Core of Coalition Games on MV-algebras[1]

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Abstract

Coalition games are generalized to semisimple MV-algebras. Coalitions are viewed as $[0,1]$-valued functions on a set of players, which enables to express a degree of membership of a player in a coalition. Every game is a real-valued mapping on a semisimple MV-algebra. The goal is to recover the so-called core: a set of final distributions of payoffs, which are represented by measures on the MV-algebra. A class of sublinear games are shown to have a non-empty core and the core is completely characterized in certain special cases. The interpretation of games on propositional formulas in Łukasiewicz logic is introduced.

1 Introduction

Foundations of cooperative game theory were laid by von Neumann and Morgenstern in [21]. They developed mathematical models of cooperation in a social environment in which coalition formation goes hand in hand with payoff negotiation at the level of individual players. Analyses of such games are usually based only on payoff opportunities available to each coalition so that neither strategies nor actions are associated with players. A coalition game is specified by three components: a set of players, a set of possible coalitions and a real mapping (so-called game) defined on the set of coalitions. An unambiguous definition of coalition is made precise below in this article. Every coalition operates on the assumption of maximizing profit (or minimizing loss) of its members, which is the amount that the members of the coalition can jointly guarantee themselves disregarding the payoff opportunities of other coalitions. There are many solution concepts for coalition games whose goal is to predict a final distribution of payoffs in the game. This article deals only with a solution in the form of a core, which counts among the most important concepts in cooperative game theory (see [16, Chapter 10–12] or [20]). Roughly speaking, the core is a set containing any payoff distribution among all the players that respects the basic criterion of collective rationality: no coalition will accept a payoff giving its players less than they are jointly able to guarantee themselves by forming the coalition. Analogously, when the game associates a loss with every coalition (instead of a profit), then its core consists of all loss distributions such that none of them inflicts on members of some coalition a loss that actually exceeds the one incurred by the coalition.

1.1 Games on various coalition structures

In the classical setting, a coalition game is determined by a finite set of *n* players together with a set of all subsets of the *n*-player set and a set function. The set function assigns a real number to
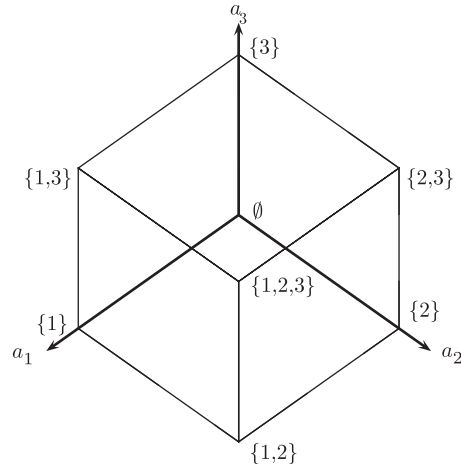

FIGURE 1. The set of all coalitions and its convex hull for three players

each coalition, which is any subset of the player set. This mathematical model captures faithfully many real-life cooperative situations ranging from voting procedures to cost allocation problems. Since solutions of these games involve combinatorial expressions, which are usually computationally expedient for a large number of players, there is a need for more analytical approaches employing functions of real variables in place of set functions. For instance, Owen [16, Chapter XII.2] considers a particular extension of an n-player game to a real function defined on the multidimensional cube $[0,1]^n$. Figure 1 depicts the underlying geometrical idea: each set of players is identified with a unique vertex of $[0,1]^n$. Hence it makes sense to claim that the original game is extended to the convex hull of all the coalitions. Interestingly, this 'convexification' technique pervades many areas of game theory: for instance, it is a crucial idea involved in the proof of existence of a solution for two person zero-sum games given by von Neumann and Morgenstern [16, Theorem II.4.1].

The n-player games modelled by real functions on $[0,1]^n$ became a standalone subject of study in the work of Aubin [1, 2]. In this framework, each coalition is identified with a point from $[0,1]^n$ in order to express a possibly partial degree of membership of a player in a coalition. On the one hand, this interpretation is especially appealing if each player possess an initial endowment (money, utility) that is to be apportioned among various projects. Then a coalition from $[0,1]^n$ can be seen as a proportional specification of the amount invested by a player in the project. This approach is used in [9]. On the other hand, it will be shown in Section 3.1 how the idea of $[0,1]$-valued coalitions naturally fits in the framework of Łukasiewicz infinite-valued propositional calculus.

Another stream of research in coalition game theory focused on games with a huge amount of players, in which no individual player is able to influence the overall outcome. Such games arise naturally as market models (for example, large mass of small investors on a stock market). Aumann and Shapley studied in [3] a class of games with the continuum of players: the players are identified with points from $[0,1]$, the set of all possible coalitions is the set of all Borel subsets of $[0,1]$ and a game is a set function on Borel sets. In order to prove the existence of the so-called value operator, which is an alternative solution concept in coalition game theory, Aumann and Shapley

also generalized this model one step further and considered the coalition structure consisting of all [0, 1]-valued Borel measurable functions on the unit interval [3, Chapter IV].

Butnariu and Klement studied in [8] the existence of a value operator for games with infinitely many players and particular subsets of [0, 1]-valued coalitions (so called tribes). The most important cases of these coalition structures correspond to ordered algebraic structures nowadays known as σ-complete MV-algebras [10]. The existence of the value operator was recently carried over to σ-complete MV-algebras and even more general algebras whose lattice reducts are σ-complete by Avallone *et al.* in [4].

A question of Dan Butnariu was whether coalition game theory can be developed as a fully fledged theory on MV-algebras. The main goal of this article is to show that this question can be answered in the affirmative as far as the core solution is concerned. The essential tool is the concept of measures on MV-algebras investigated by Mundici, Riečan, Weber and others [6, 15, 19]. In the framework of this article, a coalition game will be a real function on an MV-algebra. Every coalition game is associated with a (possibly empty) set of solutions, which is called a core. Elements of the core are measures on the MV-algebra dominated by the game. The main results of the article (Theorem 1 together with Proposition 3) suggest among others this interpretation: the measures from the core are those measures on the MV-algebra that are 'tangential' to a given game.

The article is structured as follows. Basic notions concerning MV-algebras and measures are repeated in Section 2. In Section 3, precise definitions of game and core are given. Section 3.1 introduces an interpretation of games as functionals on formulas in Łukasiewicz logic and shows that this idea fits properly into the process of 'averaging' the truth-value in Łukasiewicz logic originated in [15]. The game-theoretic meaning of measures on MV-algebras and cores is discussed in detail (Propositions 1 and 2). Theorem 1 provides a sufficient condition for non-emptiness of the core and the subsequent results show how to construct such games and recover their core elements in special cases.

2 Preliminary notions

In the article, we assume familiarity with basic definitions and results concerning MV-algebras [10], which are the Lindenbaum algebras of Łukasiewicz logic. An *MV-algebra* $M = (M, 0, \neg, \oplus)$ is an Abelian monoid $(M, 0, \oplus)$ equipped with a unary involutive operation $\neg$ such that $a \oplus \neg 0 = \neg 0$, and $a \oplus \neg(a \oplus \neg b) = b \oplus \neg(b \oplus \neg a)$, for every $a, b \in M$. Define $1 = \neg 0$ and the operation $a \odot b = \neg(\neg a \oplus \neg b)$. By X_M we denote the (non-empty) compact Hausdorff space of all maximal ideals of M [10, Section 3.4]. Only semisimple MV-algebras are considered in this work: an MV-algebra is *semisimple* iff it is isomorphic to an MV-algebra of [0, 1]-valued continuous functions over the maximal spectrum X_M [10, Corollary 3.6.8]. Without loss of generality, elements of a semisimple MV-algebra M are identified with continuous functions $X_M \to [0, 1]$. Every semisimple MV-algebra M can be thus viewed as a subset of the Banach space $C(X_M)$ of all real-valued continuous functions on the maximal spectrum X_M endowed with the supremum norm $\|\cdot\|$. For every $\alpha \in \mathbb{R}$, $a_1, a_2 \in M$, the notations αa_1 and $a_1 + a_2$ have their usual meaning in the linear space $C(X_M)$. If $a_1, a_2 \in M$ are such that $a_1 + a_2 \in M$, then $a_1 \oplus a_2 = a_1 + a_2$. Similarly, when $\alpha a \in M$ for a non-negative integer α and $a \in M$, then we have $\alpha a = \underbrace{a \oplus \cdots \oplus a}_{\alpha \text{ times}}$.

A *partition in M* is a pair $(\{a_1, \ldots, a_n\}, (\alpha_1, \ldots, \alpha_n))$, where $a_1, \ldots, a_n \in M$ and $\alpha_1, \ldots, \alpha_n$ are non-negative integers such that $\alpha_1 a_1 + \cdots + \alpha_n a_n = 1$. This notion of partition is weaker than so-called MV-partitions originally introduced by Marra in [13].

Real or $[0,1]$-valued additive functionals on M (known as 'measures' in [6] and 'states' in [19], respectively) play a role of plausible solutions to a given game in the framework introduced in this article. A *measure m* on an MV-algebra M is a mapping $M \to \mathbb{R}$ such that

(i) $m(a \oplus b) = m(a) + m(b)$,   for every $a, b \in M$ with $a \odot b = 0$,
(ii) $\sup\{|m(a)| \,|\, a \in M\} < +\infty$,
(iii) $m(0) = 0$.

A *state* on M is a non-negative measure s with $s(1) = 1$.

3   Games and cores

In the sequel we assume that a coalition structure is described by a semisimple MV-algebra. A coalition is consequently identified with a $[0,1]$-valued function $a \in M$ on the maximal spectrum X_M of M. The set X_M is viewed as a set of all players. The number $a(x) \in [0,1]$ then captures a degree of membership of the player $x \in X_M$ in the coalition $a \in M$.

The use of MV-algebras as coalition structures opens a unifying perspective on the extension procedures discussed in Section 1.1. Indeed, the coalition structure consisting of the Boolean algebra of all subsets of a finite player set gives rise to the finite direct product of standard MV-algebras by taking the convex hull (Figure 1). In case of infinitely many players an analogous assertion applies as well: the coalition structure of all Borel measurable subsets of $[0,1]$ studied by Aumann and Shapley 'generates' the MV-algebra of all $[0,1]$-valued Borel measurable functions on $[0,1]$. This MV-algebra is the right convexification of the Borel subsets of the unit interval: the result of Butnariu and Klement [8, Proposition 3.3] says that a σ-complete MV-algebra M of functions $[0,1] \to [0,1]$, whose Boolean skeleton[2] equals the set of characteristic functions of all Borel measurable subsets of $[0,1]$, is a convex subset of $[0,1]^{[0,1]}$ if and only if it consists of all Borel measurable functions.

DEFINITION 1
Let M be a semisimple MV-algebra. A *game (on M)* is a mapping $v : M \to \mathbb{R}$ such that $v(0) = 0$ and $\sup\{|v(a)| \,|\, a \in M\} < +\infty$. By $\mathcal{G}_M$ we denote the set of all games on M.

A game v is interpreted as a loss function. A number $v(a)$ is thus the total loss incurred by the players in the coalition $a \in M$ as a result of their cooperation. The goal is to find a final distribution of the loss among all the players by taking into account results of cooperation captured a priori by the loss function v. Every distribution of loss is represented by some measure on an MV-algebra. Although measures on MV-algebras are functions defined on the set of coalitions rather than on the set of players, Proposition 1 below says that every distribution of loss among all the coalitions induces a unique distribution of loss among the players. Moreover, a loss distributed in this way to each coalition $a \in M$ is precisely the mean value of the losses assigned to the individual players with weights given by the membership degrees of all the players in a.

PROPOSITION 1
If M is a semisimple MV-algebra and m is a non-zero measure on M, then there exists a unique regular Borel measure μ on Borel sets of X_M such that $m(a) = \int a \, \mathrm{d}\mu$, for every $a \in M$.

PROOF. Assume first that the measure m is non-negative. Since m is non-zero we get $m(1) > 0$. Put $s = m/m(1)$ and note that s is a state. Hence every non-negative measure is a positive multiple of a

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[2]A *Boolean skeleton* of an MV-algebra M is a set of all idempotent (Boolean) elements of M [10, p. 29].

unique state. The measure m is consequently represented by a uniquely determined non-negative Borel measure $m(1)\mu$. Indeed, the state s has integral representation with respect to a uniquely determined regular Borel probability measure μ on X_M (see [12, Corollary 29] or [17, Proposition 1.1]), which yields

$$m(a) = m(1)s(a) = m(1)\int a\,\mathrm{d}\mu = \int a\,\mathrm{d}(m(1)\mu)$$

for every $a \in M$.

Given an arbitrary measure m, employ [6, Theorem 3.1.3] to recover Jordan decomposition of the measure m. Precisely, there exists a unique pair of non-negative measures m^+, m^- on M such that $m = m^+ - m^-$. Hence $m(a) = m^+(a) - m^-(a) = \int a\,\mathrm{d}\mu^+ - \int a\,\mathrm{d}\mu^- = \int a\,\mathrm{d}(\mu^+ - \mu^-)$, where μ^+, μ^- are uniquely determined non-negative Borel measures. The proof is finished by setting $\mu = \mu^+ - \mu^-$. ∎

3.1 Games and Łukasiewicz logic

Łukasiewicz infinite-valued propositional logic (see [10, Chapter 4], for example) provides one way of thinking about a gradual membership of players to coalitions. Formulas $\varphi, \psi, \ldots$ are constructed from propositional variables $A_1, \ldots, A_k$ by applying the standard rules known in Boolean logic. The connectives are negation, disjunction and conjunction, which are denoted by $\neg$, $\oplus$ and $\odot$, respectively. This is already a complete set of connectives so that, for instance, the implication $\varphi \to \psi$ can de defined as $\neg\varphi \oplus \psi$. The set of all formulas in propositional variables $A_1, \ldots, A_k$ is denoted by Form$(A_1, \ldots A_n)$.

The algebra of truth degrees of Łukasiewicz logic is the *standard MV-algebra*, which is the unit interval $[0,1]$ endowed with the operations $\neg, \oplus, \odot$ defined as follows: $\neg a = 1-a$, $a \oplus b = \min\{a+b, 1\}$, $a \odot b = \max\{a+b-1, 0\}$. A *valuation* is a mapping $V : \text{Form}(A_1, \ldots A_k) \to [0,1]$ such that $V(\neg\varphi) = 1 - V(\varphi)$, $V(\varphi \oplus \psi) = V(\varphi) \oplus V(\psi)$ and $V(\varphi \odot \psi) = V(\varphi) \odot V(\psi)$. Formulas $\varphi, \psi \in \text{Form}(A_1, \ldots A_k)$ are called *equivalent* when $V(\varphi) = V(\psi)$, for every valuation V. The *equivalence class* of φ is denoted $[\varphi]$. The set of all such equivalence classes is an MV-algebra L_k with the operations $\neg[\varphi] = [\neg\varphi]$, $[\varphi] \oplus [\psi] = [\varphi \oplus \psi]$ and $[\varphi] \odot [\psi] = [\varphi \odot \psi]$, for every $\varphi, \psi \in \text{Form}(A_1, \ldots, A_k)$.

Let X be a set of elements interpreted as players. The propositional formulas in Form$(A_1, \ldots, A_k)$ can represent statements expressing a degree of player's participation in certain social–economic events. For example, φ can be 'I am a minor shareholder of this company' or 'About 90% of my investments are in hedge funds'. Every player $x \in X$ then assigns to each $\varphi \in \text{Form}(A_1, \ldots, A_k)$ a level to which he/she conforms with the principle substantiated by the formula φ. Precisely, a player chooses a unique valuation V so that the truth-value $V(\varphi)$ represents this level of conformity. Assume that any two players with the same level of conformity for each formula in Form$(A_1, \ldots, A_k)$ are considered to be identical. Since every valuation is uniquely determined by its restriction to the propositional variables $V \mapsto V(A_1, \ldots, A_k) \in [0,1]^k$, every player is matched with a unique point x_V from the k-dimensional unit cube $[0,1]^k$ and vice versa. Let V_x be the valuation corresponding to $x \in [0,1]^k$. Put $[\varphi](x) = V_x(\varphi)$, for every $x \in [0,1]^k$. Hence the equivalence class $[\varphi]$ of every $\varphi \in \text{Form}(A_1, \ldots, A_k)$ can be viewed as a function $[0,1]^k \to [0,1]$ and, consequently, every level of conformity $V(\varphi)$ can be thought of as a degree of membership of the player x_V to the coalition $[\varphi]$. If the player set X is large enough, it can be identified with the whole cube $[0,1]^k$. It is evident that any quantitative assessment associated with the principle represented by φ must depend only on the semantical meaning of φ. Hence a game is in accordance with Definition 1 any real mapping defined on the MV-algebra of equivalence classes of all formulas. In the light of Proposition 1, any plausible loss distribution results from averaging the losses over all valuations with respect to the truth-values of the formulas.

A direct generalization of Theorem 4.6.9 from [10] shows that every MV-algebra is isomorphic to the Lindenbaum algebra of some theory in Łukasiewicz logic as far as sufficiently many propositional variables are appended to the alphabet. Thus although we deal with the whole class of semisimple MV-algebras in this article, the above introduced logical meaning of coalitions is always preserved.

3.2   Core

One of the ways of 'solving' a game is to predict a final loss distribution among the players provided the coalitions follow certain criteria of economical rationality. The concept of core is based on two assumptions. First, the grand coalition $1 \in M$ is formed and every final distribution of loss must redistribute precisely the loss $v(1)$. Second, every coalition will accept only a final loss distribution not exceeding the loss generated by its members. Quoting Shapley [20, p. 11], 'the core is the set of feasible outcomes that cannot be improved upon by any coalition of players'. These principles motivate the following definition, which carries over the concept of core on various coalition structures (cf. [1, 3, 20]) to games on MV-algebras. Let $\mathcal{M}_M$ denotes the set of all measures on M.

DEFINITION 2
Let v be a game on a semisimple MV-algebra M. The *core (of the game v)* is a set

$$\mathcal{C}_M(v) = \{m \in \mathcal{M}_M \mid m(1) = v(1), \, m(a) \leq v(a), \text{ for every } a \in M \setminus \{1\}\}.$$

When the core of a game is empty, the coalitions are not able to arrive at an agreement concerning the final distribution of loss.

EXAMPLE 1
Let M be an MV-algebra containing at least three elements. Define

$$v(a) = \begin{cases} 1, & a = 1, \\ 0, & \text{otherwise,} \end{cases} \quad a \in M.$$

In the game v the only coalition that in fact incurs loss is the grand coalition 1. Intuitively, no distribution of the loss $v(1)$ of the grand coalition among the other coalitions is jointly acceptable since those coalitions realize no loss at all. The core of this game is empty: every element $m \in \mathcal{C}_M(v)$ is lower or equal to 0 on $M \setminus \{1\}$ while it must simultaneously satisfy $m(1) = v(1) = 1$, which is impossible.

The question of non-emptiness of the core is omnipresent in coalition game theory. This question is highly non-trivial even for games on the MV-algebra that is in fact the set of all subsets of a finite n-player set. Then the core is an intersection of a hyperplane with $2^n - 2$ halfspaces in $\mathbb{R}^n$, which amounts to solving a large linear programming problem. If an MV-algebra is a direct product of n standard MV-algebras then, a fortiori, checking the non-emptiness is hard as the core is an intersection of uncountably many halfspaces and hyperplanes in $\mathbb{R}^n$. Below we focus on the question of non-emptiness for the whole class of semisimple MV-algebras. Moreover, it will be demonstrated how the elements of core can be recovered in some special cases.

If $\mathcal{M}_M^1, \mathcal{M}_M^2$ are two subsets of $\mathcal{M}_M$ and $\alpha \in \mathbb{R}$, then their sum is $\mathcal{M}_M^1 + \mathcal{M}_M^2 = \{m_1 + m_2 \mid m_1 \in \mathcal{M}_M^1, m_2 \in \mathcal{M}_M^2\}$ and the multiple of $\mathcal{M}_M^1$ by α is $\alpha \mathcal{M}_M^1 = \{\alpha m \mid m \in \mathcal{M}_M^1\}$. Given an automorphism π of an MV-algebra M, a game $v \in \mathcal{G}_M$ and a set $\mathcal{G}_M' \subseteq \mathcal{G}_M$, put $\pi v = v \circ \pi$ and $\pi \mathcal{G}_M' = \{\pi v \mid v \in \mathcal{G}_M'\}$.

PROPOSITION 2

Let M be a semisimple MV-algebra.

(i) For every game $v \in \mathcal{G}_M$ the core $\mathcal{C}_M(v)$ is a (possibly empty) closed convex subset of $\mathcal{M}_M$ endowed with the subspace product topology of $\mathbb{R}^M$.

(ii) If $v_1, v_2 \in \mathcal{G}_M$, then $\mathcal{C}_M(v_1) + \mathcal{C}_M(v_2) \subseteq \mathcal{C}_M(v_1 + v_2)$.

(iii) If $v \in \mathcal{G}_M$ and $\alpha > 0$, then $\mathcal{C}_M(\alpha v) = \alpha \mathcal{C}_M(v)$.

(iv) If $v \in \mathcal{G}_M$ and π is an automorphism of the MV-algebra M, then

$$\mathcal{C}_M(\pi v) = \pi \mathcal{C}_M(v).$$

(v) If $v \in \mathcal{G}_M$ is such that $\mathcal{C}_M(v) \neq \emptyset$, then

$$v(1) = \inf \left\{ \sum_{i=1}^{n} \alpha_i v(a_i) \,\middle|\, (\{a_i\}_{i=1}^{n}, (\alpha_i)_{i=1}^{n}) \text{ is a partition in } M \right\}. \tag{1}$$

(vi) If M has at least three elements and $v \in \mathcal{M}_M$, then $\mathcal{C}_M(v) = \{v\}$.

PROOF. (i) The core of every game v is the intersection of sets

$$\{m \in \mathcal{M}_M \mid m(1) = v(1)\}, \quad \{m \in \mathcal{M}_M \mid m(a) \leq v(a)\}, \, a \in M,$$

each of which is easily seen to be closed in $\mathbb{R}^M$ and convex. The property (ii) results from a routine verification as well as the inclusion $\mathcal{C}_M(\alpha v) \supseteq \alpha \mathcal{C}_M(v)$ in (iii). If $\alpha > 0$ and $m \in \mathcal{C}_M(\alpha v)$, then define $m' = m/\alpha$. Since $m' \in \mathcal{C}_M(v)$, this gives $m = \alpha m' \in \alpha \mathcal{C}_M(v)$. In (iv) the inclusion $\mathcal{C}_M(\pi v) \supseteq \pi \mathcal{C}_M(v)$ follows from the definition. We will show that the reversed inclusion holds true. Let $v \in \mathcal{G}_M$ and π be an automorphism of M such that $m \in \mathcal{C}_M(\pi v)$. Put $m' = \pi^{-1} m$. Then

$$m'(1) = m(\pi^{-1}(1)) = m(1) = (\pi v)(1) = v(1),$$

and for every $a \in M \setminus \{1\}$, we have

$$m'(a) = m(\pi^{-1}(a)) \leq (\pi v)(\pi^{-1}(a)) = v(a).$$

Hence $m' \in \mathcal{C}_M(v)$ and $m = \pi m' \in \pi \mathcal{C}_M(v)$. In order to establish (v), consider an element $m \in \mathcal{C}_M(v)$ and a partition $(\{a_i\}_{i=1}^{n}, (\alpha_i)_{i=1}^{n})$ in M. The definition of a partition gives that each block a_i satisfies $a_i \odot (\alpha_i - k) a_i = 0$, for $1 \leq k \leq \alpha_i$, and each pair of blocks a_i, a_j with $i \neq j$ fulfils $\alpha_i a_i \odot \alpha_j a_j = 0$. The inequality $\leq$ in (1) then results from

$$v(1) = m(1) = m\left(\sum_{i=1}^{n} a_i \alpha_i\right) = \sum_{i=1}^{n} \alpha_i m(a_i) \leq \sum_{i=1}^{n} \alpha_i v(a_i).$$

The reversed inequality follows by taking the partition $(1,0)$ in M with multiplicities $\alpha_1 = 1, \alpha_2 = 0$. To prove (vi) observe that $v \in \mathcal{C}_M(v)$ and assume that there exists some $m \in \mathcal{C}_M(v)$ with $m \neq v$. Hence there must be $a \in M \setminus \{0, 1\}$ with the property $m(a) < v(a)$. This gives

$$m(\neg a) = m(1) - m(a) > v(1) - v(a) = v(\neg a),$$

which is a contradiction. $\blacksquare$

In particular, the properties (ii)–(iv) mean that the mapping $v \in \mathcal{G}_M \mapsto \mathcal{C}_M(v) \subseteq \mathcal{M}_M$ is superadditive, positively homogeneous, and commutes with every automorphism of M. The last property is an important 'fairness' axiom. It expresses the fact that the core of the game πv with the roles of players interchanged coincides with the π-image of the core containing the distributions of loss (measures) in the original game v. Precisely, every automorphism π of M gives rise to a homeomorphism $\pi^* : x \in X_M \mapsto \{a \in M \mid \pi(a) = x\}$ of the compact space of all players X_M [11, p. 72], so that we can write $(\pi v)(a) = v(\pi(a)) = v(a \circ \pi^*)$. The action of the automorphism group of M on the set of games $\mathcal{G}_M$ is thus completely captured by the action of the corresponding group of homeomorphisms $\{\pi^* \mid \pi \text{ is an automorphism of } M\}$ on the set of the players X_M and vice versa.

3.3   Sublinear games

A game v on M is *subadditive* if $v(a \oplus b) \leq v(a) + v(b)$, for every $a, b \in M$ with $a \odot b = 0$. The subadditivity is a time-honored condition in game theory, it captures the idea of 'l'union fait la force'. In other words, disjoint coalitions have an incentive to join their forces as this leads to a loss that is not greater than the sum of the individual losses.

The set of all subadditive games forms a convex cone in the linear space $\mathcal{G}_M$. There are numerous examples of games from this cone. We say that a real function f defined on a real linear space E is *subadditive*, when $f(x + y) \leq f(x) + f(x)$ for every $x, y \in E$. In the examples below M is any semisimple MV-algebra.

EXAMPLE 2
Let $m_1, \ldots, m_n \in \mathcal{M}_M$ and f be a subadditive function $\mathbb{R}^n \to \mathbb{R}$ vanishing at 0. The measures m_i can be viewed as potential cost allocations available in the game v a priori. Define $v(a) = f(m_1(a), \ldots, m_n(a))$, $a \in M$, and observe that v is a subadditive game.

EXAMPLE 3
Let $\mathcal{M}'_M$ be a non-empty compact subset of $\mathcal{M}_M$. Put

$$v(a) = \sup\{m(a) \mid m \in \mathcal{M}'_M\}, \quad a \in M. \tag{2}$$

The game v is subadditive: for every $a, b \in M$ with $a \odot b = 0$, we get

$$v(a \oplus b) = \sup\{m(a \oplus b) \mid m \in \mathcal{M}'_M\} = \sup\{m(a) + m(b) \mid m \in \mathcal{M}'_M\}$$
$$\leq \sup\{m(a) \mid m \in \mathcal{M}'_M\} + \sup\{m(b) \mid m \in \mathcal{M}'_M\} = v(a) + v(b).$$

Since $\mathcal{M}'_M$ is compact and the evaluation mapping $m \in \mathcal{M}'_M \mapsto m(a)$ is continuous for every $a \in M$, the supremum in (2) becomes in fact the maximum so that $v(a) = m(a)$ for some $m \in \mathcal{M}'_M$. The loss of every coalition $a \in M$ in the game is thus the most pessimistic loss distribution from $\mathcal{M}'_M$. In this particular case showing that $\mathcal{C}_M(v) \neq \emptyset$ is straightforward. There exists $m' \in \mathcal{M}'_M$ giving the grand coalition 1 its exact loss $v(1)$ and this measure m' in addition satisfies $m'(a) \leq v(a)$ due to the definition of v. Hence $m' \in \mathcal{C}_M(v)$.

Subadditivity alone does not guarantee non-emptiness of the core even when M is the set of all subsets of a finite set of players. The property (v) from Proposition 2 suggests that non-emptiness of the core can be decided in a sufficiently small neighbourhood of the point 1. This line of reasoning is confirmed by Theorem 1, which singles out a family of games whose core is non-empty. This family of games covers some well-known models: for example, convex games with finitely many players

investigated by Shapley [20] and Aubin [1]. The idea of Theorem 1 does not appear to be formulated elsewhere. In a nutshell, a sufficient condition for non-emptiness of the core is that a game v on M is minorized by a 'suitable' function defined over the space of all continuous function on the maximal spectrum of M.

The following notations are introduced. By $C^*(X_M)$ we denote the Banach space of all bounded linear functionals on $C(X_M)$, which can be identified with the space of regular Borel measures on X_M by Riesz theorem. The restriction of any functional $m^* \in C^*(X_M)$ to M then obviously defines a measure on M. A real function f defined on a real linear space E is said to be *sublinear* if it is subadditive and *positively homogeneous* (that is, $f(tx) = tf(x)$ for every $t \geq 0$ and every $x \in E$).

THEOREM 1

Let M be a semisimple MV-algebra and $w : C(X_M) \to \mathbb{R}$ be a continuous sublinear function. If $v \in \mathcal{G}_M$ is such that $w \leq v$ on M and $v(1) = w(1)$, then $\mathcal{C}_M(v)$ includes the non-empty set

$$\{m^* \restriction M \mid m^*(a-1) \leq w(a) - w(1), m^* \in C^*(X_M), a \in C(X_M)\}.$$

PROOF. Since w is positively homogeneous and subadditive, it is a convex function on $C(X_M)$. Hence the right-hand directional derivative at 1

$$\mathrm{d}^+ w(1)(a) = \lim_{t \to 0_+} \frac{w(1 + ta) - w(1)}{t}$$

exists for every $a \in C(X_M)$ [18, Lemma 1.2]. Let $\partial w(1)$ be the *subdifferential* of w at 1:

$$\partial w(1) = \{m^* \in C^*(X_M) \mid m^*(a) \leq \mathrm{d}^+ w(1)(a), a \in C(X_M)\}.$$

Since w is continuous at 1, Proposition 1.11 in [18] guarantees that the set $\partial w(1)$ is non-empty, convex and weak*-compact subset of $C^*(X_M)$. Moreover, the subdifferential $\partial w(1)$ coincides with the set

$$\{m^* \in C^*(X_M) \mid m^*(a-1) \leq w(a) - w(1), a \in C(X_M)\}. \tag{3}$$

We will show that $\mathcal{C}_M(v)$ contains the set of all elements of $\partial w(1)$ restricted to M. Let $m^* \in \partial w(1)$. If $a = \alpha \cdot 1$ for $\alpha > 0$, then (3) together with linearity of m^* and positive homogeneity of w yields

$$\alpha m^*(1) - m^*(1) \leq \alpha w(1) - w(1),$$

which is equivalent to

$$(1 - \alpha)(m^*(1) - w(1)) \geq 0. \tag{4}$$

Setting $\alpha = 1/2$ in (4) leads to $m^*(1) \geq w(1)$, while, on the other hand, if $\alpha = 3/2$, then $m^*(1) \leq w(1)$. Hence necessarily $m^*(1) = w(1) = v(1)$. Due to (3) this means also that $m^*(a) \leq w(a) \leq v(a)$, for every $a \in M$. Hence $m^* \restriction M \in \mathcal{C}_M(v)$. ∎

Loosely speaking, the elements of $\partial w(1)$ in the above proof can be visualized as all plausible candidates for the derivative of w at 1 (see Figure 2). Precisely, if $m^* \in \partial w(1)$, then there exists an affine function $A : C(X_M) \to \mathbb{R}$ such that $A \leq w$ and $A(1) = w(1)$. To see this it is enough to put $A(a) = m^*(a) + w(1) - m^*(1)$ for every $a \in C(X_M)$. Vice versa, if such a function A with $m^* \in C^*(X_M)$ exists, then m^* necessarily belongs to $\partial w(1)$.

In special cases, the core of v can be completely described by the subdifferential $\partial w(1)$ of w at 1. Moreover, it will be shown that in this situation the core is an 'additive' solution [cf. Proposition 2 (ii)] and the most 'pessimistic' solution [cf. Example 3].
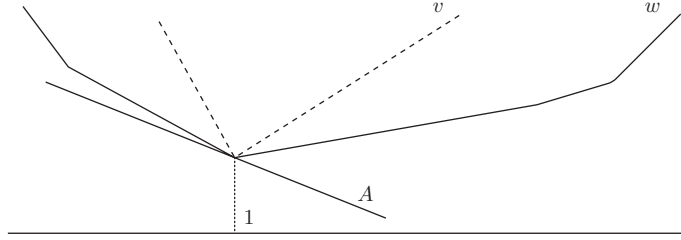
FIGURE 2. Function w and of its plausible tangents A

PROPOSITION 3

Let M be an MV-algebra of all $[0, 1]$-continuous functions on some compact Hausdorff space X.

(i) If v is a continuous sublinear function $C(X) \rightarrow \mathbb{R}$, then

$$\mathcal{C}_M(v{\restriction}M) = \{m^*{\restriction}M \mid m^* \in \partial v(1)\} \tag{5}$$

is non-empty compact and

$$v(a) = \sup\{m(a) \mid m \in \mathcal{C}_M(v{\restriction}M)\}, \quad a \in M. \tag{6}$$

(ii) If v_1, v_2 are continuous sublinear functions $C(X) \rightarrow \mathbb{R}$, then $\mathcal{C}_M(v_1{\restriction}M + v_2{\restriction}M) \neq \emptyset$ and

$$\mathcal{C}_M(v_1{\restriction}M + v_2{\restriction}M) = \mathcal{C}_M(v_1{\restriction}M) + \mathcal{C}_M(v_2{\restriction}M).$$

(iii) Let v be a continuous sublinear function $C(X) \rightarrow \mathbb{R}$ and let

$$dv(1)(a) = \lim_{t \to 0} \frac{v(1 + ta) - v(1)}{t}, \quad a \in C(X). \tag{7}$$

The limit in (7) exists for every $a \in C(X)$ if and only if

$$\mathcal{C}_M(v{\restriction}M) = \{dv(1)(\cdot){\restriction}M\}.$$

PROOF.

(i) Letting $w = v$ in Theorem 1 yields non-emptiness of $\mathcal{C}_M(v{\restriction}M)$ and the inclusion $\mathcal{C}_M(v{\restriction}M) \supseteq \{m^*{\restriction}M \mid m^* \in \partial v(1)\}$ in (5). Let $m \in \mathcal{C}_M(v{\restriction}M)$. Then there exists a unique representing Borel measure μ for m according to Proposition 1. Defining $m^*(a) = \int a \, d\mu$, for every $a \in C(X)$, gives a linear functional $m^* \in C^*(X)$ such that $m^*{\restriction}M = m$. We will show by verifying the condition (3) that $m^* \in \partial v(1)$. If $a = 0$, then $m^*(0 - 1) = -m^*(1) = -m(1) = -v(1) = v(0) - v(1)$. Let $a \in C(X)$ be non-zero. Then

$$m^*(a - 1) = m^*(a) - m^*(1) = \|a\| m^*\left(\frac{a}{\|a\|}\right) - v(1) = \|a\| m\left(\frac{a}{\|a\|}\right) - v(1)$$

$$\leq \|a\| v\left(\frac{a}{\|a\|}\right) - v(1) = v(a) - v(1).$$

Compactness of $\mathcal{C}_M(v{\restriction}M)$ follows from weak*-compactness of $\partial v(1)$ [18, Proposition 1.11] and continuity of the mapping $m^* \in \partial v(1) \mapsto m^*{\restriction}M$.

In order to show (6), let $a \in M$. Since for every $m \in \mathcal{C}_M(v{\restriction}M)$ we have $v(a) \geq m(a)$, the inequality $v(a) \geq \sup\{m(a) \mid m \in \mathcal{C}_M(v{\restriction}M)\}$ holds true. Conversely, since v is convex and continuous at every $a \in M$, then it follows from [18, Proposition 1.11] that there exists an element

$$m^* \in \partial v(a) = \{m' \in C^*(X) \mid m'(b-a) \leq v(b) - v(a), \text{ for every } b \in C(X)\}.$$

This implies $m^* \leq v$ and $m^*(a) = v(a)$, so

$$v(a) = m^*(a) \leq \sup\{m(a) \mid m \in \mathcal{C}_M(v{\restriction}M)\}.$$

(ii) This is basically Proposition 2 (ii) combined with Theorem 3.23 from [18].

(iii) Existence of the limit in (7) for every $a \in C(X)$ means by the definition Gâteaux differentiability of v at 1. If it exists, the Gâteaux derivative $\mathrm{d}v(1)(\cdot)$ is a continuous linear functional since v is continuous and convex [18, Corollary 1.7]. Gâteaux differentiability of v at 1 is also equivalent to saying that the subdifferential $\partial v(1)$ is a singleton containing only the Gâteaux derivative $\mathrm{d}v(1)(\cdot)$ [18, Proposition 1.8]. To finish the proof use (i). ∎

The next proposition shows a method for constructing games with non-empty cores on any semisimple MV-algebra. The idea is that of 'vector measure games' discussed already in Example 2.

PROPOSITION 4

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a sublinear function with $f(0) = 0$ and $m_1, \ldots, m_n \in \mathcal{M}_M$, for some $n \in \mathbb{N}$ and a semisimple MV-algebra M. Then a game $v = f \circ (m_1, \ldots, m_n)$ on M has a non-empty core.

PROOF. First the game v is extended from M to the whole Banach space $C(X_M)$. This is done as follows. For every $m_i, i = 1, \ldots, n$, there exists a unique regular Borel measure μ_i such that $m_i(a) = \int a \, \mathrm{d}\mu_i$, for every $a \in M$ (Proposition 1). Given arbitrary $a \in C(X_M)$, put $w(a) = f(\int a \, \mathrm{d}\mu_1, \ldots, \int a \, \mathrm{d}\mu_n)$. Clearly w agrees with v on M. As a consequence of linearity of integral and sublinearity of f, the function w is easily seen to be sublinear. It is also continuous everywhere in $C(X_M)$: take a sequence (a_k) in $C(X_M)$ such that $\|a_k - a\| \to 0$ for some $a \in C(X_M)$. Since the convergence in supremum norm implies the pointwise convergence, we obtain

$$\lim_{k \to \infty} w(a_k) = \lim_{k \to \infty} f(\int a_k \, \mathrm{d}\mu_1, \ldots, \int a_k \, \mathrm{d}\mu_n) = f(\lim_{k \to \infty} (\int a_k \, \mathrm{d}\mu_1, \ldots, \int a_k \, \mathrm{d}\mu_n))$$
$$= f(\int a \, \mathrm{d}\mu_1, \ldots, \int a \, \mathrm{d}\mu_n) = w(a),$$

where the second equality results from continuity of the convex function f everywhere in $\mathbb{R}^n$, and the third equality is a consequence of Lebesgue's dominated convergence theorem. To finish the proof, use Theorem 1 with v and w. ∎

Proposition 3 in conjunction with subdifferential calculus [2, 18] provide useful rules for determining the exact shape of cores for some important classes of games. In particular, the cores of sublinear games can easily be visualized in case of finitely many players with both Boolean coalitions (Example 4) and many-valued coalitions (Example 5).

EXAMPLE 4

Let M be the set of all subsets a of an n-player set X. A game v on M is said to be *submodular* if $v(a \wedge b) + v(a \vee b) \leq v(a) + v(b)$, for every $a, b \in M$. The Shapley's result [20] says that the core of every submodular game is non-empty. Namely, the core $\mathcal{C}(v)$ is a convex polytope in $\mathbb{R}^n$ with the dimension at most $n-1$ and there are at most $n!$ vertices of $\mathcal{C}_M(v)$ which can be characterized in

the following way: a vector $m \in \mathbb{R}^n$ is a vertex of $\mathcal{C}_M(v)$ if and only if there is an $(n+1)$-tuple of coalitions $a_0, \ldots, a_n \in M$ such that $\emptyset = a_0 \leq a_1 \leq \cdots \leq a_n = X$, where the cardinality of each a_i is i, and

$$m = (v(a_1), v(a_2) - v(a_1), \ldots, v(a_n) - v(a_{n-1})).$$

EXAMPLE 5

Let M be a finite direct product of n standard MV-algebras $[0, 1]$. The set of players is then $X = \{1, \ldots, n\}$. The core of any game on M is an intersection of uncountably many halfspaces and a hyperplane in $\mathbb{R}^n$ so the question arises which classes of games are tractable from the computational point of view. The result of Brânzei *et al.* [7] states that if a game v satisfies $v(a \oplus d) - v(a) \geq v(b \oplus d) - v(b)$, for every $a, b, d \in M$ with $b \odot d = 0$ and $a \leq b$, then $\mathcal{C}(v) \neq \emptyset$, the game v restricted to 2^X is submodular and $\mathcal{C}(v)$ coincides with the core of the game $v \restriction 2^X$ described in Example 4.

For the class of games from Example 5 the description of their cores is substantially improved: all the accepted loss distributions are determined by only finitely many coalitions although there are infinitely many coalitions in the game. In the next section, we investigate a somewhat analogous situation for a certain family of games on the MV-algebra L_k of equivalence classes of formulas defined in Section 3.1.

3.4   Games on free MV-algebras

The algebra L_k is known to be the free k-generated MV-algebra [10, Section 3]. According to McNaughton theorem [14], it is precisely the MV-algebra of all functions $[0, 1]^k \to [0, 1]$ that are continuous and piecewise linear, where each piece has integer coefficients. Theory of Schauder hats in L_k developed for the purely geometrical proof of McNaughton theorem in [10, Section 9.1] is briefly repeated in the next paragraph.

A *polyhedral complex* is a finite set of polyhedra $\mathcal{R}$ such that: (i) each polyhedron of $\mathcal{R}$ is included in $[0, 1]^k$, all its vertices have rational coordinates; (ii) if $P \in \mathcal{R}$ and Q is a face of P, then $Q \in \mathcal{R}$; (iii) if $P, Q \in \mathcal{R}$, then $P \cap Q$ is a face of both P and Q. The set $\bigcup_{P \in \mathcal{R}} P$ is called a *support* of $\mathcal{R}$. If all the polyhedra of a polyhedral complex $\mathcal{S}$ are simplices, then $\mathcal{S}$ is said to be a *simplicial complex*. Alternatively, a simplicial complex $\mathcal{S}$ with the support S is called a *triangulation of S*. The *denominator* $\text{den}(q)$ of a point $q \in [0, 1]^k$ with rational coordinates $(\frac{r_1}{s_1}, \ldots, \frac{r_k}{s_k})$, where $r_i \geq 0, s_i > 0$ are the uniquely determined relatively prime integers, is the least common multiple of $s_1, \ldots, s_k$. Passing to homogeneous coordinates in $\mathbb{R}^k$, put

$$\tilde{q} = \left(\tfrac{\text{den}(q)}{s_1} r_1, \ldots, \tfrac{\text{den}(q)}{s_k} r_k, \text{den}(q) \right)$$

and note that $\tilde{q} \in \mathbb{Z}^{k+1}$. A k-simplex with vertices $v^0, \ldots, v^k$ is *unimodular* if $\{\tilde{v}^0, \ldots, \tilde{v}^k\}$ is a basis of the free Abelian group $\mathbb{Z}^{k+1}$. An n-simplex with $n < k$ is *unimodular* if it is a face of some unimodular k-simplex. We say that a triangulation Σ is *unimodular* if each simplex of Σ is unimodular. If $\mathcal{R}$ is a polyhedral complex, $\text{V}(\mathcal{R})$ denotes the set of all the vertices of $\mathcal{R}$. Let Σ be a unimodular triangulation with a support $S \subseteq [0, 1]^k$. For each $x \in \text{V}(\Sigma)$, the *Schauder hat (at x over Σ)* is the uniquely determined continuous piecewise linear function $h_x : S \to [0, 1]$ which attains the value $1/\text{den}(x)$ at x, vanishes at each vertex from $\text{V}(\Sigma) \setminus \{x\}$ and is a linear function on each simplex of Σ. The *basis H_Σ (over Σ)* is the set $\{h_x \mid x \in \text{V}(\Sigma)\}$. In the sequel $\mathfrak{T}$ denotes the collection of all unimodular triangulations of $[0, 1]^k$.

The MV-algebra L_k captures a coalitional structure of a very special kind. It describes a 'completely many-valued' model of coalition formation with infinitely many players since there are precisely

two coalitions in which all the players participates with degrees of membership only from $\{0, 1\}$: the Boolean skeleton of L_K consists of the McNaughton functions 0 and 1. Yet, the structure of each coalition from L_k exhibits several features analogous to games with finitely many players such as the decomposition of every coalition as a convex combination of $\{0, 1\}$-valued coalitions (cf. Figure 1). The essential tool lies at the heart of the Mundici's proof of McNaughton theorem [10, Theorem 9.1.5]: for every $a \in L_k$ there exist $\Sigma \in \mathfrak{T}$ and the basis H_Σ such that $a = \sum_{x \in V(\Sigma)} \alpha_x h_x$, for uniquely determined non-negative integers α_x. So the coalitions in each basis H_Σ can be viewed as generators which enable to represent each coalition in L_k by a particular linear combination of Schauder hats.

A special family of games on L_k whose cores arise as the intersection of a smaller family of sets than in the original definition (Definition 2) fulfils a condition of rigidity of worth for each coalition a with respect to any decomposition via bases:

$$v(a) = \inf \Big\{ \sum_{x \in V(\Sigma)} \alpha_x v(h_x) \mid a = \sum_{x \in V(\Sigma)} \alpha_x h_x, H_\Sigma \text{ basis} \Big\}, \quad a \in L_k. \tag{8}$$

In words, the condition (8) models a situation in which the cost $v(a)$ associated with forming the coalition a is the least attainable, since splitting a and forming all the coalitions $\alpha_x h_x$ will not improve the outlook of the coalition a. A condition similar to (8) but using Boolean coalitions and convex combinations in place of hats and linear combinations is introduced also for games on $[0, 1]^n$ in [5], where it guarantees that the core remains non-empty even after any number of players 'leave' the game. Surprisingly, the condition (8) has a different role if the coalition structure is L_k. We will show that the core of any game satisfying (8) is in fact determined by the generators of all coalitions in L_k, that is, by the set of all Schauder hats. Given a basis H_Σ, put

$$\mathcal{C}_{H_\Sigma}(v) = \{m \in \mathcal{M}_{L_k} \mid m(h) \leq v(h), \text{ for every } h \in H_\Sigma\}.$$

PROPOSITION 5
If v is a game on L_k having the property (8), then

$$\mathcal{C}_{L_k}(v) = \bigcap_{H_\Sigma \text{ basis}} \mathcal{C}_{H_\Sigma}(v) \cap \{m \in \mathcal{M}_{L_k} \mid m(1) = v(1)\}. \tag{9}$$

In particular, if v is a sublinear function $C([0, 1]^k) \to \mathbb{R}$, then the core of the game $v \upharpoonright L_k$ is non-empty and satisfies (9).

PROOF. The inclusion $\subseteq$ is trivial. Let m be a measure from the set on the right-hand side of (9) and $a \in L_k$ be a McNaughton function. Let H_Σ be any basis such that $a = \sum_{x \in V(\Sigma)} \alpha_x h_x$. Since every basis is a partition in L_k, we obtain

$$m(a) = m\Big(\sum_{x \in V(\Sigma)} \alpha_x h_x \Big) = \sum_{x \in V(\Sigma)} \alpha_x m(h_x) \leq \sum_{x \in V(\Sigma)} \alpha_x v(h_x).$$

The condition (8) yields $m(a) \leq v(a)$ and thus $m \in \mathcal{C}_{L_k}(v)$. The second part of the proposition results from Proposition 3 (i) and the fact that every restriction to L_k of a sublinear function fulfils (8). ∎

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References

[1] J.-P. Aubin. Coeur et valeur des jeux flous à paiements latéraux. *Comptes Rendus de l'Academie des Sciences paris Série A*, **279**, 891–894, 1974.

[2] J.-P. Aubin. Optima and equilibria. An introduction to nonlinear analysis. In *Graduate Texts in Mathematics*, vol. 140, second edn. Springer-Verlag, 1998.

[3] R. J. Aumann and L. S. Shapley. A rand corporation research study. *Values of Non-atomic Games*. Princeton University Press, 1974.

[4] A. Avallone, A. Basile, and P. Vitolo. Positive operators à la Aumann-Shapley on spaces of functions on D-lattices. *Positivity*, **10**, 701–719, 2006.

[5] Y. Azrieli and E. Lehrer. On some families of cooperative fuzzy games. *International Journal of Game Theory*, **36**, 1–15, 2007.

[6] G. Barbieri and H. Weber. Measures on clans and on MV-algebras. In *Handbook of Measure Theory*, vol. I, II, pp. 911–945. North-Holland, 2002.

[7] R. Brânzei, D. Dimitrov, and S. Tijs. Convex fuzzy games and participation monotonic allocation schemes. *Fuzzy Sets and Systems*, **139**, 267–281, 2003.

[8] D. Butnariu and E. P. Klement. *Triangular Norm Based Measures and Games with Fuzzy Coalitions*. Kluwer, 1993.

[9] D. Butnariu and T. Kroupa. Shapley mappings and the cumulative value for n-person games with fuzzy coalitions. *European Journal of Operational Research*, **186**, 288–299, 2008.

[10] R. L. O. Cignoli, I. M. L. D'Ottaviano, and D. Mundici. Algebraic foundations of many-valued reasoning. *Trends in Logic—Studia Logica Library*, vol. 7. Kluwer Academic Publishers, 2000.

[11] A. Di Nola, R. Grigolia, and G. Panti. Finitely generated free MV-algebras and their automorphism groups. *Studia Logica*, **61**, 65–78, 1998.

[12] T. Kroupa. Every state on semisimple MV-algebra is integral. *Fuzzy Sets and Systems*, **157**, 2771–2782, 2006.

[13] V. Marra. *Non-Boolean Partitions. A Mathematical Investigation Through Lattice-ordered Abelian Groups and MV-algebras*. PhD Thesis, Università degli studi di Milano, 2002.

[14] R. McNaughton. A theorem about infinite-valued sentential logic. *Journal of Symbolic Logic*, **16**, 1–13, 1951.

[15] D. Mundici. Averaging the truth-value in Łukasiewicz logic. *Studia Logica*, **55**, 113–127, 1995.

[16] G. Owen. *Game Theory*, Third edn. Academic Press Inc., 1995.

[17] G. Panti. Invariant measures in free MV-algebras. *Communications in Algebra*, **6**, 2849–2861, 2008.

[18] R. R. Phelps. *Convex Functions, Monotone Operators and Differentiability*, second edn. vol. 1364 of *Lecture Notes in Mathematics*. Springer-Verlag, 1993.

[19] B. Riečan and D. Mundici. Probability on MV-algebras. In *Handbook of Measure Theory*, vol. I, II, pp. 869–909. North-Holland, 2002.

[20] L. S. Shapley. Cores of convex games. *International Journal of Game Theory*, **1**, 11–26, 1972.

[21] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.