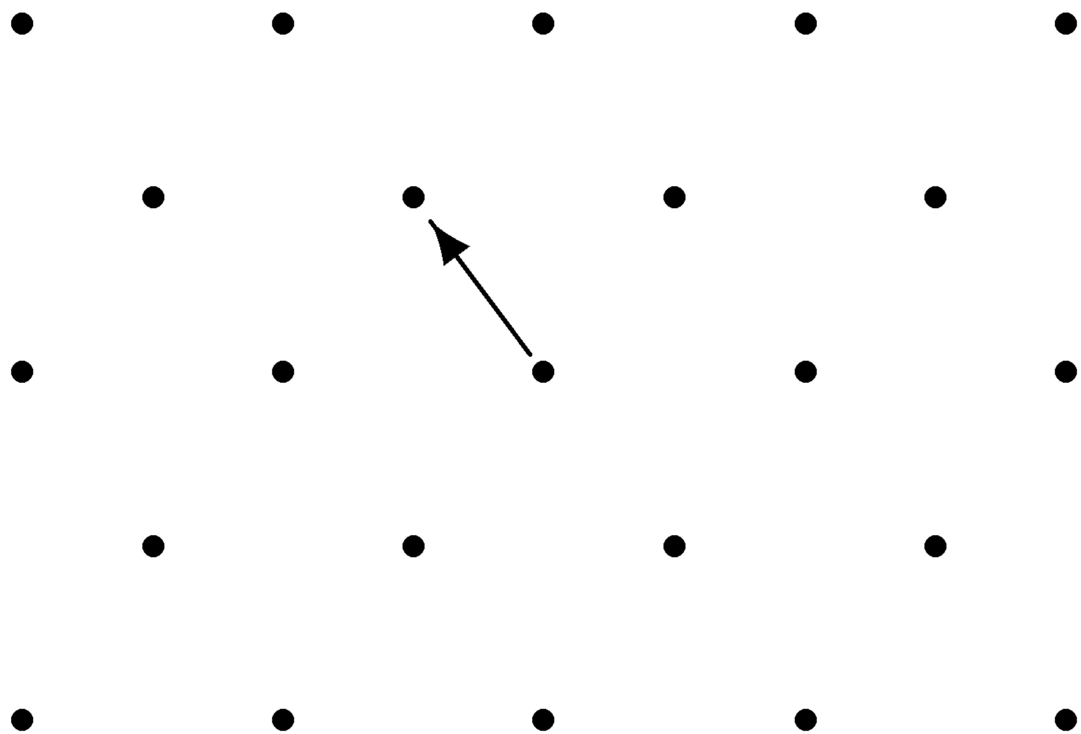
The diagonal square lattice



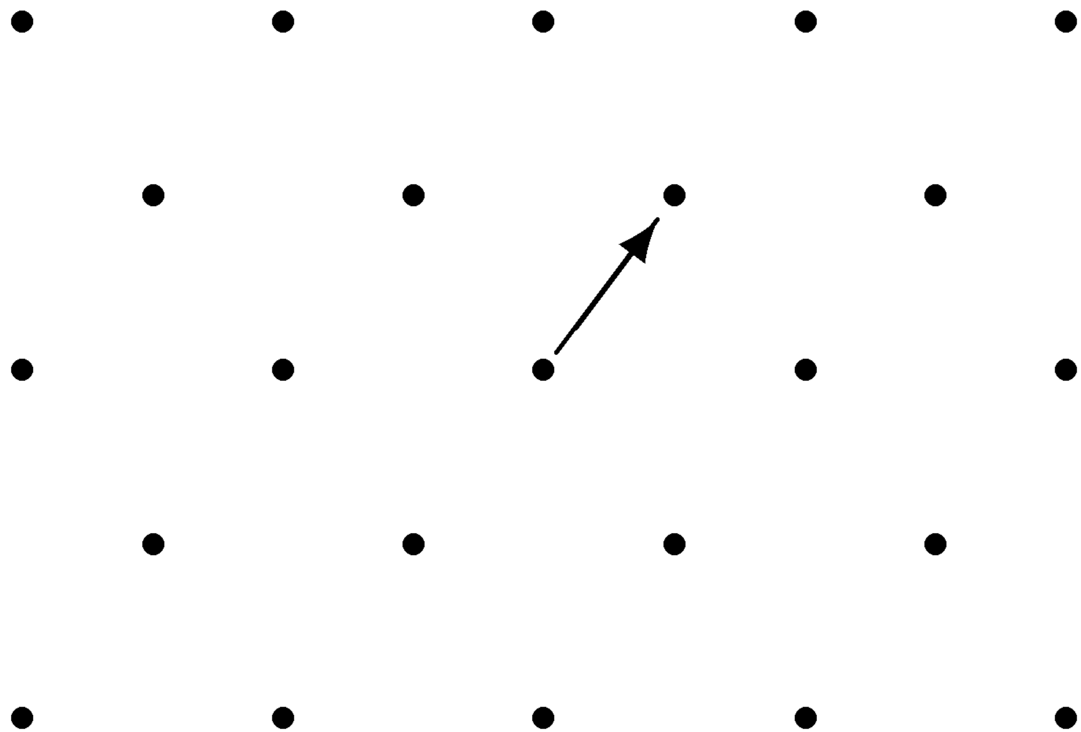
We start with the diagonal square lattice.

Arrows



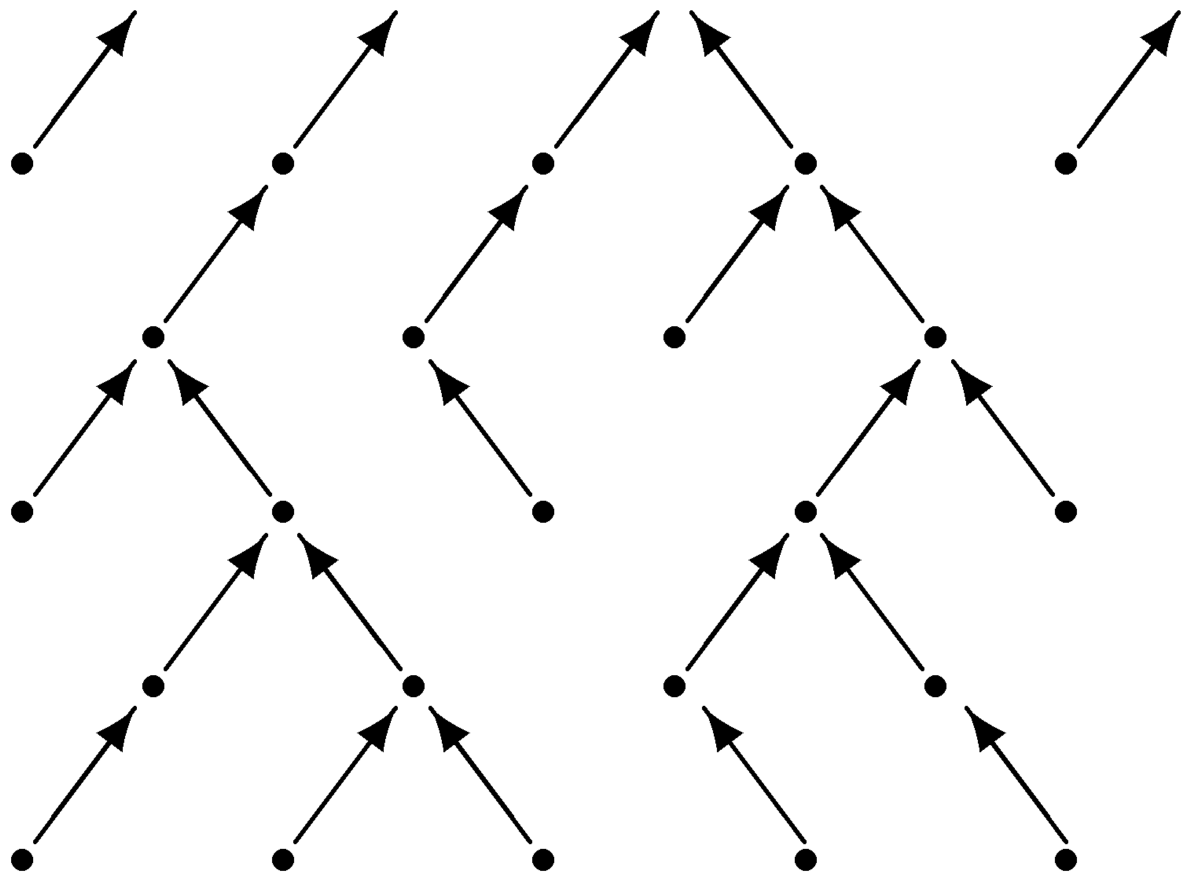
With probability $1/2$ we draw an arrow to the left. . .

Arrows



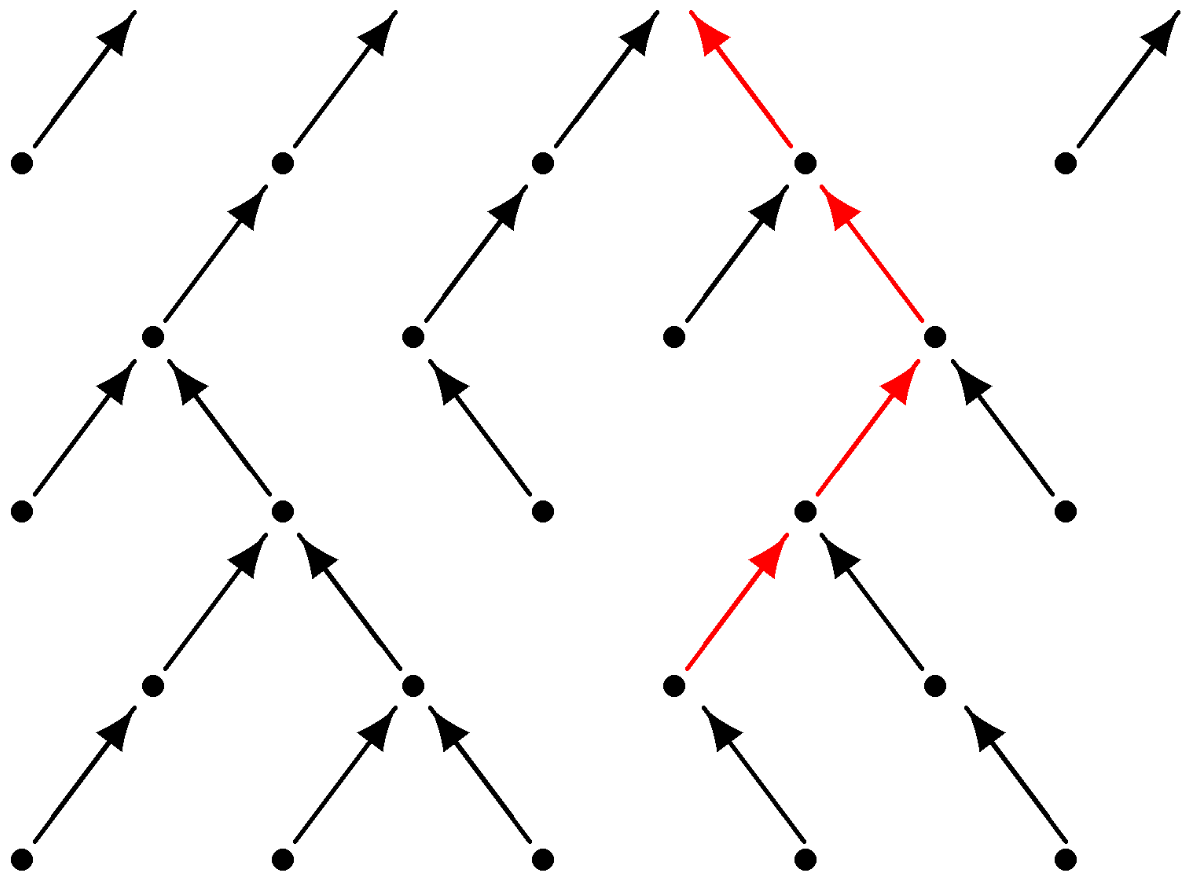
. . . and with the same probability to the right.

Arrows



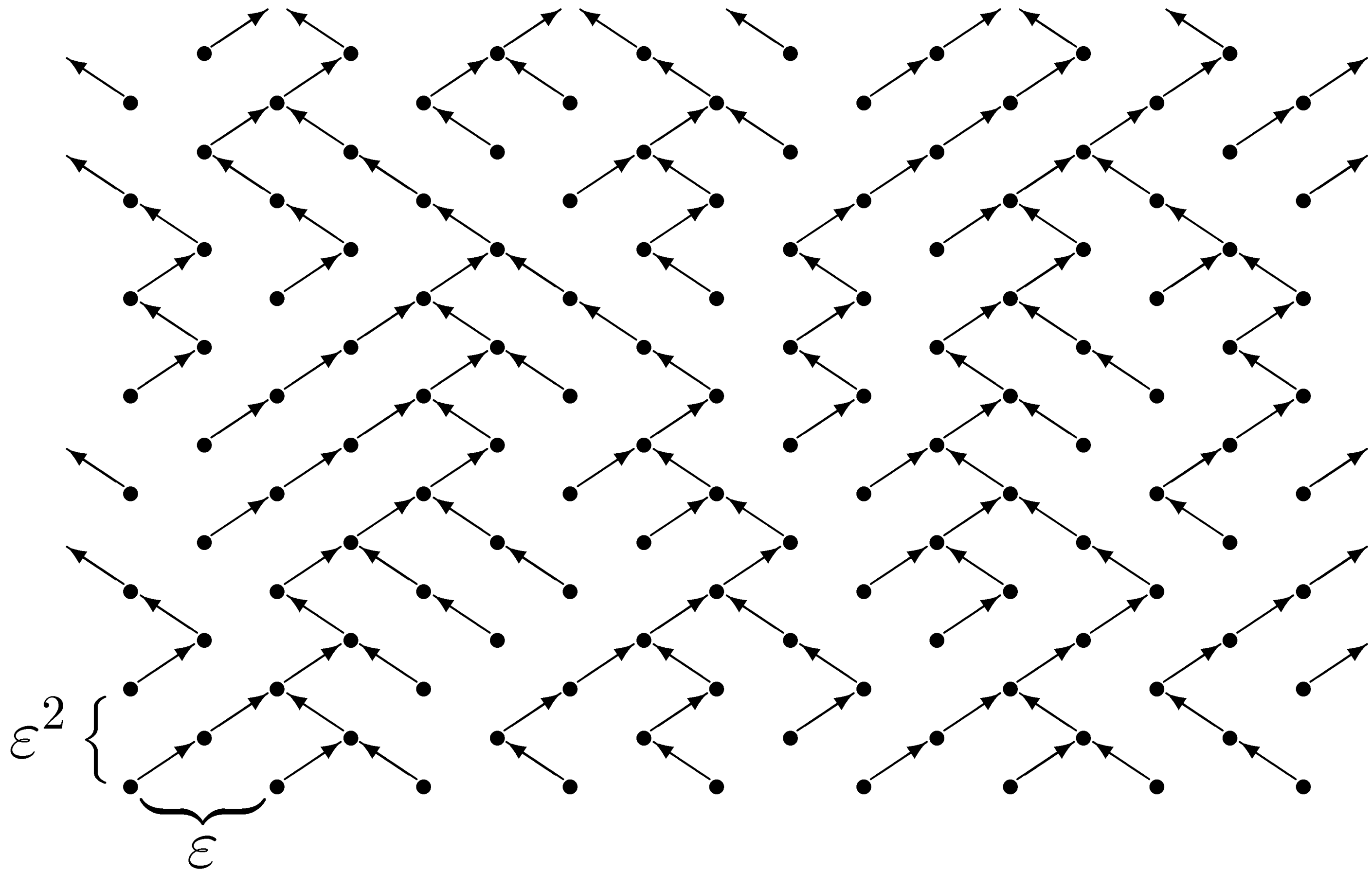
We do this for every point.

Paths



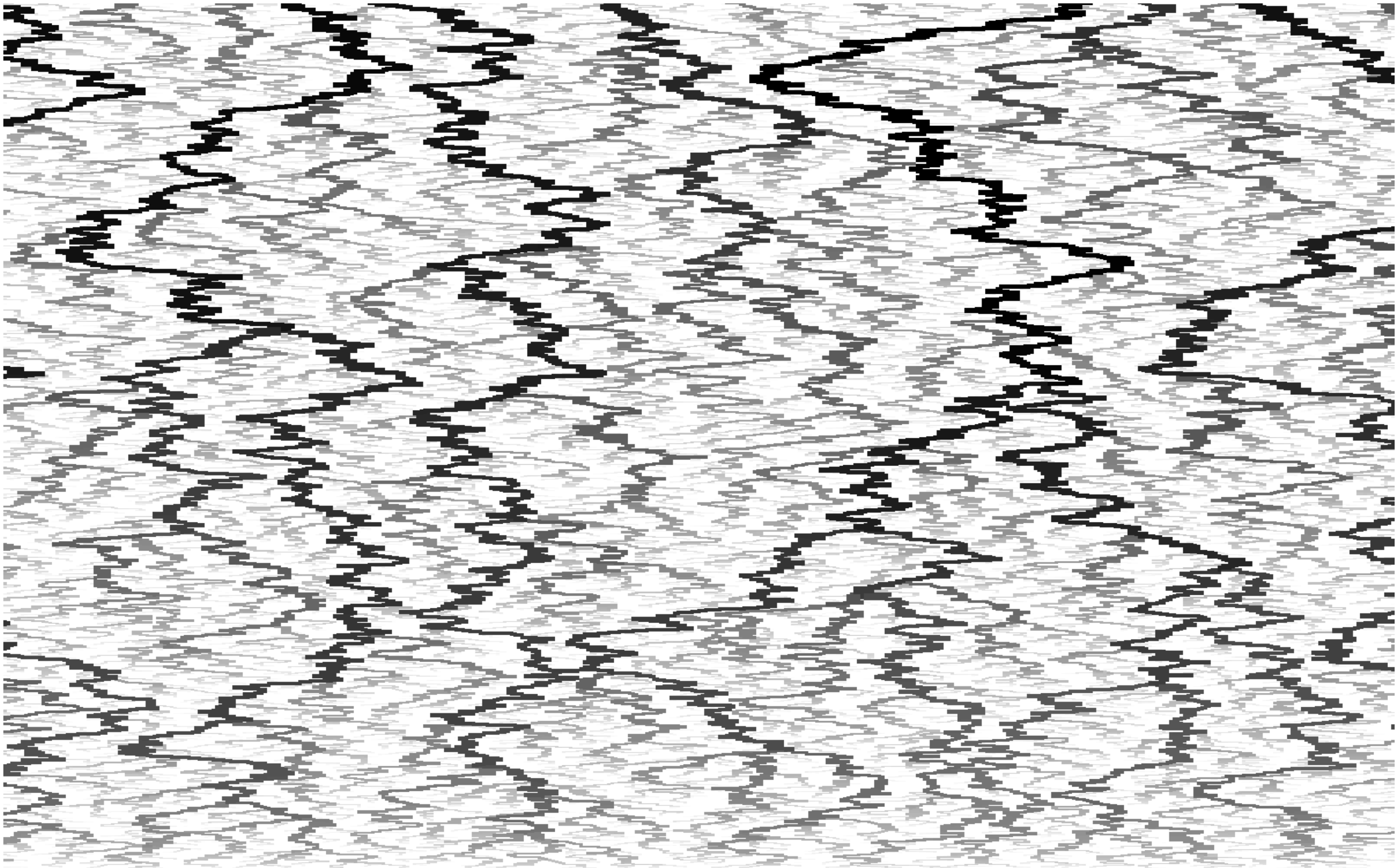
We are interested in paths along arrows.

Diffusive scaling



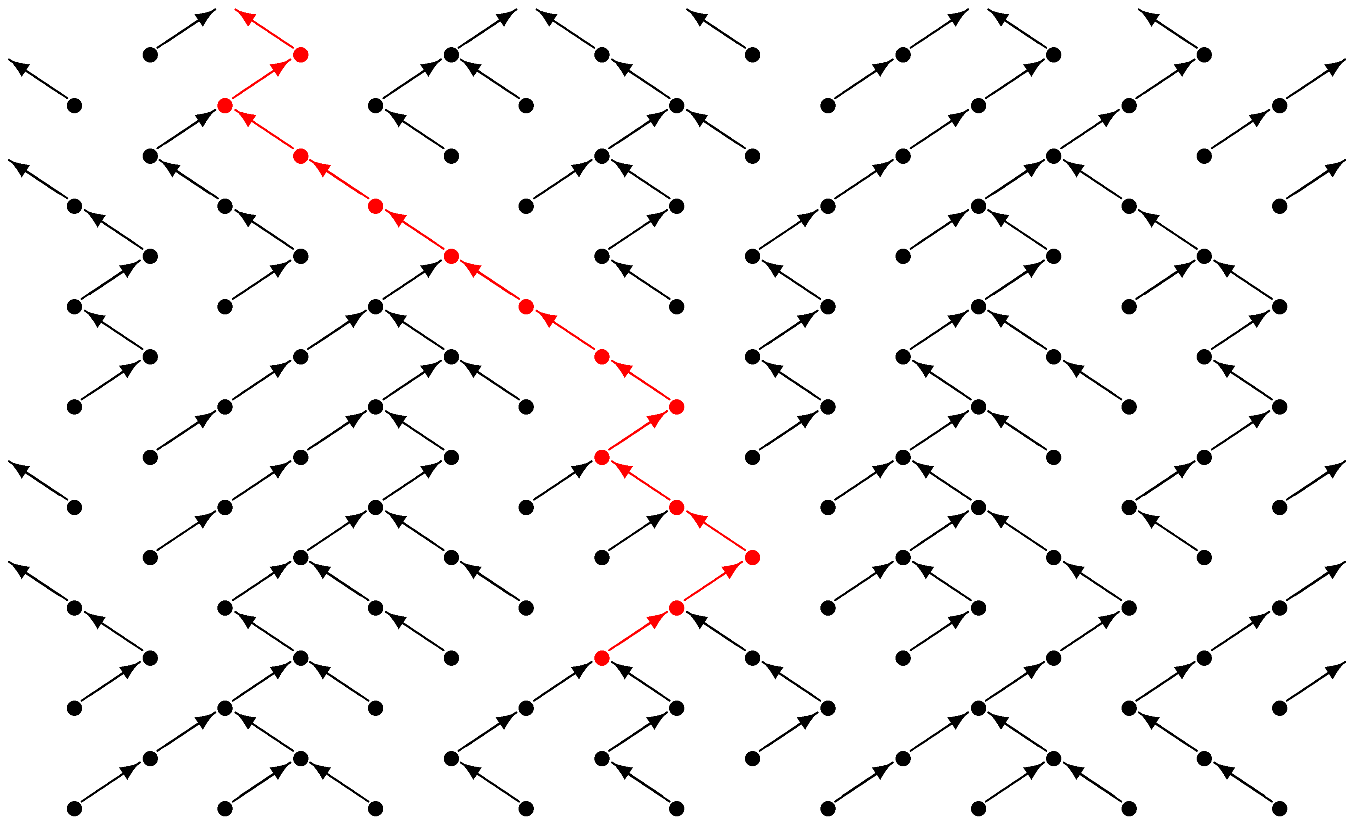
We scale space with ε, time with ε^2, and let $\varepsilon \to 0$.

The Brownian web



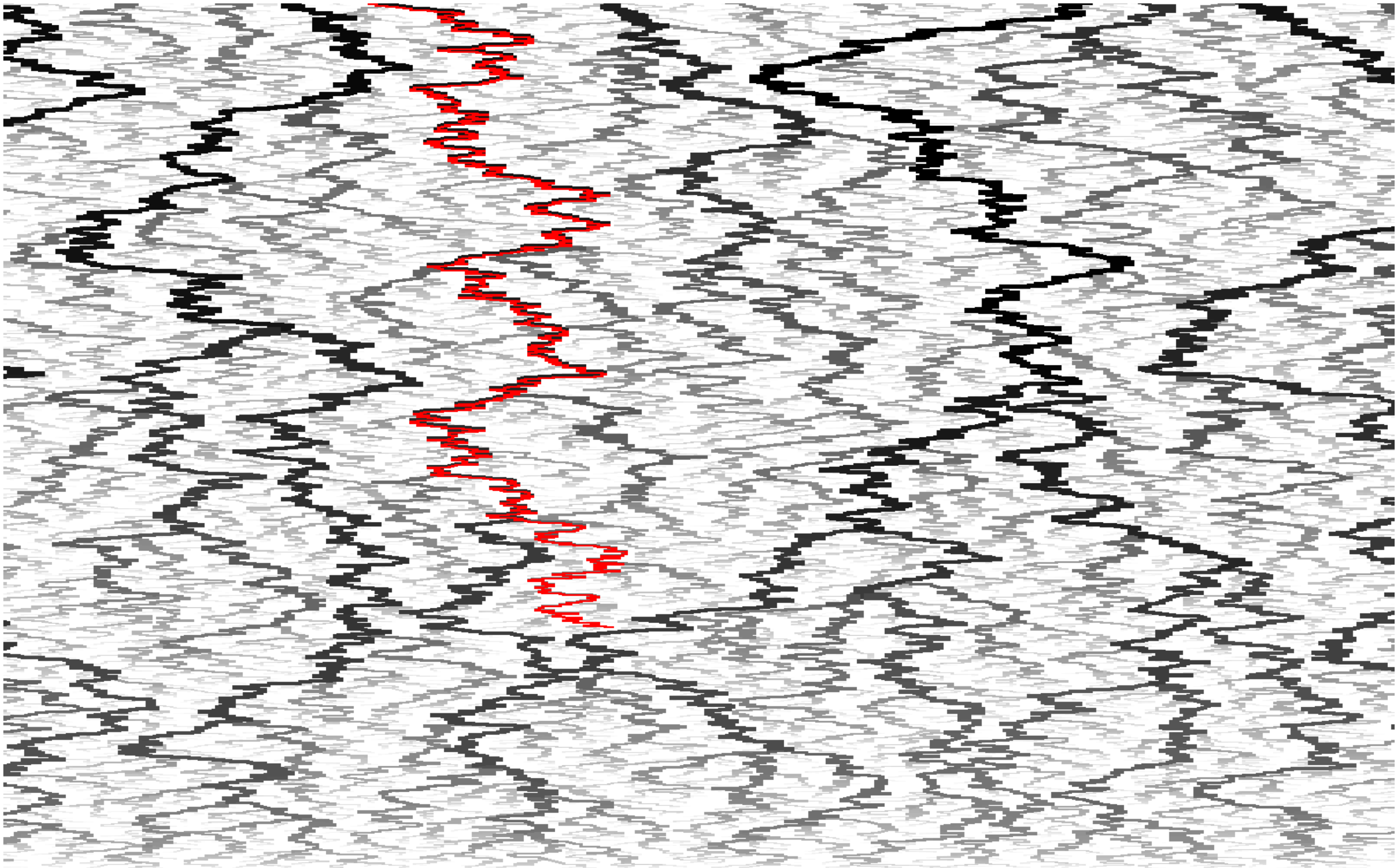
In the limit we obtain the Brownian web.

Random walk paths



Discrete paths along arrows are random walks.

Brownian paths

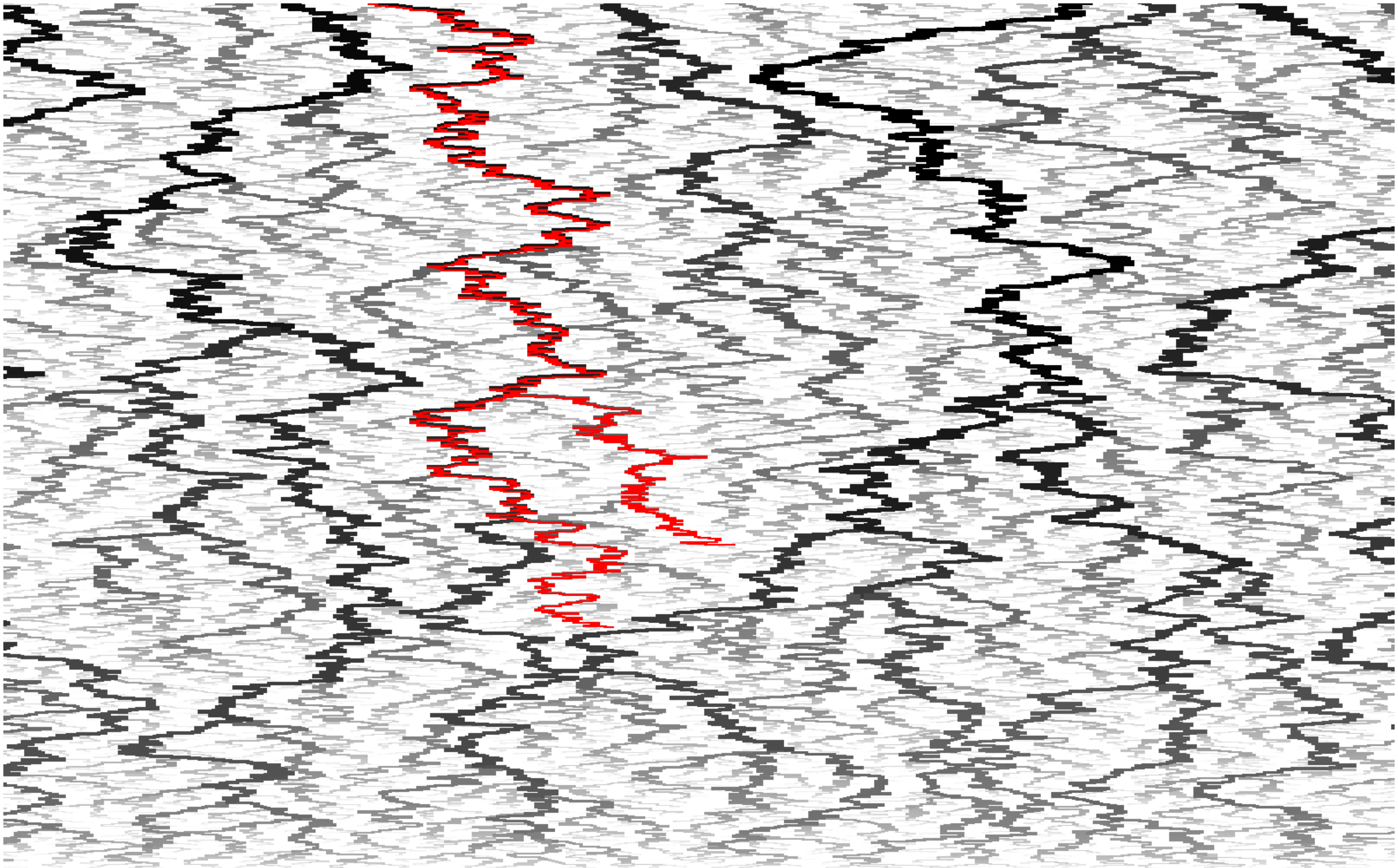


In the Brownian web, paths are Brownian motions.

Brownian motion

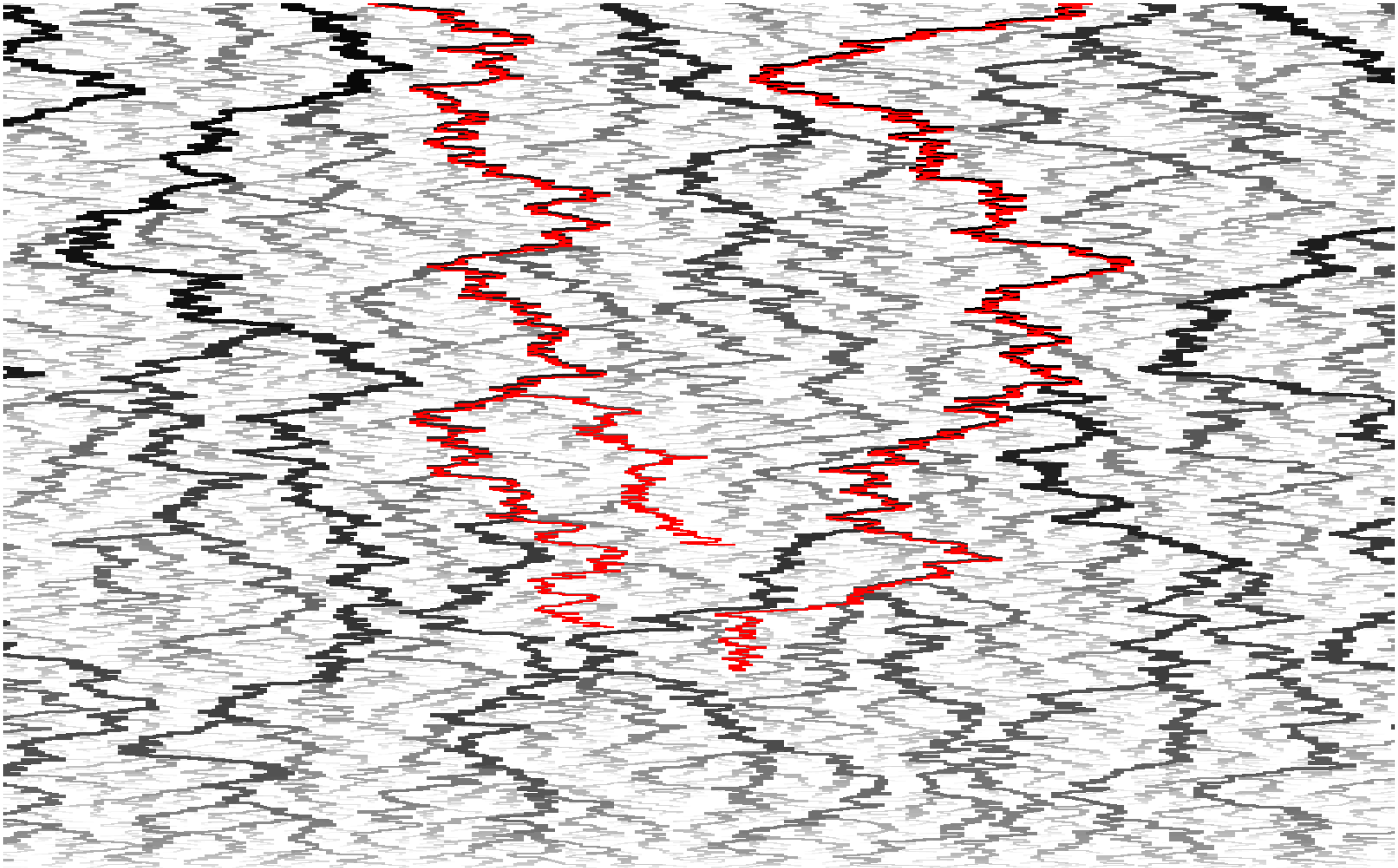
- 1827: The botanist Robert Brown observes the irregular motion of small particles submerged in water.
- 1905: Albert Einstein explains the motion as being caused by the collision of water molecules with the particle. In good approximation, the motion should be a Markov process with normally distributed increments.
- Early 1920-ies: Norbert Wiener constructs a probability measure on the space of continuous paths which has the properties postulated by Einstein.
- Present: (mathematical) Brownian motion is one of the cornerstones of modern probability theory.

Coalescing Brownian motions



Paths started at different points coalesce.

Coalescing Brownian motions



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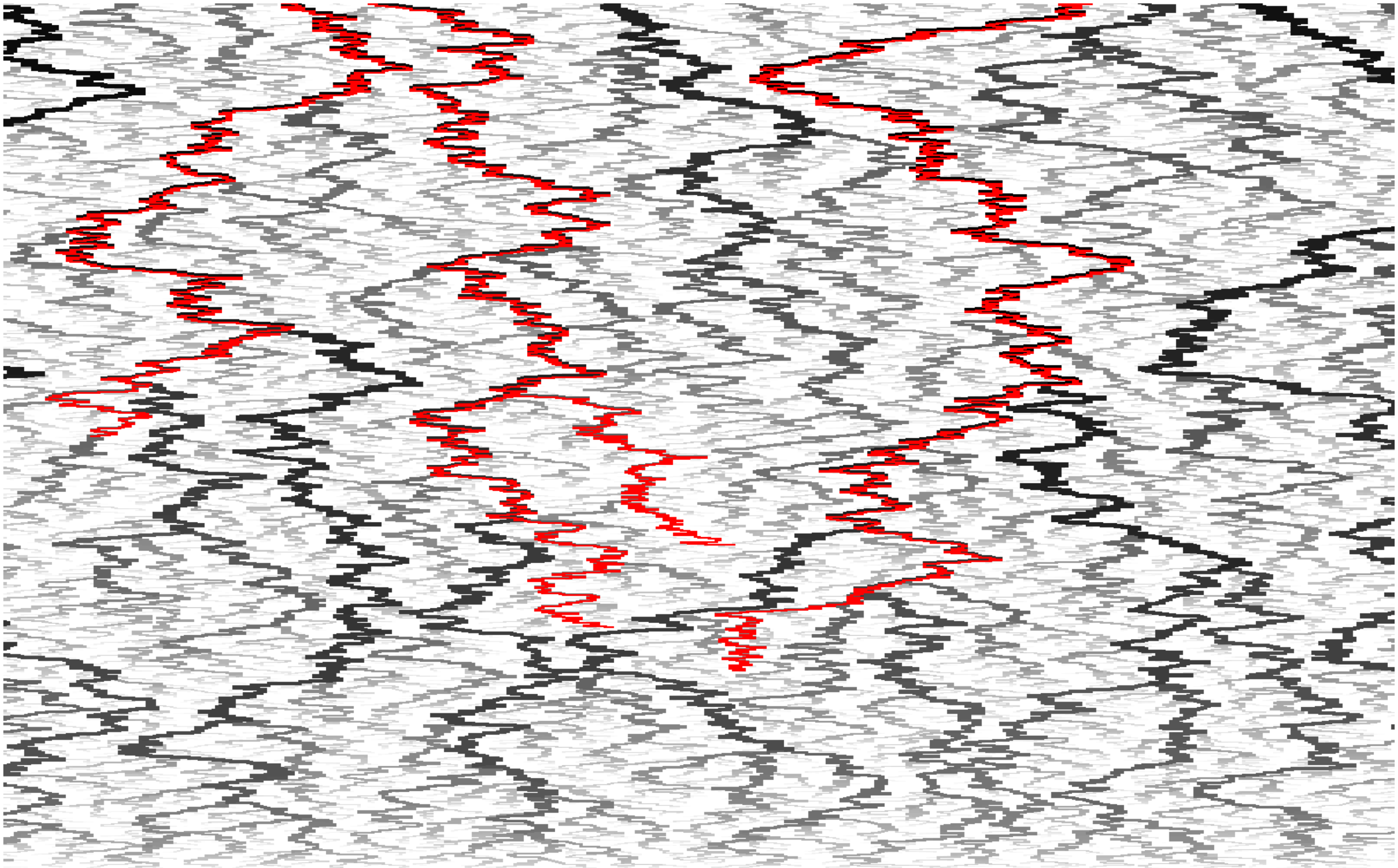

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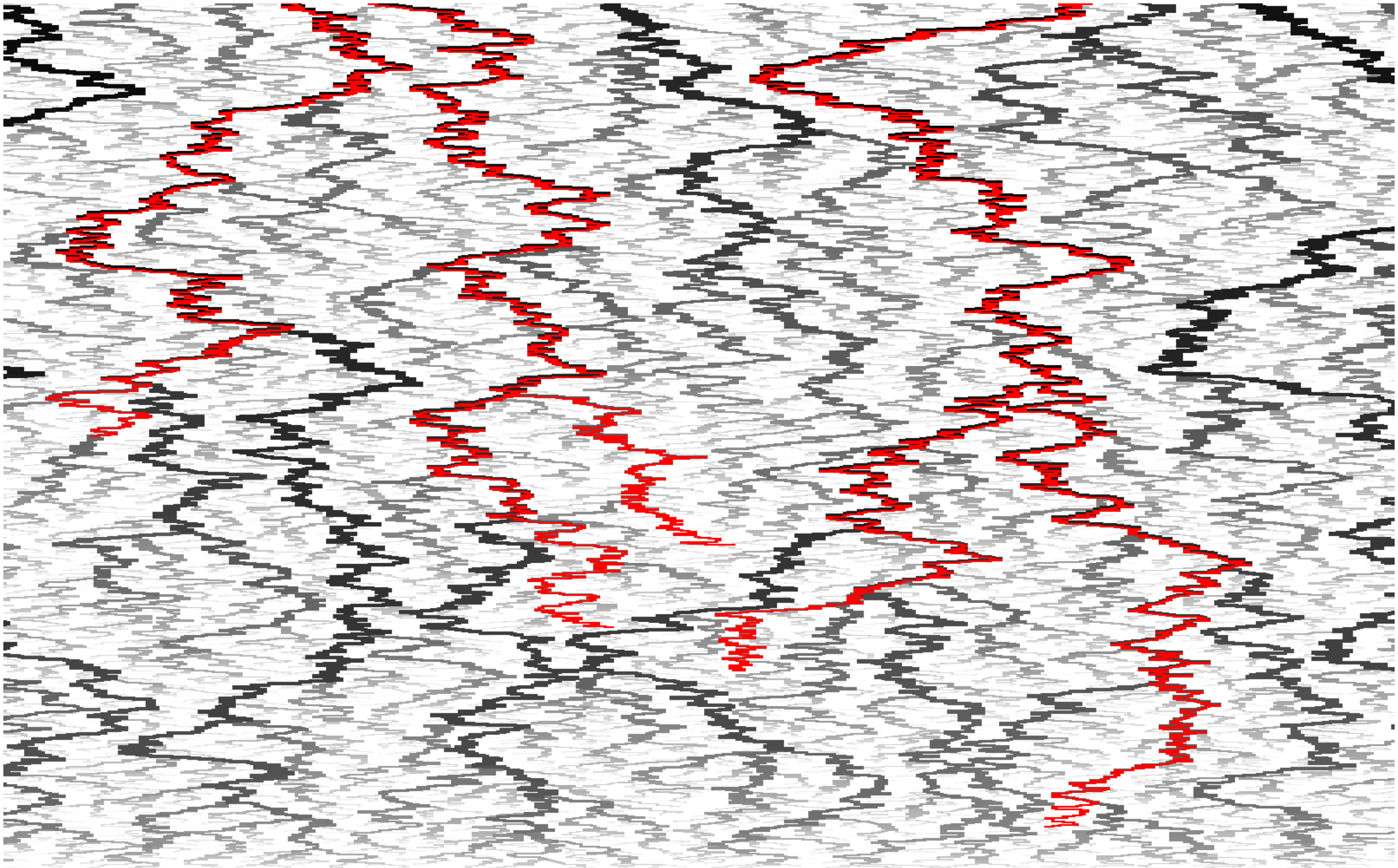
Coalescing Brownian motions



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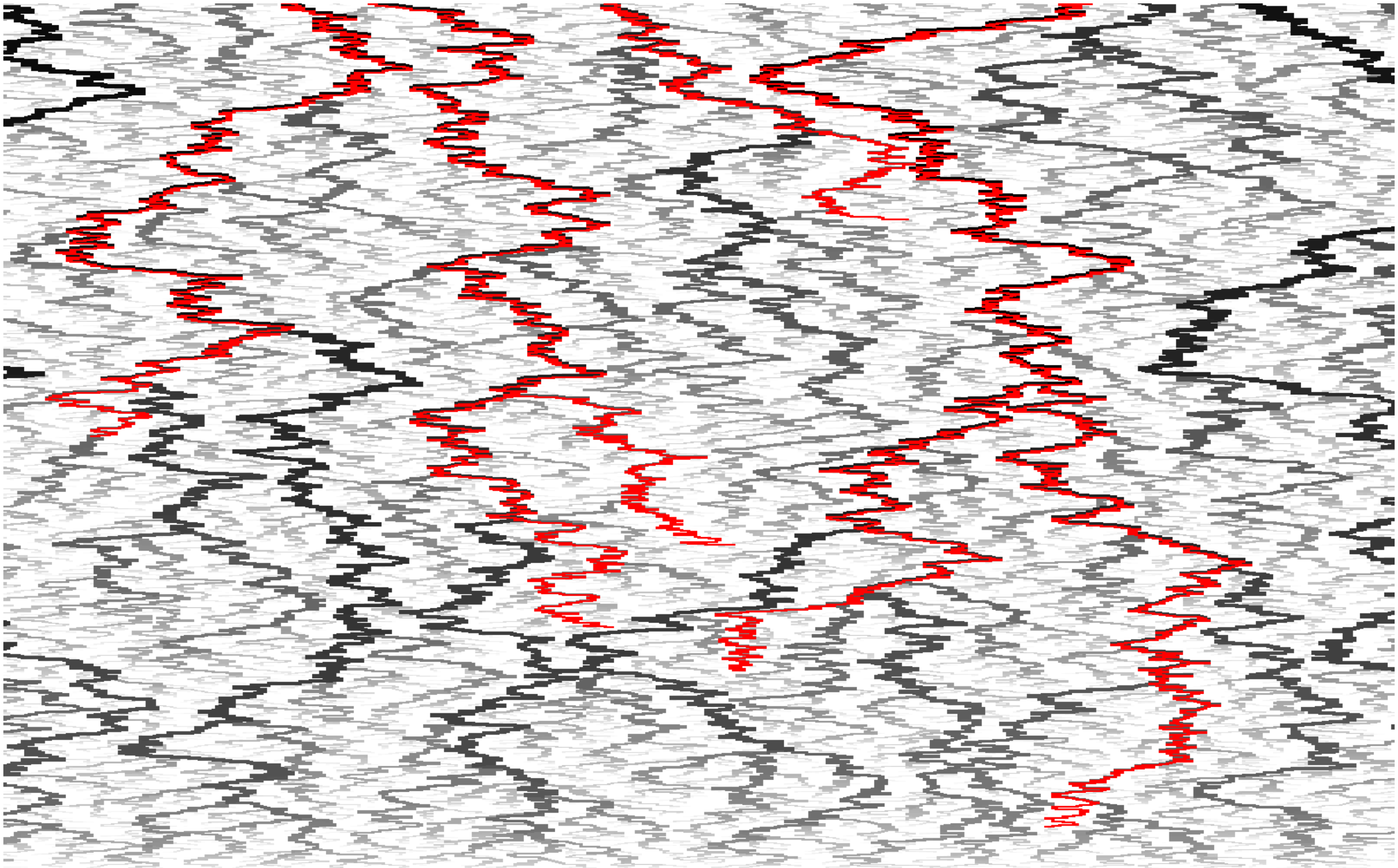
Coalescing Brownian motions



Paths started at different points coalesce.

Construction

The Brownian web can be constructed in two steps:

- Construct an infinite sequence of coalescing Brownian paths whose starting points fill up space.
- Take all paths that occur as the limits of these paths.

The result can be described as "coalescing Brownian motions, started from every point in space and time".

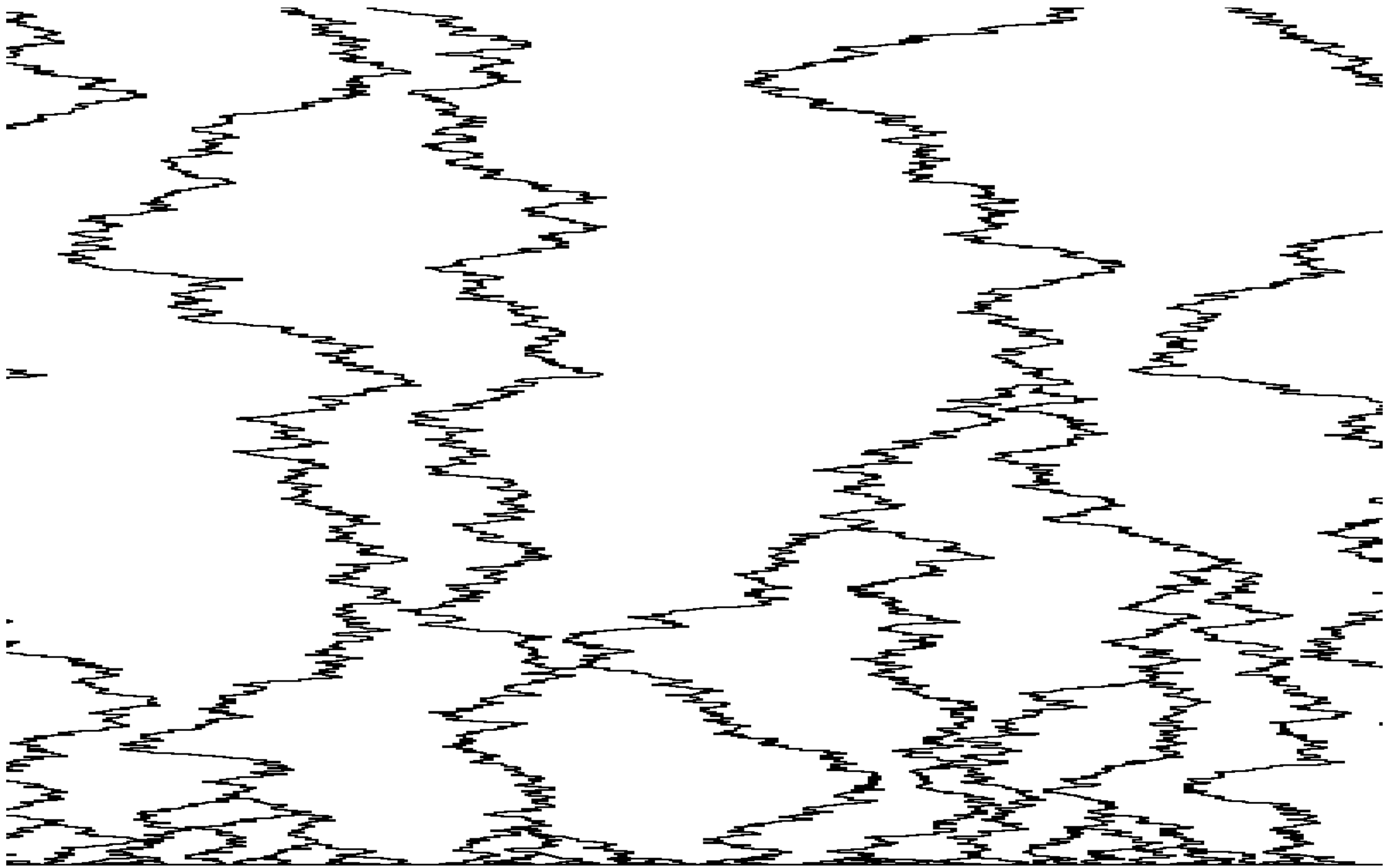
Applications

- River systems
- Lines of descent
- Evolution of interfaces
- Coalescing particles

History of the Brownian web

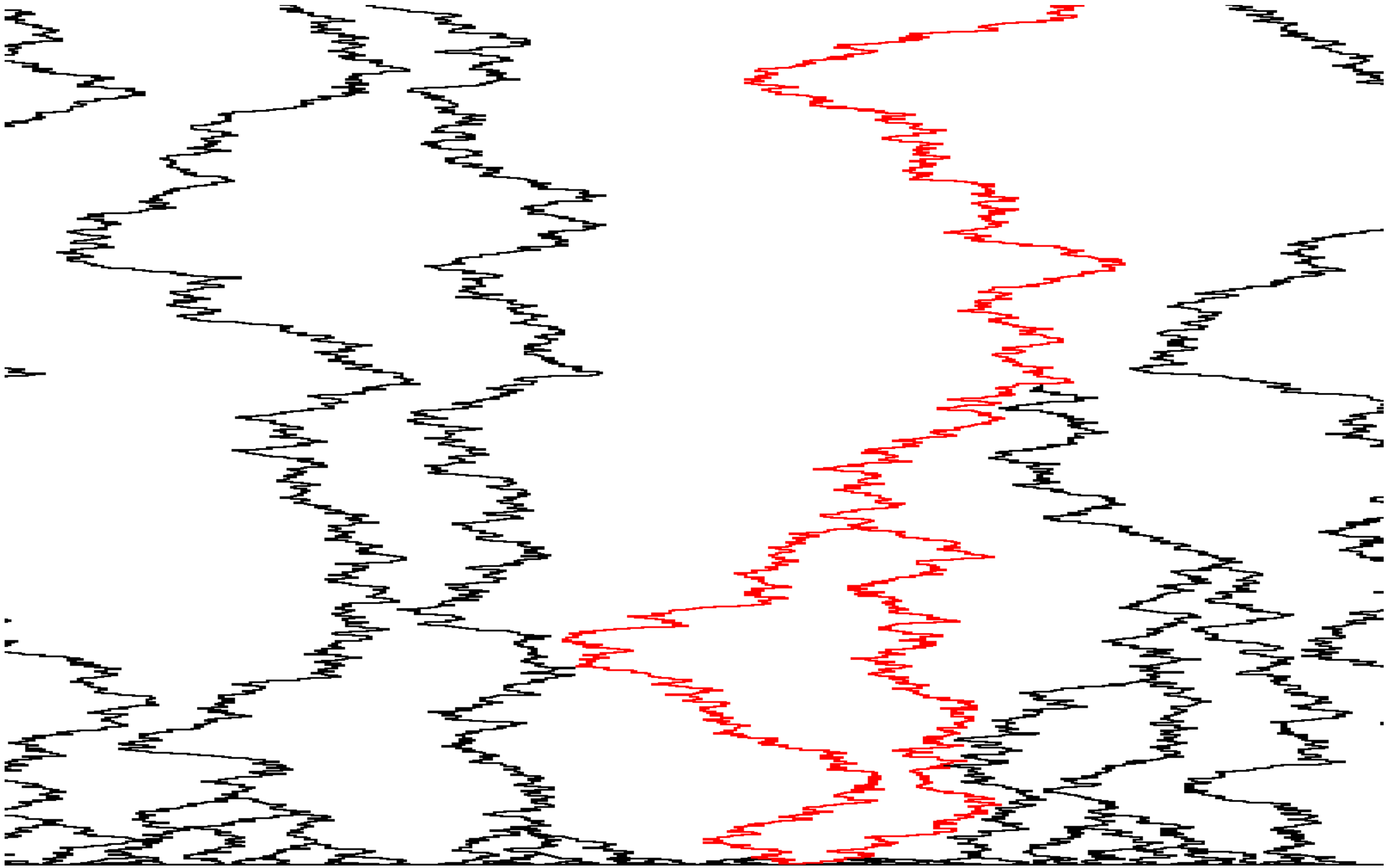
- R. Arratia (1979): one-dimensional voter model.
- B. Tóth and W. Werner (1998): true self-repellent motion
- L.R. Fontes, L.G.R. Isopi, C.M. Newman and K. Ravishankar (2002–): one-dimensional Potts model.

Paths started at time zero



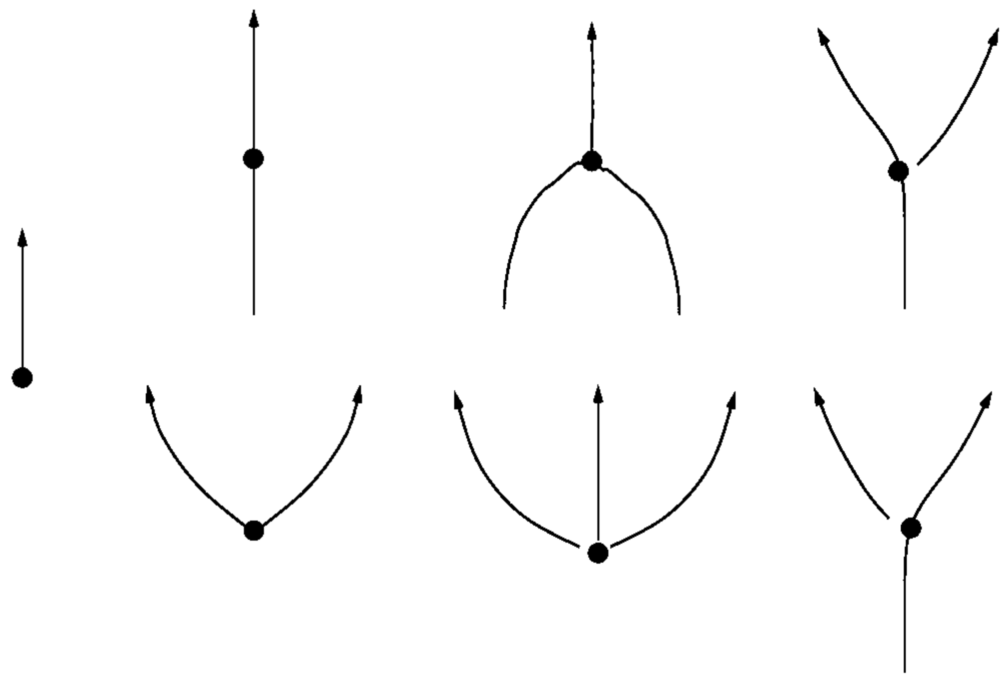
Coalescing Brownian motions started everywhere on the line.

Point with two outgoing paths



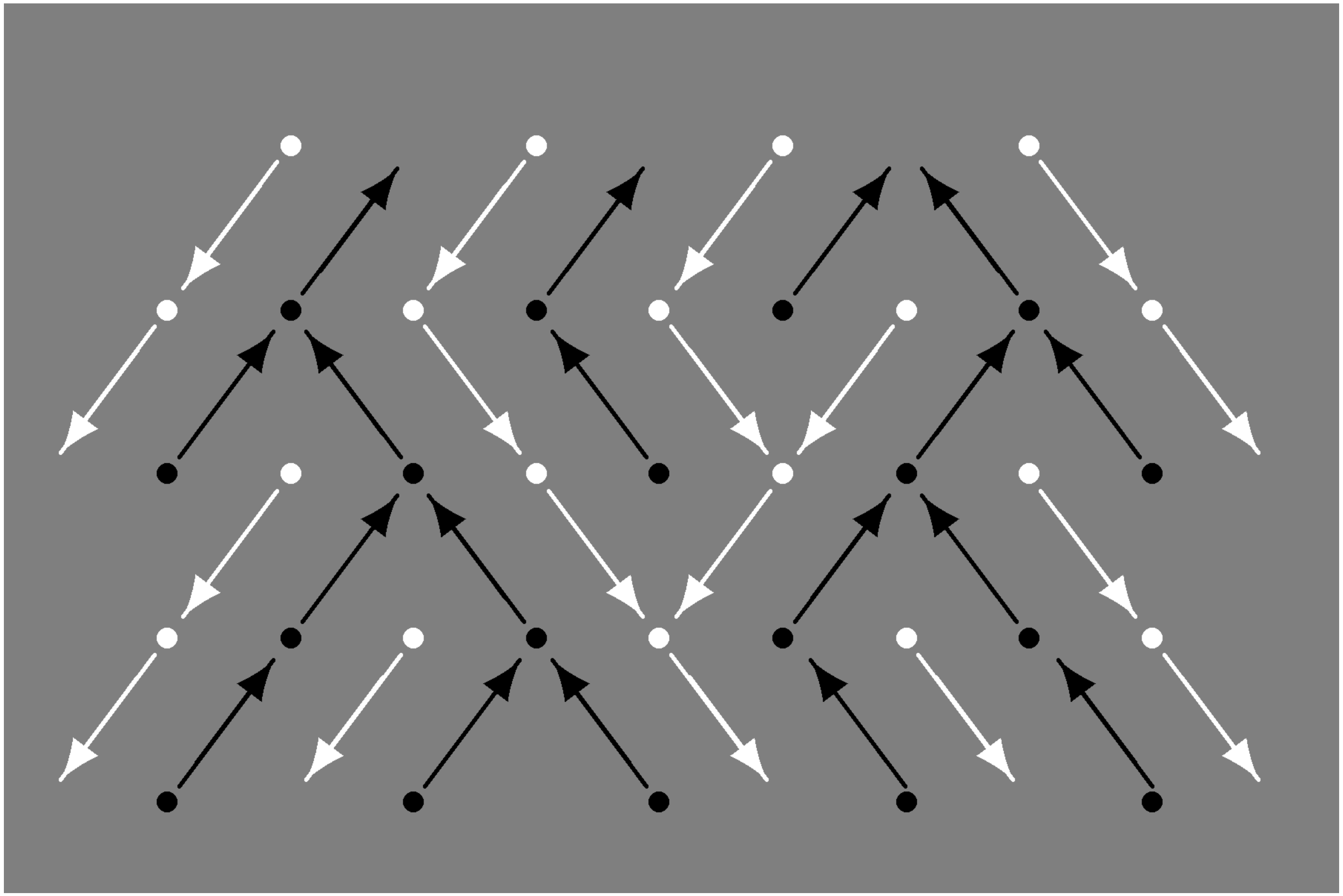
There exist random points from where there starts more than one path.

Special points



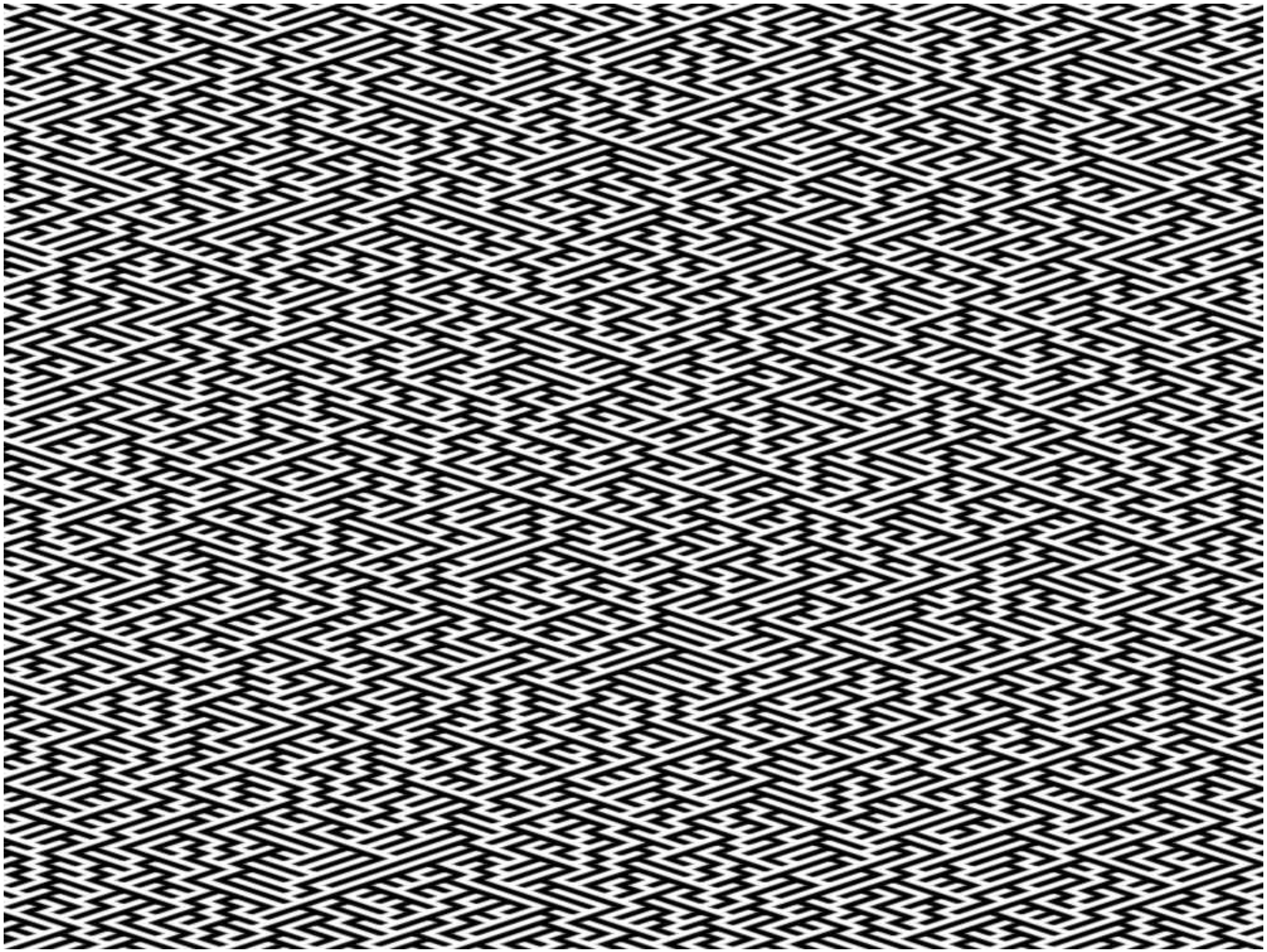
Special points are classified according to the number of incoming and outgoing paths. There exists 7 types of special points.

Dual arrows



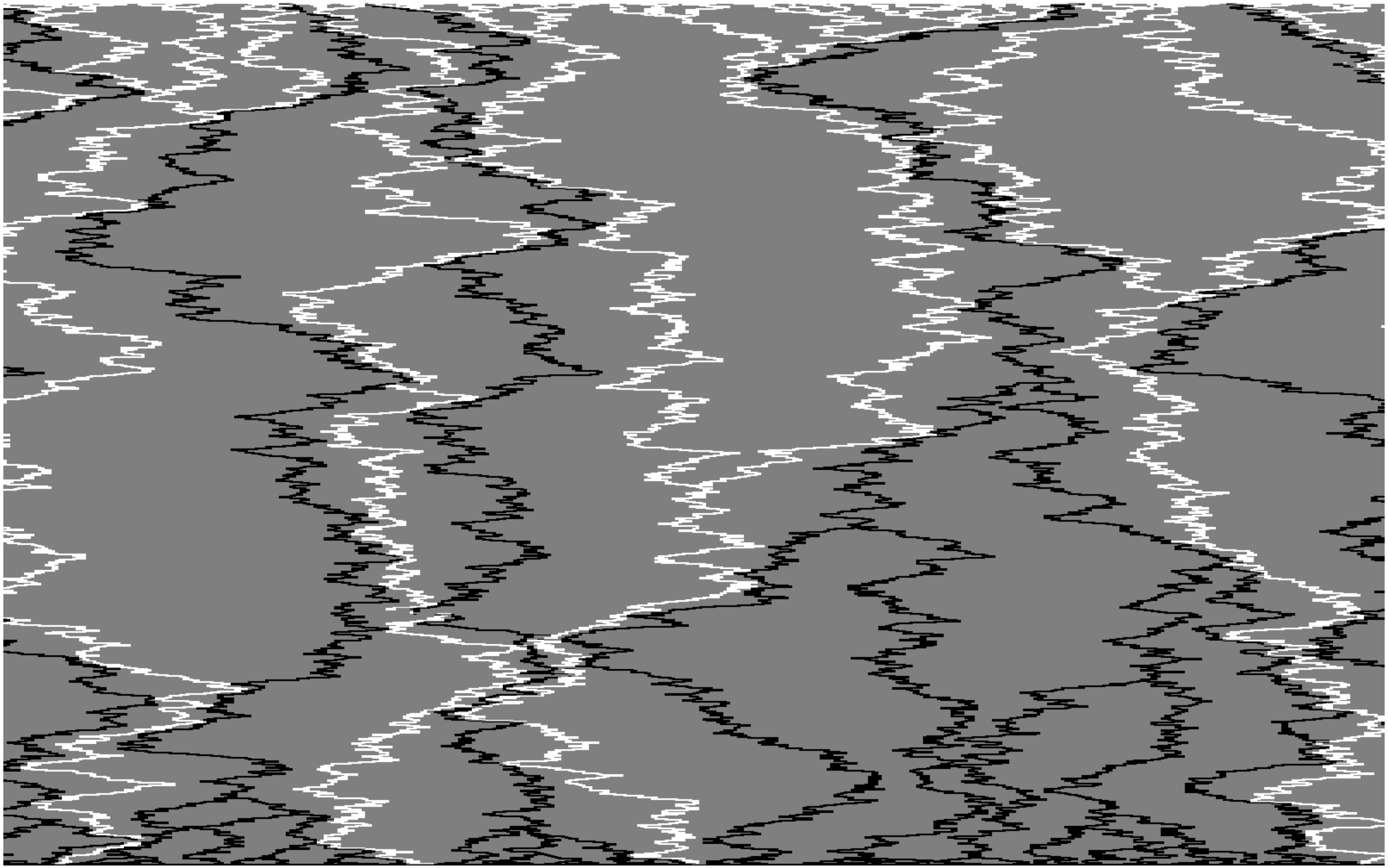
Forward and dual arrows.

Dual Brownian web



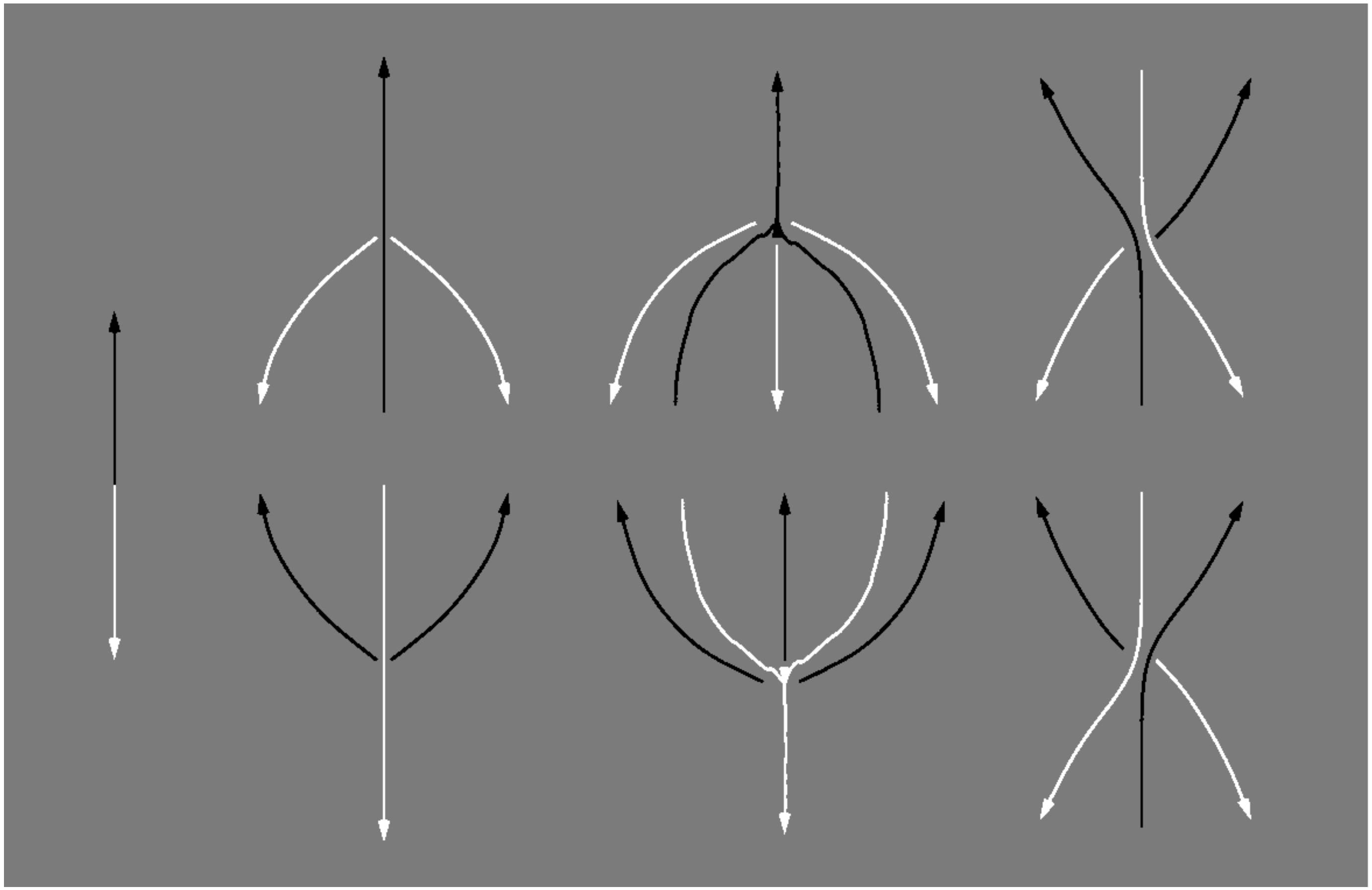
Approximation of the forward and dual Brownian web.

Dual Brownian web



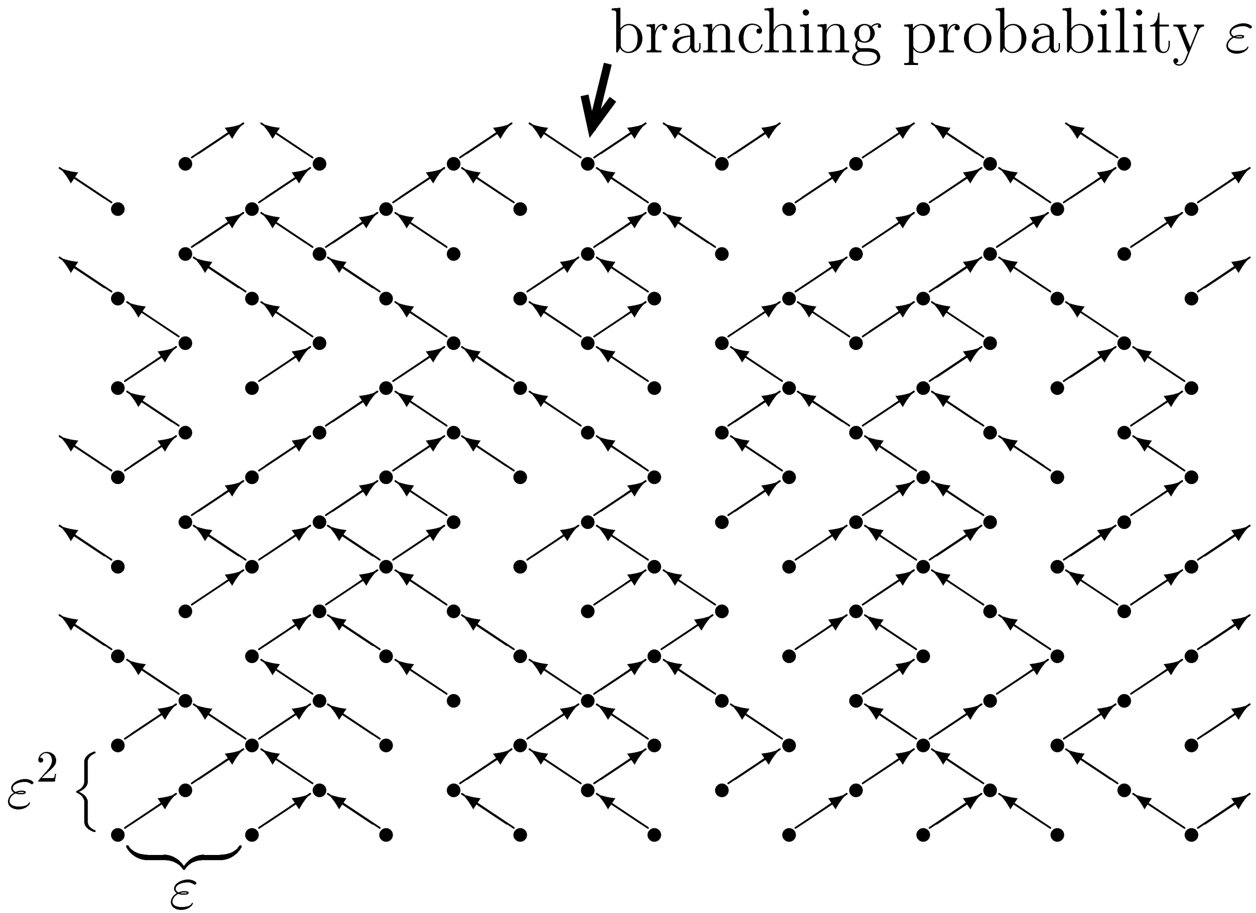
Forward and dual paths started from fixed times.

Special points revisited



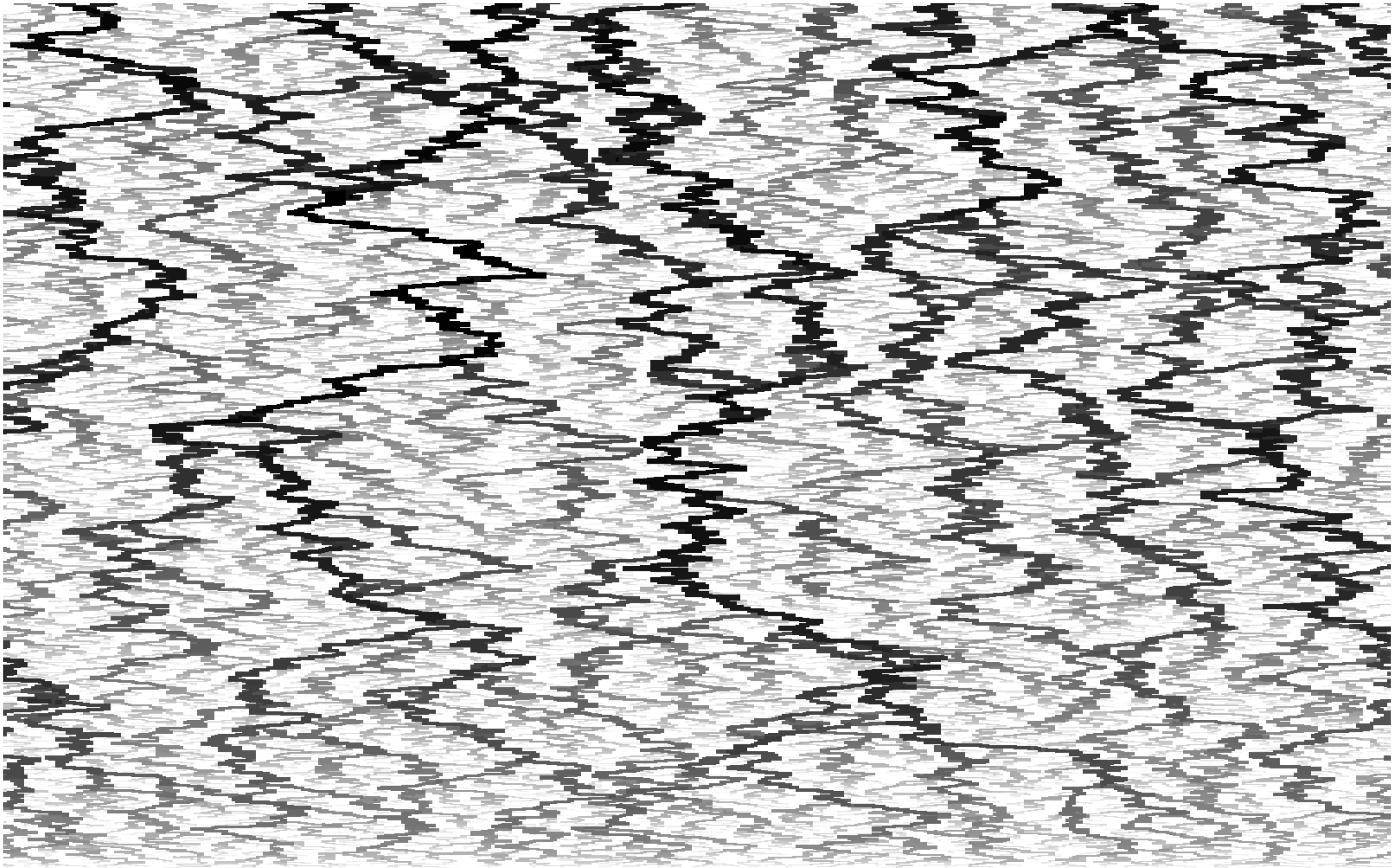
Structure of dual paths at special points.

Adding branching



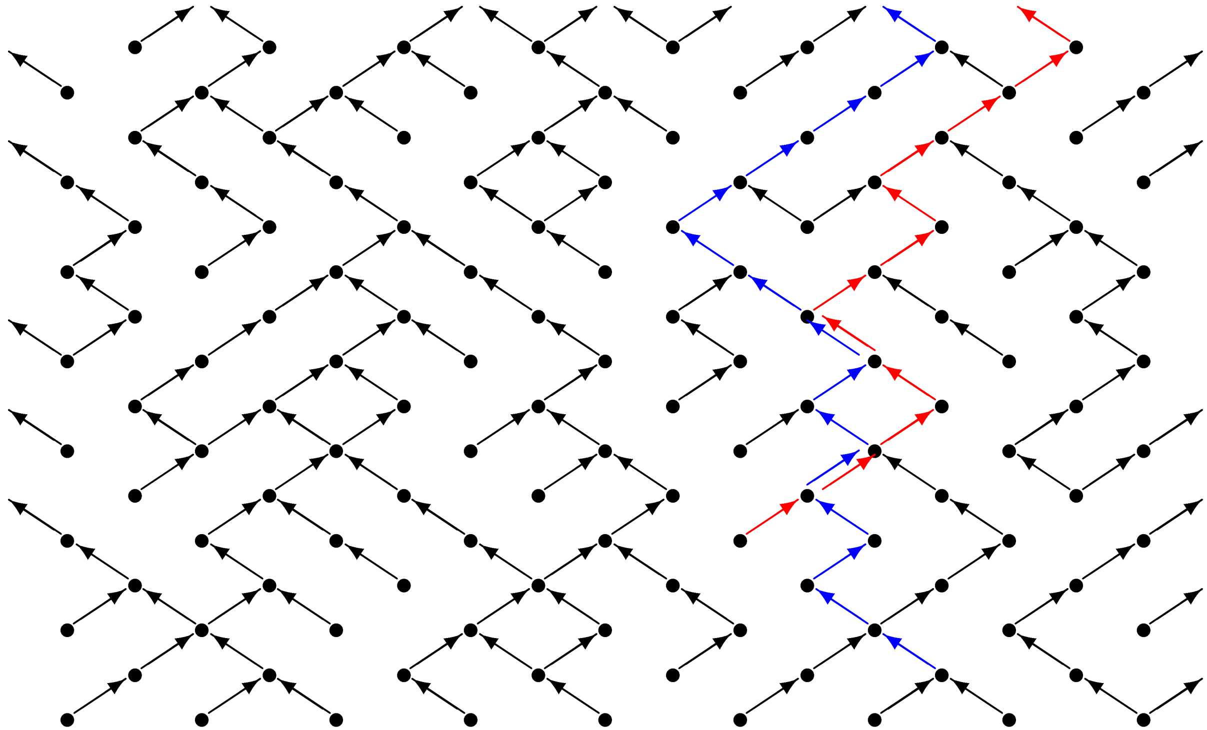
With probability ε, we draw two arrows at a point. We scale diffusively and let $\varepsilon \to 0$.

The Brownian net



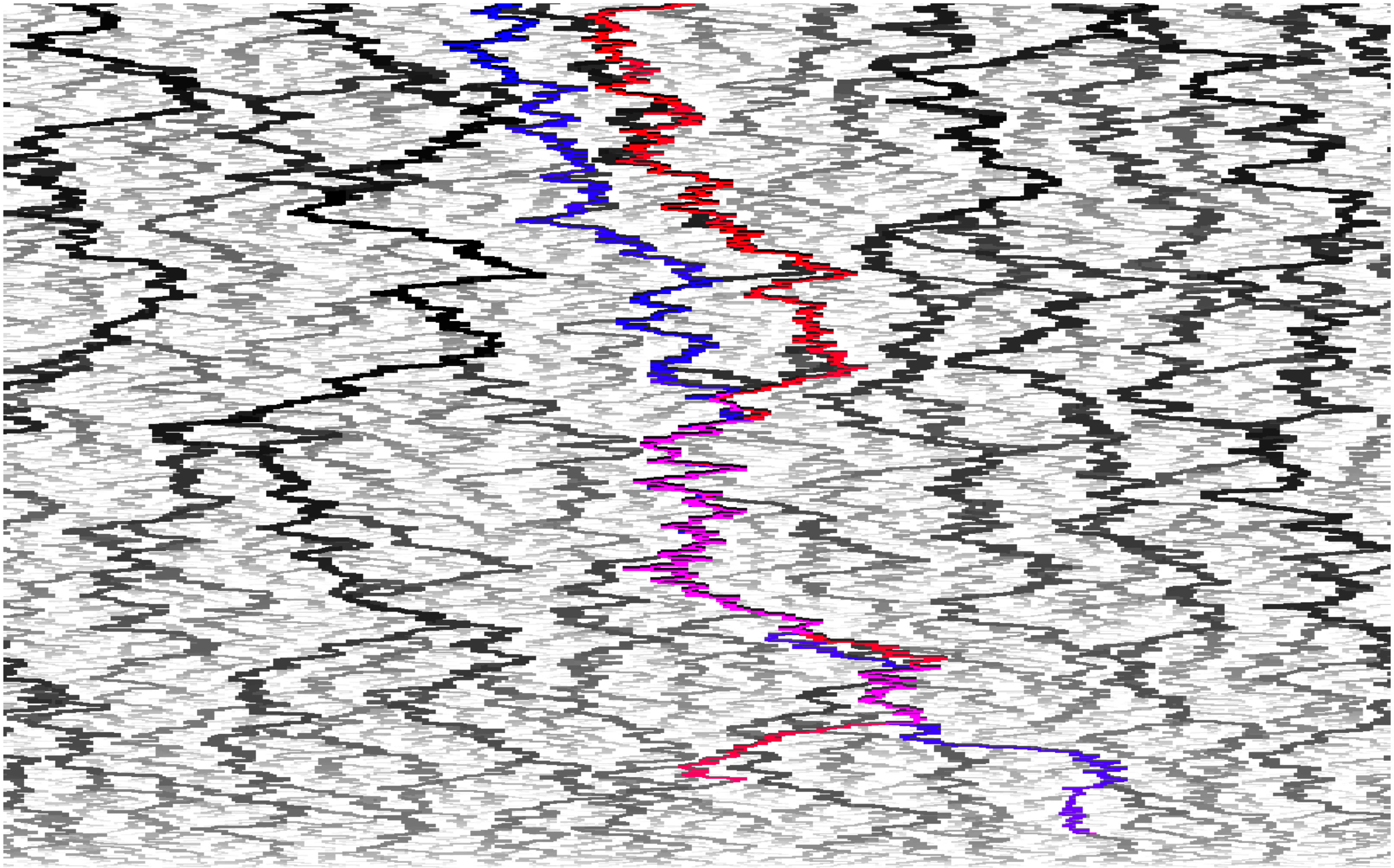
In the limit we obtain the Brownian net.

Left and right random walks



In the discrete approximation, we draw left-most paths in blue and right-most paths in red.

Left and right paths



In the limit, left-most and a right-most paths are Brownian motions with drift -1 and $+1$, respectively.

Left and right paths

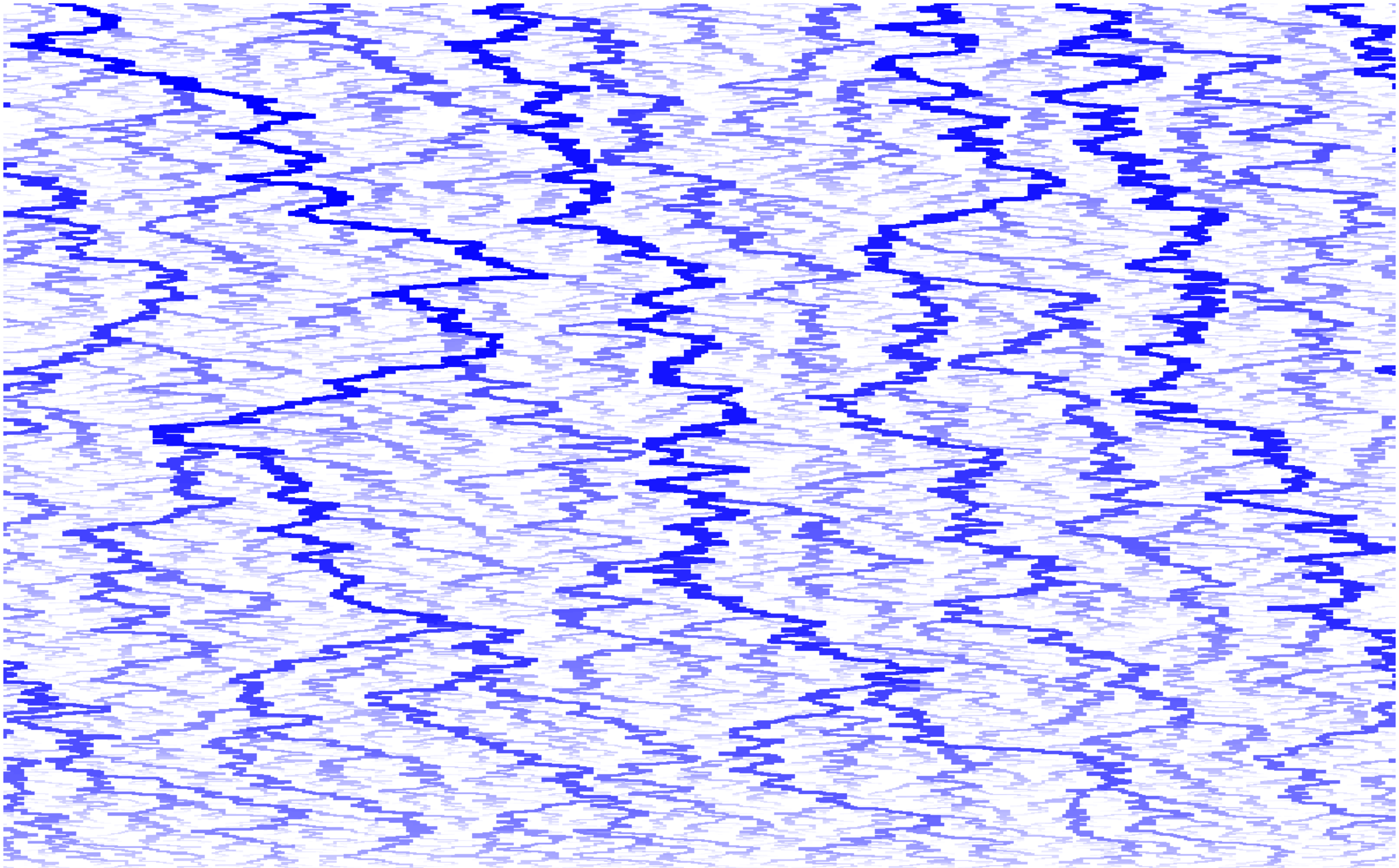
The interaction between left-most and right-most paths is described by the stochastic differential equation (SDE):

$$\mathrm{d}L_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{l}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t - \mathrm{d}t,$$

$$\mathrm{d}R_t = 1_{\{L_t \neq R_t\}} \mathrm{d}B^{\mathrm{r}}_t + 1_{\{L_t = R_t\}} \mathrm{d}B^{\mathrm{s}}_t + \mathrm{d}t,$$

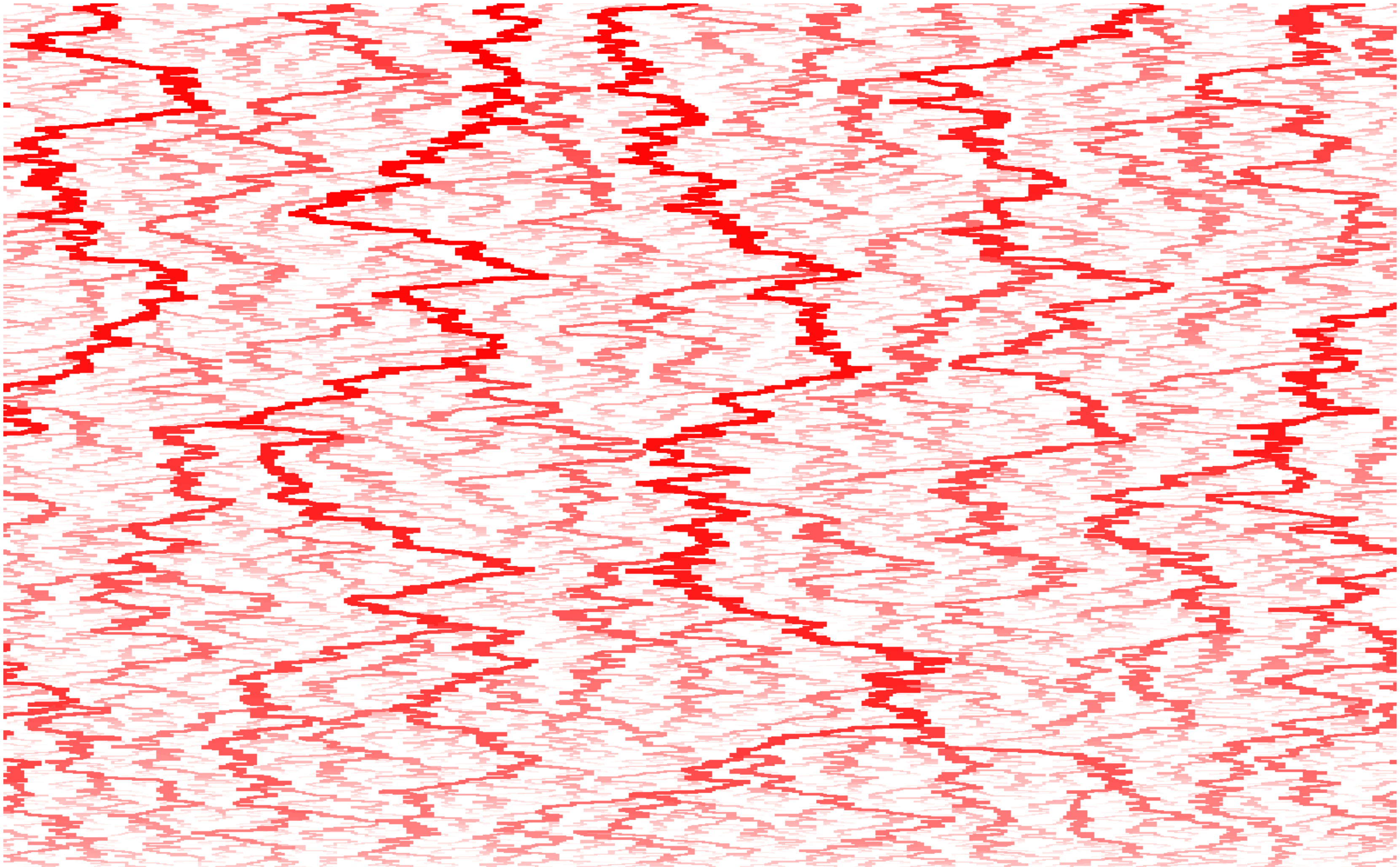
where $B^{\mathrm{l}}_t, B^{\mathrm{r}}_t, B^{\mathrm{s}}_t$ are independent Brownian motions, and L_t and R_t are subject to the constraint that $L_t \leq R_t$ for all $t \geq T := \inf\{u \geq 0 : L_u \leq R_u\}$.

Left Brownian web



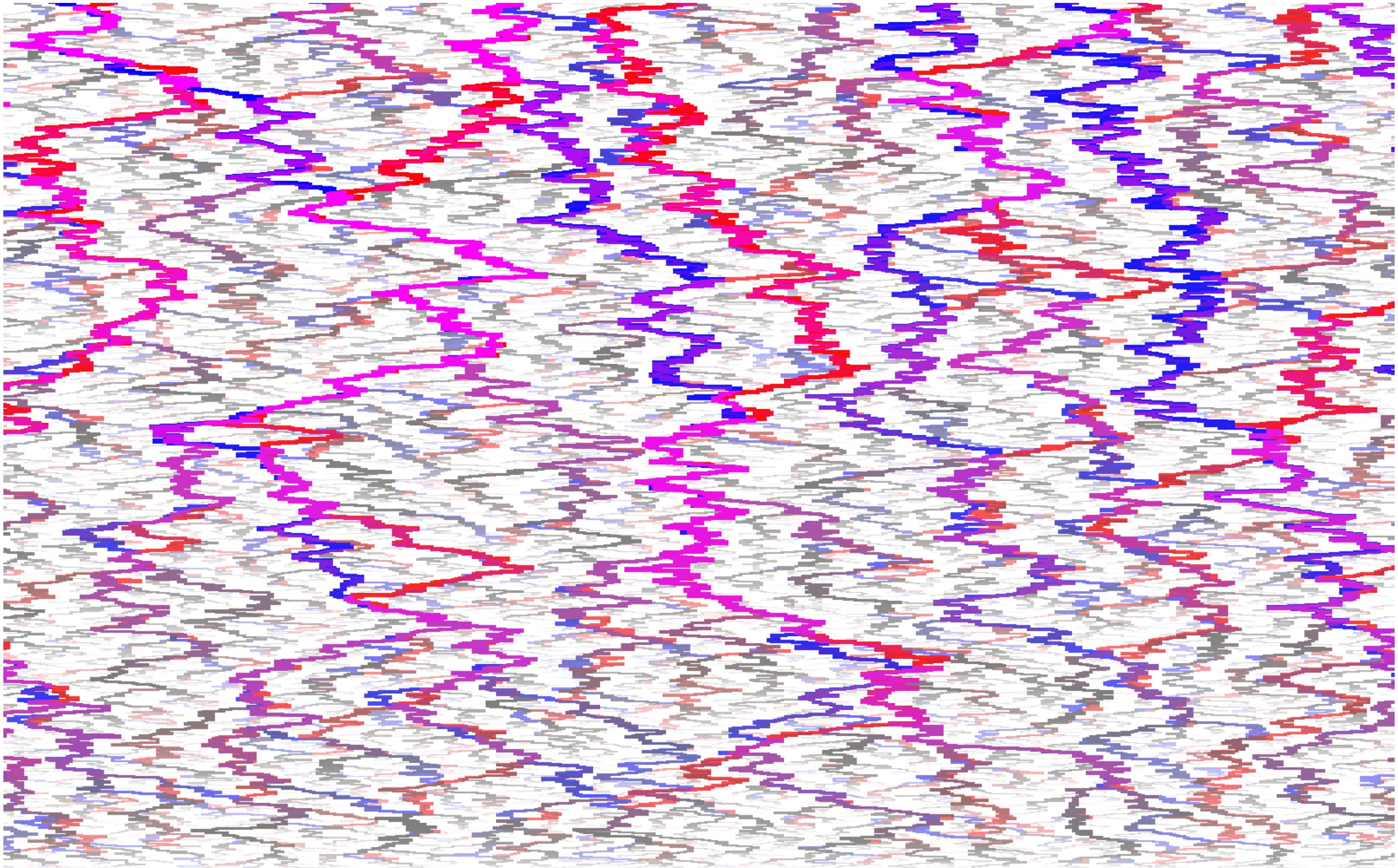
The left-most paths form a left-most Brownian web…

Right Brownian web



. . . and the right-most paths form a right-most Brownian web.

Left-right Brownian web



Together, they are known as the “left-right Brownian web”.

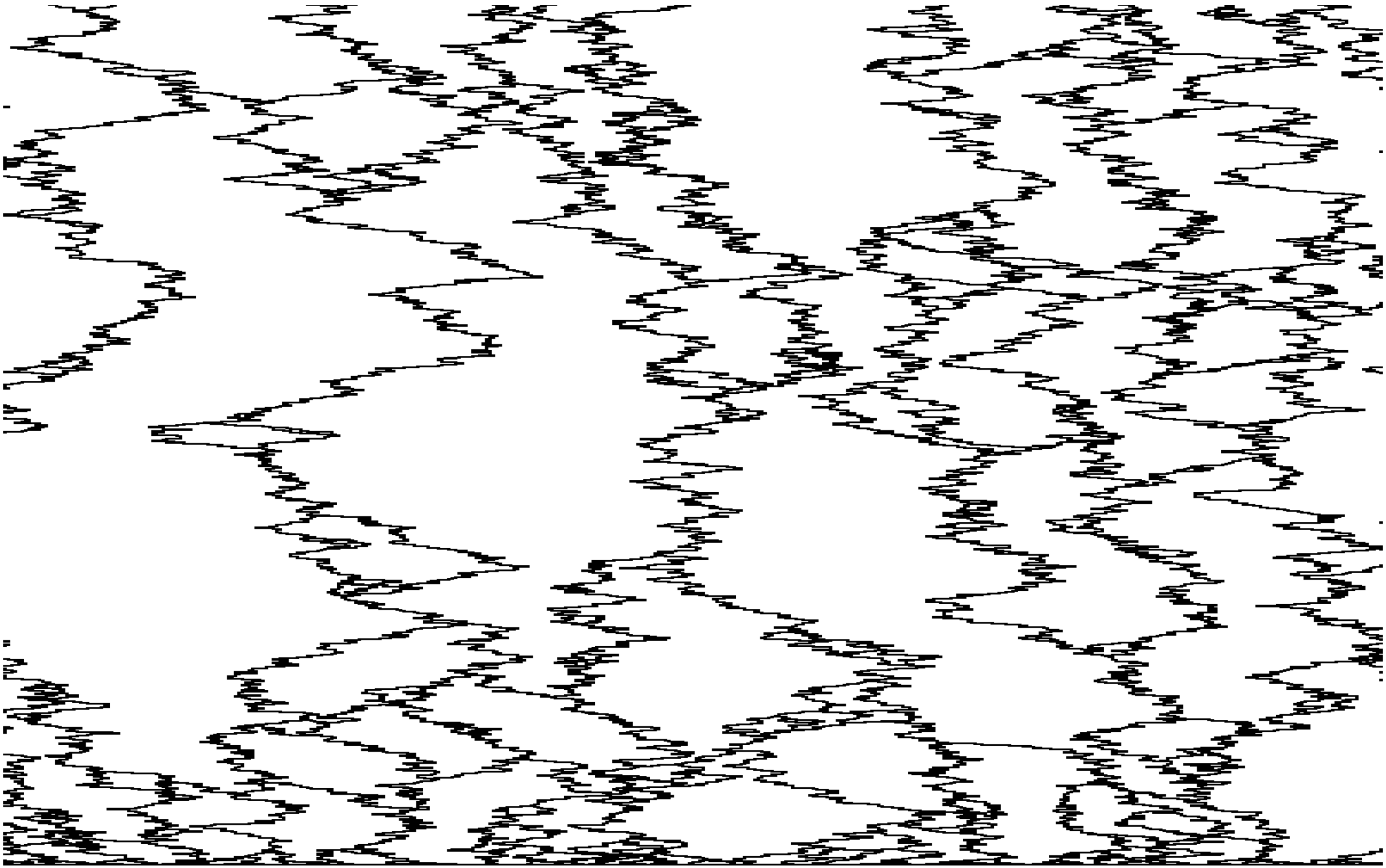
Construction of the Brownian net

'Hopping construction' [Sun & S. 2008]:

1. Couple a left and right Brownian web using the "left-right SDE".
2. Add all finite concatenations of left-most and right-most paths.
3. Add all limits of paths from step 2.

Alternative constructions: 'wedge construction', 'mesh construction' [Sun & S. 2008]; 'marking construction' [Newman, Ravishankar & Schertzer 2008].

The branching-coalescing point set

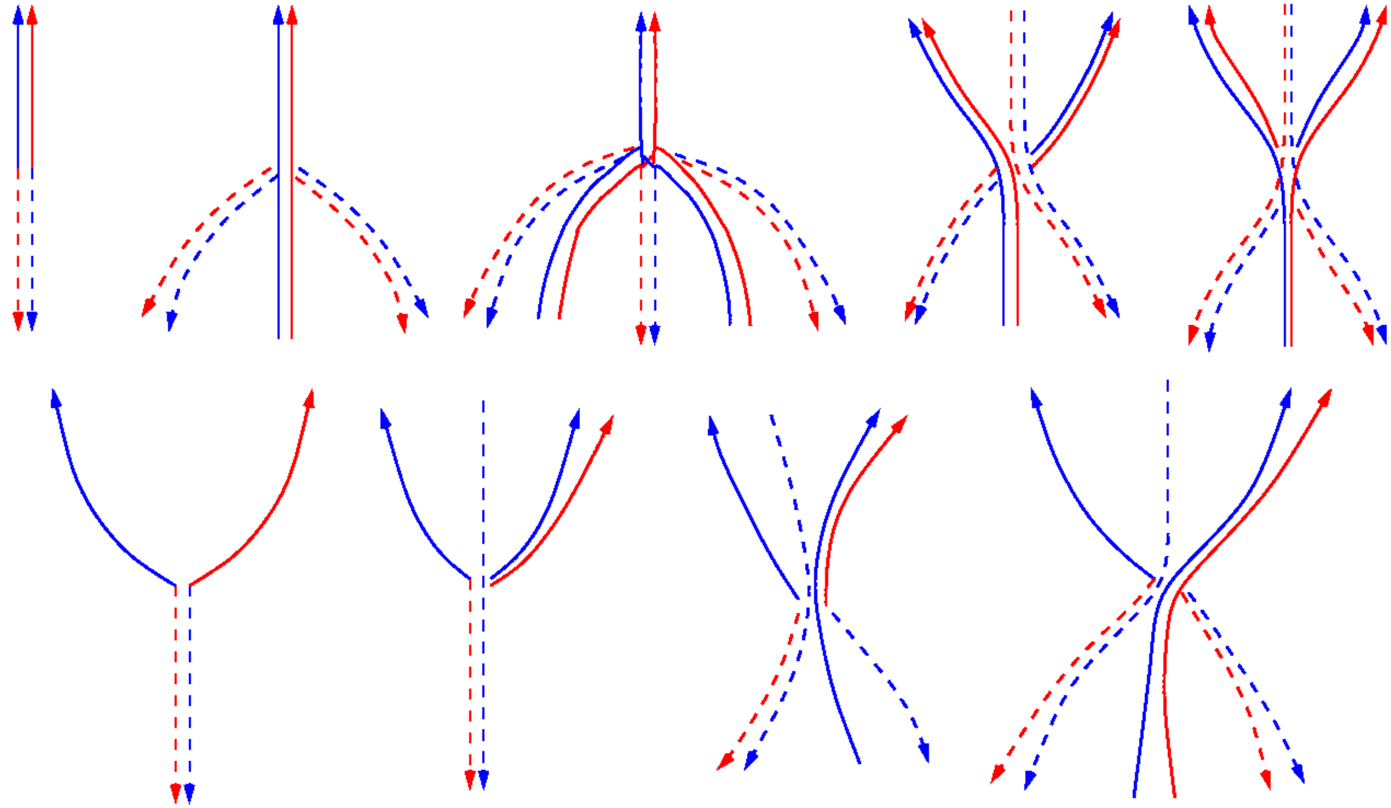


The paths in the Brownian net started at time zero form the 'branching-coalescing point set'.

The branching-coalescing point set

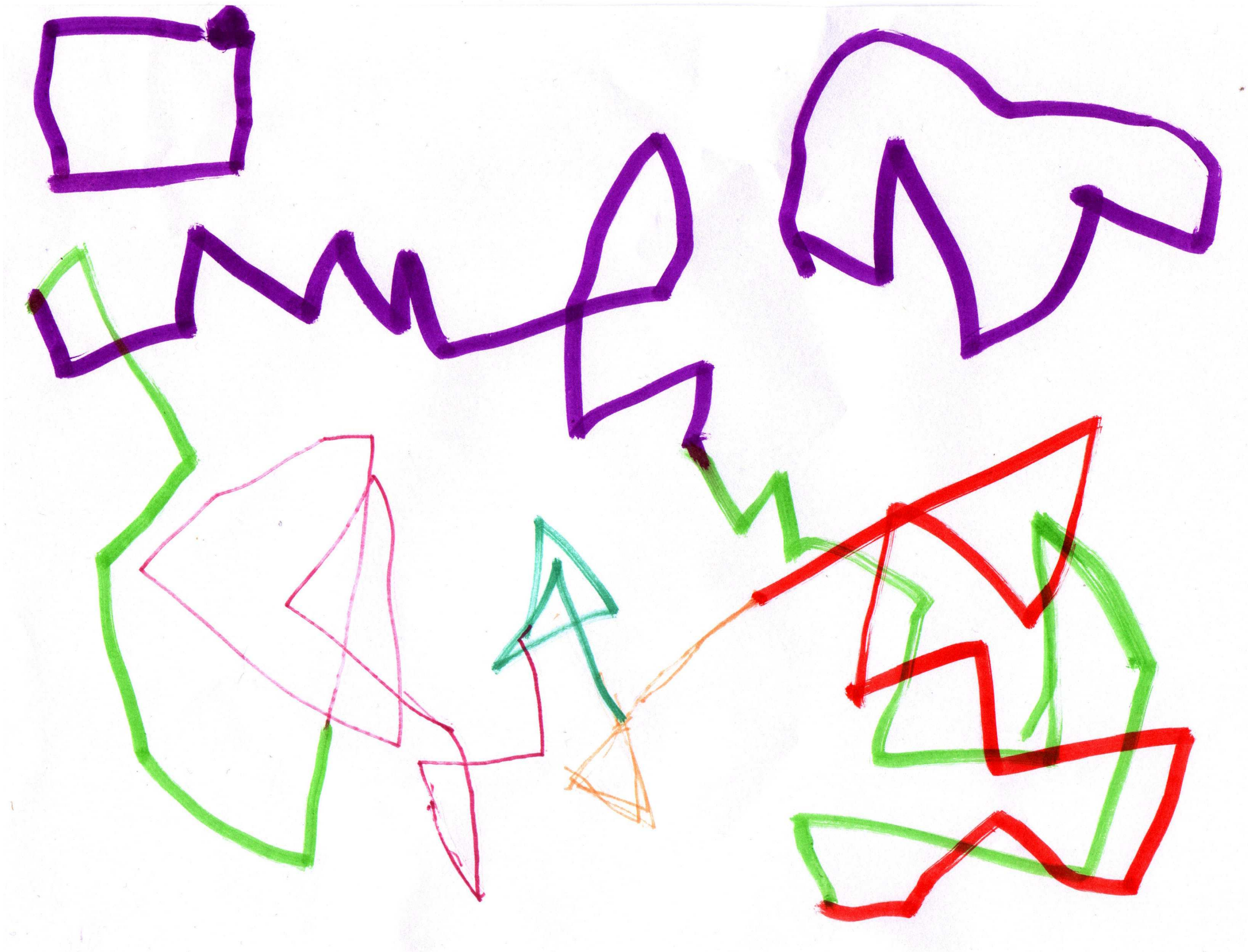
- Markov process taking values in the closed subsets of the real line.
- Equilibrium law: Poisson point set with intensity 2.
- At each deterministic time $t > 0$ locally finite.
- There exists random times when the process is not locally finite.

Special points of the Brownian net



Modulo symmetry, there exist 9 types of special points of the Brownian net. [Schertzer, Sun & S. 2009].

The End



Thank you!