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On Metric and Calmness Qualification Conditions in Subdifferential Calculus

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Abstract The paper contains two groups of results. The first are criteria for calmness/subregularity for set-valued mappings between finite-dimensional spaces. We give a new sufficient condition whose subregularity part has the same form as the coderivative criterion for "full" metric regularity but involves a different type of coderivative which is introduced in the paper. We also show that the condition is necessary for mappings with convex graphs. The second group of results deals with the basic calculus rules of nonsmooth subdifferential calculus. For each of the rules we state two qualification conditions: one in terms of calmness/subregularity of certain set-valued mappings and the other as a metric estimate (not necessarily directly associated with aforementioned calmness/subregularity property). The conditions are shown to be weaker than the standard Mordukhovich–Rockafellar subdifferential qualification condition; in particular they cover the cases of convex polyhedral set-valued mappings and, more generally, mappings with semi-linear graphs. Relative strength of the conditions is thoroughly analyzed. We also show, for each of the calculus rules, that the standard qualification conditions are equivalent

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to “full” metric regularity of precisely the same mappings that are involved in the subregularity version of our calmness/subregularity condition.

Keywords Subdifferential calculus · Qualification conditions · Metric estimates · Calmness · Metric subregularity

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1 Introduction

Qualification conditions is the backbone of subdifferential calculus, both in convex and nonsmooth analysis. In the finite dimensional calculus with a necessary amount of compactness automatically guaranteed, a qualification condition is the only assumption in any calculus rule. Since 1976, the research in this area has been deeply influenced by the pioneering work of B. Mordukhovich. He was the first to use unconvexified limiting normal cones, subdifferentials and coderivatives which now occupy the most prominent position in the technical arsenal of modern variational analysis.

The finite dimensional calculus of limiting subdifferentials developed in [11] still used qualification conditions formulated in terms of the Clarke–Rockafellar directional derivative (an equivalent formulation of such conditions in terms of Clarke subdifferentials was introduced in [31]). It was again Mordukhovich who completed shortly afterwards the developments by introducing the now standard qualification conditions in [22]. They are based on limiting subdifferentials and perfectly compatible with the proofs given in [11]. Applied to the calculation of the limiting normal cone to a set having a standard constraint structure (e.g. defined by finitely many smooth equalities and inequalities), these conditions reduce to the Mangasarian–Fromowitz constraint qualification. In convex analysis, however, a weaker condition is well-known (relative interiors of the intersecting sets have a common point). But even this condition does not cover the polyhedral case for which no qualification condition is needed.

There is also a “gray zone” between the polyhedral case and the domain of the mentioned relative interior qualification condition in convex analysis. Consider for instance the intersection of the following sets in $\mathbb{R}^2$: $\{(x, y) : x \geqslant |y|\}$ and $\{(x, y) : (x+1)^2 + y^2 \leqslant 1\}$. The normal cone to the intersection is the entire plane and obviously equal to the sum of the normal cones to the sets at zero. But this fact cannot be deduced either from the corresponding calculus rule of convex analysis (relative interiors of the sets do not meet) or from its polyhedral counterpart. A similar gray zone can be found also in nonsmooth analysis.

In this paper we discuss qualification conditions for finite dimensional limiting subdifferentials which are weaker then the standard qualification conditions. These conditions are automatically satisfied when the sets or graphs of the mappings involved in the operations are “semilinear”, that is to say, unions of convex polyhedral sets. The conditions therefore provide for a substantial unification which excludes the necessity to consider separately the class of semilinear sets and mappings, at least in the context of subdifferential calculus (and in particular the class of polyhedral sets and functions in calculus of convex subdifferentials). Moreover, the conditions cover a good part of the mentioned gray zone even when applied to the convex case.

Unlike standard qualification conditions usually stated in terms of subdifferentials, one group of our conditions is formulated as certain metric estimates and is usually referred to as *metric qualification conditions*. They were first developed in a series of papers by Ioffe, Jourani, Penot and Thibault [12, 13, 15–17] for the general situation of subsets of, functions on, and set-valued mappings between Banach spaces. The choice of the finite dimensional case in this paper is only dictated by our desire to fully concentrate on the conditions rather than on the corresponding calculus rules and not to obstruct the discussion of the first by shifting much of the attention to the discussion of the compactness properties, unavoidable in infinite dimensional case.

It was shown in [13] that some metric qualification conditions are naturally associated with the *metric subregularity* of certain set-valued mappings at the point of interest. (The metric subregularity property was introduced in [8] for single-valued maps in the context of necessary optimality conditions. In [13] this property was called "regularity at a point" and the terminology "metric subregularity" was suggested in [4].) This naturally leads to an idea to consider qualification conditions based on the latter property. A close approach based on the concept of *calmness* was suggested also in [6].

A multifunction $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ is *metrically subregular* at $(\bar{x}, \bar{y}) \in \text{Graph}\, F$, provided there exists a neighborhood $\mathcal{U}$ of $\bar{x}$ and a real number $K \geqslant 0$ such that

$$d(x, F^{-1}(\bar{y})) \leqslant K d(\bar{y}, F(x)) \quad \text{for all} \;\; x \in \mathcal{U},^{1}$$

where $d(x, A)$ stands for the distance of a point x to a set A. It is said that a multifunction $M[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ is *calm* at $(\bar{y}, \bar{x})$ if there is an $L \geqslant 0$ such that for any $x,\; y$ sufficiently close to $\bar{x}$ and $\bar{y}$ and such that $x \in M(y)$ one has

$$d(x, M(\bar{y})) \leqslant L d(y, \bar{y}).$$

It is easily verified that F is metrically subregular at $(\bar{x}, \bar{y})$ if and only if $M = F^{-1}$ is calm at $(\bar{y}, \bar{x})$ with the same constant.

A fundamental and somewhat more familiar concept of the "full" metric regularity property is obtained if we replace $\bar{y}$ by y in the above inequality and require in addition that the inequality holds also for all y of a neighborhood of $\bar{y}$. We see that the standard subdifferential qualification conditions precisely correspond to the metric regularity property of the same set-valued mappings that appear in the subregularity qualification conditions.

Calmness as defined above is closely related to the calmness property of the value function introduced in [3]. This property was also used as an a posteriori qualification condition in the theory of necessary optimality conditions. Calmness/metric subregularity also plays an important role in the theory of weak sharp minima and error bounds, cf., e.g., [29, 33, 34]. For these reasons there is a growing interest in verifiable calmness criteria. Some progress has been already achieved in ([5–7, 18]); in

[1] The definition of metric subregularity in [4] contains in the right-hand side the distance $d(\bar{y}, F(x) \cap \mathcal{V})$, where $\mathcal{V}$ is a neighborhood of $\bar{y}$. But it can be easily verified that this and our conditions are equivalent.

particular, for convex multifunctions a complete characterization of calmness/metric subregularity in both primal and dual terms has been obtained in [35].

The aim of this paper is twofold. First, we provide a sufficient condition for calmness/metric subregularity in terms of new, specially tailored derivative-like objects which we call *outer subdifferential* and *outer coderivative*. Earlier (and weaker) versions of the condition can be found in [9, 34]. In Section 2 we further examine the relationship of our criterion with some other known calmness/metric subregularity criteria and show that under additional assumptions our condition is also necessary.

Section 3 contains a detailed discussion of metric and calmness/subregularity qualification conditions for a sufficiently complete list of basic calculus rules for sets, functions and set-valued mappings. The metric qualification conditions all have a form of certain estimates for distances to the originally given sets, epigraphs of the functions and graphs of the set-valued mappings. The calmness/subregularity qualification conditions are stated in terms of set-valued mappings that, although of course closely connected with the original data, may not appear explicitly in the statements of the corresponding calculus rules. We usually present the latter conditions in two forms: as a calmness condition of a certain mapping and as a subregularity condition for the inverse mapping.

As follows from the definition, the calmness/subregularity qualification conditions can also be formulated in terms of certain metric estimates. These estimates always imply the estimates in the metric qualification conditions. An interesting and important circumstance is that the converse implication does not hold in certain cases which makes metric qualification conditions better (weaker) than their calmness/subregularity counterparts. This circumstance has not been noticed so far, although metric estimates of both kinds appeared in earlier publications (estimates associated with the metric qualification conditions in [13–15], estimates associated with the calmness/subregularity qualification conditions—without a mention of the latter—in [17]).

As well-known, composite functions of maps can, in general, be defined in different ways with the help of different operations. For example a sum of functions $f_1(x) + ... + f_k(x)$ can also be viewed as a composition $g \circ F$, where F is the diagonal mapping $x \mapsto (x, ..., x)$ and $g(x_1, ..., x_k) = f_1(x_1) + ... + f_k(x_k)$. Another interesting fact to be mentioned in this connection is that, applying our results to different representation of the same set, function or mapping, we typically get different qualification conditions. All of them, however, will be better than the standard qualification conditions. Thus, along with some natural qualification condition, we actually propose an approach that gives enough flexibility to find a qualification condition that looks most suitable in one or another situation.

Since it does not lead to any confusion, we will omit the adverb "metrically", whenever we speak about (metrically) subregular mappings and the adjective "metric", if we speak about (metric) subregularity qualification conditions. Our notation is basically standard. If f is an extended-real-valued function on $\mathbb{R}^n$, then $[f \leqslant \alpha]$ stands for the level set $\{x \in \mathbb{R}^n : f(x) \leqslant \alpha\}$ and $|\nabla| f(x) = \limsup_{\substack{u \to x \\ u \neq x}} \dfrac{(f(x) - f(u))^+}{\|x - u\|}$ is the *slope* of f at x.

For the reader's convenience, let us recall now the definitions of limiting subdifferentials, normal cones and coderivatives (see [26, 32] for details). Let f be an extended-real-valued lower semicontinuous function on $\mathbb{R}^n$ which is finite at x. A *Fréchet subgradient* of f at x is any $x^* \in \mathbb{R}^n$ which satisfies

$$f(x+h) - f(x) \geqslant \langle x^*, h\rangle + o(\|h\|).$$

The set $\partial_F f(x)$ of all Fréchet subgradients of f at x is called the *Fréchet subdifferential* of f at x. The *limiting (Mordukhovich) subdifferential* $\partial f(x)$ of f at x, introduced in [20], is the upper (outer) limit of the Fréchet subdifferentials with respect to the f-convergence. This means that $x^* \in \partial f(x)$ if and only if there are sequences $x_k \underset{f}{\to} x$ (i.e. $x_k \to x$ and $f(x_k) \to f(x)$) and $x_k^* \to x^*$ with $x_k^* \in \partial_F f(x_k)$.

Let $S \subset \mathbb{R}^n$ be a closed set and $x \in S$. The *indicator* δ_S of S is the function equal to zero on S and ∞ outside of S. The (Fréchet, limiting) subdifferential of δ_S at x is a closed cone called the *(Fréchet, limiting) normal cone* to S at x and denoted by $N_F(S, x)$, $N(S, x)$, respectively.

Finally, let $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ be a set-valued mapping with closed graph and $y \in F(x)$. In [21], Mordukhovich introduced the set-valued mapping (from $\mathbb{R}^m$ into $\mathbb{R}^n$) which is defined by

$$y^* \;\mapsto\; D^*F(x|y)(y^*) := \{x^* \in \mathbb{R}^n :\ (x^*, -y^*) \in N(\text{graph} F, (x, y))\}.$$

It is called the *(limiting) coderivative* of F at (x, y). For properties of all above notions the reader is referred to [32] and [26].

As we have mentioned, we shall use a new concept of *outer subdifferential* to be denoted $\partial^> f$. It is defined by

$$\begin{aligned}\partial^> f(x) :&= \left\{ \lim_{k\to\infty} x_k^* :\ \exists\, x_k \underset{f}{\to} x,\ \ f(x_k) > f(x),\ \ x_k^* \in \partial f(x_k)\right\}\\ &= \left\{ \lim_{k\to\infty} x_k^* :\ \exists\, x_k \underset{f}{\to} x,\ \ f(x_k) > f(x),\ \ x_k^* \in \partial_F f(x_k)\right\}.\end{aligned}$$

2 A Calmness Criterion

All results in this section are derived on the basis of the following statement.

Theorem 2.1 *Let* $f[\mathbb{R}^n \to \overline{\mathbb{R}}]$ *be lower semicontinuous in a neighborhood of* $\bar{x} \in \mathbb{R}^n$ *and* $f(\bar{x}) = 0$. *Fix a* $\bar{\gamma} > 0$ *and consider the following properties:*

(a) *For any* $\gamma < \bar{\gamma}$ *there is a* $\delta > 0$ *such that* $d(x, [f \leqslant 0]) \leqslant \gamma^{-1} f^+(x)$ *if* $\|x - \bar{x}\| < \delta$*;*
(b) *For any* $\gamma < \bar{\gamma}$ *there is a* $\delta > 0$ *such that* $\liminf \|h_k\|^{-1} f(x_k + h_k) \geqslant \gamma$ *whenever* $f(x_k) \leqslant 0$, $\|x_k - \bar{x}\| \leqslant \delta$ *and* $h_k \to 0$, $h_k \in N_F([f \leqslant 0], x_k) \setminus \{0\}$*;*
(c) *For any* $\gamma < \bar{\gamma}$ *there is a* $\delta > 0$ *such that* $\|x^*\| \geqslant \gamma$ *if* $x^* \in \partial f(x)$ *for some* x *satisfying* $\|x - \bar{x}\| < 2\delta$ *and* $0 < f(x) < \delta\gamma$*;*
(d) $\|x^*\| \geqslant \bar{\gamma}$ *if* $x^* \in \partial^> f(\bar{x})$*;*
(e) *For any* $\gamma < \bar{\gamma}$ *there is a* $\delta > 0$ *such that* $|\nabla f|(x) \geqslant \gamma$ *if* $\|x - \bar{x}\| < 2\delta$ *and* $0 < f(x) < \delta\gamma$.

Then (e)$\Rightarrow$(d)$\Leftrightarrow$(c)$\Rightarrow$(a)$\Leftarrow$(b).

Proof The equivalence of (c) and (d) is immediate from the definition. The implication (e)$\Rightarrow$(d) is well known [13, Proposition 3.2]; it follows from the fact that

$|\nabla f|(x) \leqslant \|x^*\|$ if x^* is a Fréchet subgradient of f at x. Hence we only have to prove that (c) implies (a) and (b) implies (a).

Let (c) hold and take an x satisfying $\|x - \bar{x}\| < \delta$. If $f(x) \geqslant \gamma\delta$, then

$$d(x, [f \leqslant 0]) \leqslant \|x - \bar{x}\| \leqslant \delta \leqslant \gamma^{-1} f(x).$$

So we assume that $0 < f(x) < \gamma\delta$ and thus $\gamma^{-1} f(x) < \delta$. Put $g := f^+$. Applying the variational principle of Ekeland to g with $\varepsilon = f(x)$ and an arbitrary $\lambda \in (\gamma^{-1}\varepsilon, \delta)$, we find a w such that $f(w) \leqslant f(x)$, $\|w - x\| \leqslant \lambda$ and the function $g(\cdot) + (\varepsilon/\lambda)\| \cdot - w\|$ attains minimum at w. If $f(w) > 0$, then g and f coincide in a neighborhood of w and we have to conclude that there is an $x^* \in \partial f(w)$ such that $\|x^*\| \leqslant \varepsilon/\lambda < \gamma$ in contradiction with the assumptions as $\|w - \bar{x}\| \leqslant \|w - x\| + \|x - \bar{x}\| < 2\delta$ and $f(w) < \gamma\delta$. Thus $f(w) \leqslant 0$ and therefore $d(x, [f \leqslant 0]) \leqslant \|w - x\| \leqslant \lambda$. Since this is true for each $\lambda \in (\gamma^{-1}\varepsilon, \delta)$ (with a suitable w), one also has the (sharper) inequality $d(x, [f \leqslant 0]) \leqslant \gamma^{-1} f(x)$. This proves the implication (c)⇒(a).

Assume finally that (b) holds. If (a) were not valid, then for some $\gamma < \bar{\gamma}$ we could find a sequence (u_k) converging to $\bar{x}$ such that $\gamma d(u_k, [f \leqslant 0]) > f^+(u_k)$. It follows that $u_k \notin [f \leqslant 0]$, and we arrive at a contradiction with (b) by taking x_k to be a closest point to u_k in $[f \leqslant 0]$ and $h_k = u_k - x_k$. □

Note that a slightly different form of the implication (b)⇒(a) was introduced in [7]. The implication (c)⇒(a) cannot be reversed. Indeed, consider e.g. $f[\mathbb{R} \to \mathbb{R}]$ given by

$$f(x) = \begin{cases} 1 \ \text{ for } \ x \leqslant -1 \\ i^{-1} \ \text{ for } \ x \in \left[-\frac{1}{i}, -\frac{1}{i+1}\right), \ i = 1, 2, \ldots \\ 0 \ \text{ otherwise} \end{cases}$$

with $\bar{x} = 0$. The same is true for (b)⇒(a): consider the function $f(x) = d(x, Q)$ (on $\mathbb{R}^2$), where $Q = Q_1 \cup Q_2$ with $Q_1 = \{x = (\xi, \eta) : \ \eta \geqslant \xi^2\}$ and $Q_2 = -Q_1$. Let $\bar{x} = (0, 0)$. Then (a) is fulfilled, but (b) is violated. Indeed, it suffices to consider, e.g., the sequences $x_k = \left(\frac{1}{k}, \frac{1}{k^2}\right) \in Q_1$ and $h_k \in N_F(Q_1, x_k)$ such that $x_k + h_k \in Q_2$ for all $k \in \mathbb{N}$.

On the other hand (d)⇒(e) if f is continuous (as is immediate from Proposition 3.1 of [13]).

We also note that the two examples in the previous paragraph show that the conditions (c), (d) and (e), on the one hand, and (b), on the other hand, are independent: the first three are satisfied in the second example but not in the first, whereas (b) holds in the first example but not in the second one.

As in the general theory of metric regularity (e.g. [2, 13]), we can use Theorem 2.1 as a basis for sufficient condition for calmness/subregularity properties of a set-valued mapping $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$. All we need is to apply Theorem 2.1 to the function $f(x) = d(\bar{y}, F(x))$ and notice that

(a) $[f \leqslant 0] = F^{-1}(\bar{y})$;
(b) If $\bar{y} \notin F(x)$, then $\partial f(x)$ belongs to the union of $D^*F(x|y)(y^*)$ over all pairs (y, y^*) such that $y \in F(x)$, $\|y - \bar{y}\| = d(\bar{y}, F(x))$ (that is to say, $y \in \operatorname{Proj}_{F(x)}(\bar{y})$) and $y^* = \|y - \bar{y}\|^{-1}(y - \bar{y})$ [26, Theorem 1.105].

To state the corresponding result, it is convenient to introduce the mentioned "outer coderivative" of a set-valued mapping F, in the spirit of the outer subdifferential $\partial^{>}$ defined in the Introduction.

Definition Let F map $\mathbb{R}^n$ into (subsets of) $\mathbb{R}^m$ and let $\bar{y} \in F(\bar{x})$. The *outer coderivative* $D^*_> F(\bar{x}|\bar{y})$ of F at $(\bar{x}, \bar{y})$ is the multifunction from $\mathbb{R}^m$ into (subsets of) $\mathbb{R}^n$, whose graph consists of all pairs (y^*, x^*) such that there is a sequence of quadruples (x_n, y_n, x_n^*, y_n^*) converging to $(\bar{x}, \bar{y}, x^*, y^*)$ and such that

$$\bar{y} \notin F(x_n), \quad y_n \in \mathrm{Proj}_{F(x_n)}(\bar{y}), \quad y_n^* = \lambda_n(y_n - \bar{y}), \lambda_n > 0, \quad x_n^* \in D^* F(x_n|y_n)(y_n^*).$$

Thus the paragraph preceding the definition justifies the following two statements.

Proposition 2.2 *Let $F[R^n \rightrightarrows \mathbb{R}^m]$ be a multifunction with closed graph containing $(\bar{x}, \bar{y})$. Assume that there is a $\gamma > 0$ such that $\|x^*\| \geqslant \gamma \|y^*\|$ whenever $x^* \in D^*_> F(\bar{x}|\bar{y})(y^*)$. Then F is subregular at $(\bar{x}, \bar{y})$ with modulus not exceeding γ^{-1}.*

Proposition 2.3 *Let $M[R^m \rightrightarrows \mathbb{R}^n]$ be a multifunction with closed graph containing $(\bar{y}, \bar{x})$. Assume that there is a $\gamma > 0$ such that $\|x^*\| \geqslant \gamma \|y^*\|$ whenever $y^* \in D^*_> M(\bar{y}|\bar{x})(x^*)$. Then M is calm at $(\bar{y}, \bar{x})$ with modulus not exceeding γ^{-1}.*

Observe that

$$y^* \in D^*_> M(\bar{y}|\bar{x})(x^*) \Leftrightarrow -x^* \in D^*_> F(\bar{x}|\bar{y})(-y^*) \quad \text{whenever} \quad M = F^{-1}.$$

For example, consider the multifunction $F[\mathbb{R} \rightrightarrows \mathbb{R}]$ defined by $F(x) = [f(x), +\infty)$, where $f(x) = x^+$. Put $(\bar{x}, \bar{y}) = (0, 0)$. By Propositions 2.2, 2.3 we have to consider only sequences $x_n \to 0$ with $x_n > 0$. For such sequences $y_n = x_n$ and $x_n^* \in D^* F(x_n|y_n)(y_n^*)$ iff $x_n^* = y_n^* \geqslant 0$. Thus, we have

$$D^*_> F(0|0)(y^*) = \begin{cases} y^*, & \text{provided } y^* \geqslant 0 \\ \emptyset, & \text{otherwise,} \end{cases}$$

$$D^*_> M(0|0)(x^*) = \begin{cases} -x^*, & \text{provided } x^* \leqslant 0 \\ \emptyset, & \text{otherwise.} \end{cases}$$

It is to be emphasized that if in assumptions of the propositions we replace the outer coderivative by the usual limiting coderivative D^*, we get a necessary and sufficient condition for metric regularity of F and Aubin property (Lipschitz-like property) of M, respectively. The latter could be expected to be a strictly stronger assumption and the example following Proposition 2.4 confirms this.

A useful specification of Propositions 2.2, 2.3 arises when we consider the "perturbational" multifunction $M[\mathbb{R}^m \rightrightarrows \mathbb{R}^n]$ given by

$$M(y) := \{x \in \mathbb{R}^n : \varphi(x) + y \in Q\}, \tag{2.1}$$

where $\varphi[\mathbb{R}^n \to \mathbb{R}^m]$ is continuously differentiable and $Q \subset \mathbb{R}^m$ is closed. Set $\bar{y} = 0$ and choose an $\bar{x} \in M(0)$. This multifunction can be viewed as a perturbation of the constraint set $\{x \in \mathbb{R}^n : \varphi(x) \in Q\}$ which typically appears in mathematical programming. The inverse of M is the "feasibility mapping"

$$F(x) = M^{-1}(x) = Q - \varphi(x). \tag{2.2}$$

For any $(x, y) \in \text{Graph } F$ we have

$$D^*F(x|y)(y^*) = \begin{cases} (J\varphi(x))^T y^*, & \text{if } y^* \in N(Q, (\varphi(x) + y)), \\ \emptyset, & \text{otherwise,} \end{cases} \tag{2.3}$$

where $J\varphi(x)$ stands for the Jacobian matrix of φ at x. By Proposition 2.2 we must consider only those y^* which also satisfy $y^* = \lambda y$, where $\lambda > 0$ and $y \in \text{Proj}_{F(x)}(0)$. The latter means that $y = z - \varphi(x)$, where $z \in \text{Proj}_Q(\varphi(x))$. Taking into account the definition of the outer coderivative, we arrive at

Proposition 2.4 *Let M be given by (2.1), and let $\bar{x} \in M(0)$. Assume that there is a positive γ such that $\|x^*\| \geqslant \gamma\|y^*\|$ whenever there is a sequence (x_n, y_n, y_n^*) converging to $(\bar{x}, 0, y^*)$ and such that $\varphi(x_n) \notin Q$,*

$$y_n \in \text{Proj}_Q(\varphi(x_n)) - \varphi(x_n), \;\; y_n^* = \lambda_n y_n, \;\; \lambda_n > 0, \;\; \text{and} \;\; x^* = \lim_{n\to\infty} (J\varphi(x_n))^T y_n^*.$$

Then M is calm at $(0, \bar{x})$ with modulus not greater than γ^{-1}.

The subdifferential characterization of the metric regularity of F/Aubin property of M is provided by the (generalized) Mangasarian–Fromowitz constraint qualification (MFCQ)

$$\left.\begin{array}{ll} 0 & = (J\varphi(\bar{x}))^T w^* \\ w^* & \in N(Q, \varphi(\bar{x})) \end{array}\right\} \Rightarrow w^* = 0, \tag{2.4}$$

cf. [32, Example 9.44]. Let us compare condition (2.4) with the qualification condition of Propositions 2.2–2.4. By virtue of (2.3), condition (2.4) says that $\text{Ker} D^*F(\bar{x}|\bar{y}) = \{0\}$ which is equivalent to the existence of a $\gamma > 0$ such that

$$\|x^*\| \geqslant \gamma\|y^*\| \;\text{ if }\; x^* \in D^*F(\bar{x}|\bar{y})(y^*). \tag{2.5}$$

We conclude:

Proposition 2.5 [24, Theorem 3.2]. *Inequality (2.5) is equivalent to metric regularity of F near $(\bar{x}, \bar{y})$.*

Thus the condition of Proposition 2.4 is all the more satisfied if MFCQ holds. The converse implication does not hold already in a very simple case of convex constraint systems (see also [7], Example 4 of which the example below is a substantial simplification).

Example Let $\varphi[\mathbb{R} \to \mathbb{R}^2]$ be defined by $\varphi(x) = (x, x)$, let $Q = \{(\xi, \eta) \in \mathbb{R}^2 : \xi \geqslant 0, \; \eta \leqslant 0\}$ (the fourth quadrant of the plane), and let $\bar{x} = 0$. Then $\text{Ker } (J\varphi(0))^T = \{(\xi, \eta) : \; \xi + \eta = 0\}$ and the normal cone to Q at zero is the second quadrant, so that, e.g., $(-1, 1)$ belongs to both sets and the MFCQ fails to hold. On the other hand, $\varphi(x) \notin Q$ for $x \neq 0$ and the projection of $\varphi(x)$ onto Q is either $(x, 0)$ if $x > 0$ or $(0, x)$ if $x < 0$. In either case $\|x^*\| = \|y^*\|$ for both normals y^* in question and so the condition of Proposition 2.4 holds true with $\gamma = 1$.

Under a strengthening of the assumptions imposed in Theorem 2.1 on f the properties (a), (c) and (d) turn out to be equivalent. This follows from the next statement.

Proposition 2.6 *Suppose the existence of a neighborhood $\mathcal{U}$ of $\bar{x}$ such that*

(1) $\mathrm{bd}[f \leqslant 0] \cap \mathcal{U} \subset f^{-1}(0)$;
(2) *There exists a positive real ϑ such that for all $x \in \mathcal{U}$ with $0 < f(x) < \vartheta$ and for all $x^* \in \partial f(x)$ one has*

$$f(x') - f(x) \geqslant \langle x^*, x' - x\rangle - o(\|x' - x\|) \ \ \textit{for all} \ \ x',$$

where the function o does not depend on x.

Then for the properties (a), (d), stated in Theorem 2.1*, one has (a) $\Rightarrow$ (d).*

Proof Assume that there exists a sequence $\left(x_k, x_k^*\right)$ such that $x_k \underset{f}{\to} \bar{x}$, $f(x_k) > 0$, $x_k^* \in \partial f(x_k)$ and $\|x_k^*\| \leqslant \bar{\gamma} - \varepsilon$ for some $\varepsilon > 0$. Let $\bar{x}_k \in \mathrm{Proj}_{[f \leqslant 0]}(x_k)$ so that $\bar{x}_k \to \bar{x}$ and, by assumption (1), $f(\bar{x}_k) = 0$ for all k sufficiently large. By virtue of assumption (2) for these indices

$$f(\bar{x}_k) - f(x_k) = -f(x_k) \geqslant \langle x_k^*, \bar{x}_k - x_k\rangle - o(\|\bar{x}_k - x_k\|).$$

Evidently, to ε we can find $\bar{k}$ such that

$$o(\|\bar{x}_k - x_k\|) < \varepsilon/2\|\bar{x}_k - x_k\| \ \ \text{for all} \ \ k \geqslant \bar{k}.$$

Thus, for $k \geqslant \bar{k}$ and sufficiently large for (1) to apply, one has

$$f(x_k) \leqslant \left(\|x_k^*\| + \varepsilon/2\right) \|\bar{x}_k - x_k\|.$$

Consequently,

$$d(x_k, [f \leqslant 0]) = \|x_k - \bar{x}_k\| \geqslant \frac{1}{\|x_k^*\| + \varepsilon/2} f(x_k) > (1/\bar{\gamma}) f(x_k),$$

which contradicts (a). □

Condition (2) is automatically fulfilled if f is convex. It holds true, however, e.g. for a much broader class of the so-called weakly convex functions, cf. [1, Theorem 4.1].

If f is convex, we have also the following result.

Proposition 2.7 *In the setting of Proposition* 2.6 *let f be convex. Then properties (a) and (b) of Theorem* 2.1 *are equivalent.*

Proof Assume the existence of $\gamma := \bar{\gamma} - \varepsilon$ with some $\varepsilon > 0$, and sequences $x_k \to \bar{x}$, $h_k \to 0$ with $x_k \in [f \leqslant 0]$, $h_k \in N_F([f \leqslant 0], x_k) \setminus \{0\}$ such that

$$\lim_{k\to\infty} \|h_k\|^{-1} f(x_k + h_k) \leqslant \gamma.$$

Thus, one can find an index $\bar{k}$ such that for all $k \geqslant \bar{k}$ one has $x_k \in \mathcal{U}$ [specified in the assumption (1) of Proposition 2.6] and

$$\|h_k\|^{-1} f(x_k + h_k) \leqslant \gamma + \varepsilon/2. \tag{2.6}$$

Since the normals h_k are nonzero, $x_k \in \mathrm{bd}[f \leqslant 0]$ and thus $f(x_k) = 0$ for all $k \geqslant \bar{k}$. Moreover, by convexity of $[f \leqslant 0]$, for these indices k one has $f(x_k + h_k) > 0$ and

$$\|h_k\| = d(x_k + h_k, [f \leqslant 0]).$$

From (2.6) it follows that

$$d(x_k + h_k, [f \leqslant 0]) \geqslant \frac{f(x_k + h_k)}{\gamma + \varepsilon/2} > \frac{f(x_k + h_k)}{\bar{\gamma}},$$

which contradicts (a) and we are done. □

On the basis of Propositions 2.2, 2.4 we now easily arrive at the following statement.

Theorem 2.8 *Let $F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ be a multifunction with closed and convex graph and let $(\bar{x}, \bar{y}) \in \mathrm{Graph} F$. Then F is subregular at $(\bar{x}, \bar{y})$ if and only if there is a $\gamma > 0$ such that*

$$\|x^*\| \geqslant \gamma \|y^*\| \quad \textit{whenever} \quad x^* \in D^*_{>} F(\bar{x}|\bar{y})(y^*). \tag{2.7}$$

Proof Under the posed assumptions the function $f(x) = d(\bar{y}, F(x))$ is convex and one has

$$\mathrm{bd}[f \leqslant 0] \subset [f \leqslant 0] = f^{-1}(0).$$

In this way $d(\bar{y}, F(x))$ fulfills all assumptions of Proposition 2.6 and so the respective properties (a) and (d) are equivalent. Since (d) amounts to condition (2.7), the result follows. □

Another condition, necessary and sufficient for subregularity was obtained in [35, Theorem 3.1]. It involves normals to the graph of F at all points $(x, \bar{y})$ with $x \in \mathrm{bd} F^{-1}(\bar{y})$ and close to $\bar{x}$. This, of course, does not mean that the condition in Theorem 2.8 is always easier to verify. We wish, however, to draw the reader's attention to the fact that by replacing the outer coderivative in Eq. 2.7 by the usual limiting coderivative we get the necessary and sufficient condition for metric regularity (valid for all, not just convex-graph) mappings.

On the basis of Proposition 2.6 we can also obtain the following global error bound result.

Proposition 2.9 *Let f be proper convex, lower semicontinuous and $\mathrm{bd}[f \leqslant 0] \subset f^{-1}(0)$. Then the following two conditions are equivalent.*

(1) *For any $\gamma < \bar{\gamma}$ one has $d(x, [f \leqslant 0]) \leqslant \gamma^{-1} f^+(x)$ for all $x \in \mathbb{R}^n$;*
(2) $\|x^*\| \geqslant \bar{\gamma}$ *whenever* $x^* \in \partial^> f(x)$, $x \in \mathrm{bd}[f \leqslant 0]$.

3 Metric and Calmness/Subregularity Qualification Conditions in Subdifferential Calculus

In this section we present a set of basic calculus rules together with the corresponding "weak" qualification conditions. It is divided into three subsections, devoted to sets, real-valued functions and multifunctions. Rules from these subsections are denoted by $(\mathbf{S}_1)$, $(\mathbf{S}_2)$,..., $(\mathbf{F}_1)$, $(\mathbf{F}_2)$,..., and $(\mathbf{M}_1)$, $(\mathbf{M}_2)$,..., respectively.

For each case we subsequently

(a) State the rule and two qualification conditions: a metric qualification condition and two equivalent forms of the calmness/subregularity condition;
(b) Discuss the relationship between the qualification conditions, showing in each case that the metric qualification condition follows from the calmness/subregularity condition and, wherever possible, giving either a proof of their equivalence or a contradictory example;
(c) Prove that the metric qualification condition (hence the calmness/subregularity qualification condition) imply the calculus rule;
(d) Show that metric regularity of the mapping that appears in the definition of the calmness/subregularity condition is equivalent to the standard qualification condition (that can be found e.g. in [23, 26, 32]).

The key element in proofs of the calculus rules is the following well-known property of the limiting subdifferential of the distance function.

Proposition 3.1 *Let $C \subset \mathbb{R}^n$, and let f be an extended-real-valued and lower semicontinuous function which possess the following two properties:*

(1) $f(x) = 0, \ \forall x \in C;$
(2) $f(x) \geqslant d(x, C), \ \forall x.$

Then $\partial d(x, C) \subset \partial f(x), \ \forall x \in C.$

Proof By [32, Example 8.53], (see [12, Lemmas 3 and 5], for a more general original result)

$$\partial d(x, C) = \limsup_{u \underset{C}{\to} x} \partial_F d(u, C).$$

As follows from properties (1) and (2), $\partial_F d(u, C) \subset \partial_F f(u)$ for all $u \in C$. Therefore

$$\partial d(x, C) = \limsup_{u \underset{C}{\to} x} \partial_F d(u, C) \subset \limsup_{u \to x} \partial_F f(u) = \partial f(x)$$

as claimed. □

Other facts that will be used in subsequent discussion consist of "elementary" calculus rules. Herewith we mean e.g. the inclusion

$$\partial(f_1 + ... + f_k)(x) \subset \partial f_1(x) + ... + \partial f_k(x)$$

if all functions are Lipschitz near x, and the inclusion

$$\partial(\varphi \circ F)(x) \subset \bigcup_{y^* \in \partial\varphi(F(x))} D^* F(x)(y^*)$$

if F is a continuous mapping and φ is a Lipschitz function [26, 32].[2] A nice methodical consequence of this is that the most general calculus rules are actually consequences of their elementary (Lipschitz) counterparts.

3.1 Normal Cones to Sets

($\mathbf{S}_1$) Intersection: $C = \bigcap_{i=1}^{k} C_i, \quad C_i \subset \mathbb{R}^n$;

Assumptions: C_i are closed sets, $\bar{x} \in C$;

Calculus rule:

$$N(C, \bar{x}) \subset N(C_1, \bar{x}) + ... + N(C_k, \bar{x}); \tag{3.1}$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0$ and $\delta > 0$ such that
$$d(x, C) \leqslant \gamma\big[d(x, C_1) + ... + d(x, C_k)\big], \tag{3.2}$$
provided $\|x - \bar{x}\| < \delta$;
- *Calmness qualification condition*: the set-valued mapping
$$M[\mathbb{R}^{kn} \rightrightarrows \mathbb{R}^n] : \quad M(x_1, ..., x_k) = \{x\colon\ x + x_i \in C_i,\ i = 1..., k\} \tag{3.3}$$
is calm at $((0, ..., 0), \bar{x})$;
- *Subregularity qualification condition*: the set-valued mapping
$$F[\mathbb{R}^n \rightrightarrows \mathbb{R}^{nk}] : \quad F(x) = (C_1 - x) \times ... \times (C_k - x) \tag{3.4}$$
is subregular at $(\bar{x}, (0, ..., 0))$.

The mappings M and F are mutually inverse, so the last two conditions are equivalent forms of the same condition. To see that the metric qualification condition is also equivalent to them, we notice that $x \in M(0, ..., 0)$ means that $0 \in C_i - x$ for all i, i.e., $x \in C$. On the other hand (if we take the sum norm in the product of k copies of $\mathbb{R}^n$)

$$d((0, ..., 0), F(x)) = d((x, \dots, x), C_1 \times \dots \times C_k) = \sum_{i=1}^{k} d(x, C_i),$$

so (3.2) is precisely the subregularity condition for F at $(\bar{x}, (0, ..., 0))$.

Proposition 3.2 *Inequality* (3.2) $\Rightarrow$ *inclusion* (3.1).

Proof Set $f(x) = \gamma[d(x, C_1) + ... + d(x, C_k)]$. By Proposition 3.1 and the Lipschitz sum rule

$$\partial d(\bar{x}, C) \subset \partial f(\bar{x}) \subset \gamma[\partial d(\bar{x}, C_1) + ... + \partial d(\bar{x}, C_k)].$$

[2] In the cited monographs the results are obtained as consequences of more general calculus rules based on the standard qualification conditions. However the elementary Lipschitz versions of the rules were established earlier in [10, 19], and admit simplified independent proofs, see [26].

It remains to remember that the normal cone is precisely the cone generated by the subdifferential of the distance function. □

The standard subdifferential qualification condition for inclusion (3.1) is

$$\left.\begin{array}{l} x_i^* \in N(C_i, \bar{x}),\ i = 1, ..., k \\ x_1^* + ... + x_k^* = 0 \end{array}\right\} \Rightarrow x_1^* = ... = x_k^* = 0, \tag{3.5}$$

(see [26, 32]).

Proposition 3.3 *Condition* (3.5) *is equivalent to metric regularity of F (given by* (3.4)*) near* $(\bar{x}, 0, ..., 0)$.

Proof We can write $F(x)$ as $Q - \varphi(x)$, where $Q = C_1 \times ... \times C_k$ and $\varphi(x) = (x, ..., x)$ is the canonical mapping of $\mathbb{R}^n$ into the diagonal of $(\mathbb{R}^n)^k$. Equation 3.5 amounts then exactly to the respective MFCQ (2.4). □

Remark Observe that our qualification conditions are satisfied in the example mentioned below Proposition 2.5 whereas condition (3.5) is not.

($\mathbf{S}_2$) **Constraint set**. Let $\varphi[\mathbb{R}^n \to \mathbb{R}^m]$, $P \subset \mathbb{R}^n$, $Q \subset \mathbb{R}^m$ and

$$C = \{x \in P : \ \varphi(x) \in Q\} = P \cap \varphi^{-1}(Q);$$

Assumptions: φ satisfies the Lipschitz condition near $\bar{x}$; P and Q are closed sets.
Calculus rule:

$$N(C, \bar{x}) \subset \bigcup_{y^* \in N(Q, \varphi(\bar{x}))} D^*\varphi(\bar{x})(y^*) + N(P, \bar{x}); \tag{3.6}$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0$ and $\delta > 0$ such that

$$d(x, C) \leqslant \gamma\big(d(x, P) + d(\varphi(x), Q)\big), \tag{3.7}$$

  provided $\|x - \bar{x}\| < \delta$;
- *Calmness qualification condition*: the set-valued mapping

$$M[\mathbb{R}^m \rightrightarrows \mathbb{R}^n] : \ M(y) = P \cap \varphi^{-1}(Q - y)$$

  is calm at $(0, \bar{x})$;
- *Subregularity qualification condition*: the set-valued mapping

$$F[\mathbb{R}^n \rightrightarrows \mathbb{R}^m] : \ F(x) = \begin{cases} Q - \varphi(x), & \text{if } x \in P \\ \emptyset & \text{otherwise} \end{cases}$$

  is subregular at $(\bar{x}, 0)$.

The calmness and the subregularity qualification conditions are again the same as they involve mutually inverse mappings. Furthermore we notice that $M(0) = C$ and

$$d(0, F(x)) = \begin{cases} d(\varphi(x), Q), & \text{if } x \in P \\ \infty & \text{otherwise.} \end{cases}$$

Hence, calmness of M at $(0, \bar{x})$ means that there is a $\gamma > 0$

$$d(x, C) \leqslant \gamma d(\varphi(x), Q) \tag{3.8}$$

for all $x \in P$ sufficiently close to $\bar{x}$. Clearly (3.7) implies (3.8). On the other hand, if Eq. 3.8 holds and $x \notin P$, then, taking a closest element of P to x, say x', we have

$$d(x, C) \leqslant d(x, P) + d(x', C) \leqslant d(x, P) + \gamma d(\varphi(x'), Q)$$

$$\leqslant (1 + \gamma(1 + L))(d(x, P) + d(\varphi(x), Q)),$$

where L is the Lipschitz constant of φ near $\bar{x}$. Thus the metric qualification condition in this case is also equivalent to the calmness/subregularity condition.

Next we show that inequality (3.7) implies the desired calculus rule.

Proposition 3.4 *Inequality* (3.7) $\Rightarrow$ *inclusion* (3.6).

Proof Set $f(x) = \gamma(d(x, P) + d(\varphi(x), Q))$. Then by Proposition 3.1 one has $\partial d(\bar{x}, C) \subset \partial f(x)$. The function $x \mapsto d(\varphi(x), Q)$ is a composition of a Lipschitz function $d(\cdot, Q)$ and a Lipschitz mapping φ. Therefore the limiting subdifferential of the function at $\bar{x}$ lies in the union of $D^*\varphi(\bar{x})(y^*)$ over all $y^* \in \partial d(\cdot, Q)(\varphi(\bar{x}))$. The rest of the argument is identical to that in the proof of Proposition 3.2. □

The standard subdifferential qualification condition for inclusion (3.6) is (see [26, 32]):

$$\left.\begin{array}{l} D^*\varphi(\bar{x})(y^*) \cap -N(P, \bar{x}) \neq \emptyset \\ y^* \in N(Q, \varphi(\bar{x})) \end{array}\right\} \Rightarrow\ y^* = 0. \tag{3.9}$$

Proposition 3.5 *Condition* (3.9) *implies metric regularity of F near* $(\bar{x}, 0)$.

The statement follows directly from [23, Theorem 6.10] combined with the standard coderivative criterion of metric regularity/Aubin property, cf. [32, Theorem 9.40], [26, Theorem 4.10].

Remark Results concerning intersections of sets considered in the beginning of the section are consequences of the just established facts. Take for simplicity the case of two sets $P, Q \subset \mathbb{R}^n$; if φ is the identity map, then $\varphi^{-1}(Q) = Q$, $D^*\varphi(x)(y^*) = y^*$ etc.

3.2 Subdifferentials of Functions.

Here we for the first time encounter a situation when a "natural" metric qualification condition is strictly better than an "equally natural" calmness/subregularity condition.

($\mathbf{F}_1$) Sums of functions: $f(x) = f_1(x) + ... + f_k(x),\ x \in \mathbb{R}^n$;

Assumptions: all functions are extended-real-valued, lower semicontinuous and finite at $\bar{x}$;

Calculus rule:

$$\partial f(\bar{x}) \subset \partial f_1(\bar{x}) + ... + \partial f_k(\bar{x}),\ \partial^\infty f(\bar{x}) \subset \partial f_1^\infty(\bar{x}) + ... + \partial f_k^\infty(\bar{x}); \tag{3.10}$$

Here $\partial^\infty f(x) = \{x^*:\ (x^*, 0) \in N(\operatorname{epi} f, (x,\ f(x)))\}$ is the collection of vectors corresponding to "horizontal" normals to the epigraph of f.

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0,\ \delta > 0$ such that

$$d((x,\alpha), \operatorname{epi} f) \leqslant \gamma \sum_{i=1}^{k} d((x,\alpha_i), \operatorname{epi} f_i), \tag{3.11}$$

  provided $\|x - \bar{x}\| < \delta$, $|\alpha_i - f_i(\bar{x})| < \delta$ and $\alpha_1 + ... + \alpha_k = \alpha$.
  To state the other qualification conditions, we set for $i = 1, \dots, k$

$$C_i := \{(x, \alpha_1, ...\alpha_k) \in \mathbb{R}^n \times \mathbb{R}^k :\ \alpha_i \geqslant f_i(x)\}$$

  (in other words, up to permutation of components, $C_i = \operatorname{epi} f_i \times \mathbb{R}^{k-1}$).
- *Calmness qualification condition*: the set-valued mapping

$$M[(\mathbb{R}^n \times \mathbb{R}^k)^k \rightrightarrows \mathbb{R}^n \times \mathbb{R}^k] :$$
$$M\big((x_1, \alpha_{11}, ..., \alpha_{1k}), ..., (x_k, \alpha_{k1}, ..., \alpha_{kk})\big) \ =\ \bigcap_{i=1}^{k}(C_i - (x_i, \alpha_{i1}, ..., \alpha_{ik}))$$

  is calm at $\big(\big((0, 0, ..., 0), ..., (0, 0, ..., 0)\big), (\bar{x},\ f_1(\bar{x}), ...,\ f_k(\bar{x}))\big)$;
- *Subregularity qualification condition*: the set-valued mapping

$$F[\mathbb{R}^n \times \mathbb{R}^k \rightrightarrows (\mathbb{R}^n \times \mathbb{R}^k)^k] :$$
$$F(x, \alpha_1, ..., \alpha_k) \ =\ (C_1 - (x, \alpha_1, ..., \alpha_k)) \times ... \times (C_k - (x, \alpha_1, ..., \alpha_k))$$

  is subregular at $\big((\bar{x},\ f_1(\bar{x}), ...,\ f_k(\bar{x})), \big((0, 0, ..., 0), ..., (0, 0, ..., 0)\big)\big)$.

The set-valued mappings F and M appear naturally in the context of summation of functions. To see that, we notice that

$$C := \bigcap_{i=1}^{k} C_i = \{(x, \alpha_1, ..., \alpha_k) : \alpha_i \geqslant f_i(x),\ i = 1, ..., k\} = M\big((0, 0, ..., 0), ..., (0, 0, ..., 0)\big),$$

so that $\operatorname{epi} f$ is the image of C under the linear mapping

$$T:\ (x, \alpha_1, ..., \alpha_k) \to (x, \alpha_1 + ... + \alpha_k)$$

from $\mathbb{R}^n \times \mathbb{R}^k$ onto $\mathbb{R}^n \times \mathbb{R}$.

The latter, along with the fact that the only point in C which belongs to the preimage of $(x, f(x))$ under T is $(x, f_1(x), ..., f_k(x))$ implies that (x^*, β) belongs to the normal cone to epi f at $(x, f(x))$ precisely when $(x^*, \beta, ..., \beta)$ belongs to the normal cone to C at $(x, f_1(x), ..., f_k(x))$.

On the other hand,

$$N(C_i, (x, \alpha_1, ..., \alpha_k)) = \left\{(x^*, \beta_1, ..., \beta_k) : (x^*, \beta_i) \in N(\operatorname{epi} f_i, (x, \alpha_i)),\ \beta_j = 0,\ j \neq i\right\}.$$

Therefore the desired calculus rule will follow from the inclusion $N(C, \cdot) \subset \sum N(C_i, \cdot)$, where the dot stands for $(\bar{x}, f_1(\bar{x}), ..., f_k(\bar{x}))$. Next we observe that M and F are the set-valued mappings that appear if $(\mathbf{S}_1)$ applies to these specific sets C_i.

Calmness of M at $((0, ..., 0), ..., (0, ..., 0))$ (if we take the sum norm in the range space of F) means that

$$d((x, \alpha_1, ..., \alpha_k), C) \leqslant L \sum d((x, \alpha_1, ..., \alpha_k), C_i) = L \sum d((x, \alpha_i), \operatorname{epi} f_i) \tag{3.12}$$

for all $(x, \alpha_1, ..., \alpha_k)$ close to $(\bar{x}, f_1(\bar{x}), ..., f_k(\bar{x}))$.

It is an easy matter to see that

$$d((x, \alpha), \operatorname{epi} f)) \leqslant d((x, \alpha_1, ..., \alpha_k), C)$$

for any $(x, \alpha, \alpha_1, ..., \alpha_k)$ with $\sum_{i=1}^{k} \alpha_i = \alpha$, that is (3.12) implies (3.11). The converse implication does not hold, which suggests the possibility that for the operation of summation of functions the metric qualification condition is strictly better than the calmness/subregularity qualification condition. The following example shows that this is indeed the case.

Example Consider the following two functions on $\mathbb{R}$:

$$f_1(x) = \begin{cases} -\sqrt{1 - (1 - |x|)^2}, & \text{if } -2 \leqslant x \leqslant 0, \\ 0, & \text{if } x \geqslant 0 \text{ or } x \leqslant -2 \end{cases}$$

and

$$f_2(x) = \begin{cases} -\sqrt{1 - (1 - |x|)^2}, & \text{if } 2 \geqslant x \geqslant 0, \\ 0, & \text{if } x \leqslant 0 \text{ or } x \geqslant 2. \end{cases}$$

Then $f(x) = f_1(x) + f_2(x) = \min\{f_1(x), f_2(x)\}$, so that $\operatorname{epi} f = \operatorname{epi} f_1 \cup \operatorname{epi} f_2$. We claim that the metric qualification condition (3.11) is satisfied in a neighborhood of $\bar{x} = 0$. Indeed, if $\alpha < f(x)$ (the only case of interest) and hence $\alpha < 0$, then

$$d((x, \alpha), \operatorname{epi} f) = \sqrt{((1 - |x|)^2 + \alpha^2} - 1 \cong \alpha^2/2 - \left(|x| - x^2/2\right)$$

(where $\cong$ means equality up to higher order terms).

In calculating the right-hand part of (3.11), we can assume due to symmetry that, say, $x \leqslant 0$, that is $f_1(x) = f(x)$, $f_2(x) = 0$. We have (again assuming that α_1 and α_2 are non-positive – otherwise all is trivial)

$$d((x, \alpha_1), \text{epi} f_1) = \sqrt{(1 - |x|)^2 + \alpha_1^2} - 1;$$

$$d((x, \alpha_2), \text{epi} f_2) = \min\left\{\sqrt{(1 + |x|)^2 + \alpha_2^2} - 1, |\alpha_2|\right\}.$$

If the minimum in the last equality is attained at the first quantity, then (recall that $\alpha_1 + \alpha_2 = \alpha$, that is one of α_i must not exceed $\alpha/2$)

$$d((x, \alpha_1), \text{epi} f_1) + d((x, \alpha_2), \text{epi} f_2) \cong x^2 + \frac{\alpha_1^2 + \alpha_2^2}{2} \geqslant x^2 + \alpha^2/4 \geqslant 1/2 d((x, \alpha), \text{epi} f).$$

Consider now the case when $d((x, \alpha_2), \text{epi} f_2) = -\alpha_2$. As we are interested in a small neighborhood of zero, we may assume that $\alpha > -1$. We have

$$\begin{aligned} -\alpha_2 + d((x, \alpha_1), \text{epi} f_1) &= -\alpha_2 + d((x, \alpha - \alpha_2), \text{epi} f_1) \\ &= -\alpha_2 + \sqrt{(1 - |x|)^2 + (\alpha - \alpha_2)^2} - 1 \\ &\cong -\alpha_2 + \frac{x^2 + (\alpha - \alpha_2)^2}{2} - |x| \geqslant \frac{x^2 + \alpha^2}{2} - |x| \end{aligned}$$

and we again arrive at the desired inequality.

On the other hand, the calmness/subregularity condition (3.12) does not hold. Indeed, take the point $(0, \alpha)$ with $\alpha < 0$ and $\alpha_1 = \alpha_2 = \alpha/2$. Then for any $(u, \beta_1, \beta_2) \in C$ either $\beta_1 \geqslant 0$ or $\beta_2 \geqslant 0$. It follows that

$$d\left((0, \alpha/2, \alpha/2), C\right) \geqslant -\alpha/2,$$

that is $d((0, \alpha/2), \text{epi} f_1) + d((0, \alpha/2), \text{epi} f_2) = o(d((0, \alpha/2, \alpha/2), C))$.

As the metric qualification condition holds, the calculus rule (3.10) is, of course, valid. One may directly verify that $\partial f(0) = \emptyset$ and

$$\partial^\infty f(0) = \mathbb{R} = \partial^\infty f_1(0) + \partial^\infty f_2(0).$$

The example can be easily modified to make $\partial f(0) \neq \emptyset$. To this end we take a small $\varepsilon > 0$ and replace both functions f_i by lower semicontinuous functions g_i such that $|f_i(x) - g_i(x)| \leqslant \varepsilon |f_i(x)|$ and the negative part of the epigraph of g_i is bounded by line segments which are either vertical or have the same slope ξ for g_1 and $-\xi$ for g_2. In this case the relationship between the distance quantities remain unchanged but $\partial g(0) = (-\infty, -\xi] \cup [\xi, \infty)$.

Proposition 3.6 ([14]). *Inequality* (3.11) $\Rightarrow$ *inclusions* (3.10).

Proof Set

$$\psi(x, \alpha, \alpha_1, ..., \alpha_{k-1}) := \sum_{i=1}^{k} \psi_i(x, \alpha, \alpha_1, ..., \alpha_{k-1})$$

with

$$\psi_i(x, \alpha, \alpha_1, ..., \alpha_{k-1}) := d((x, \alpha_i), \text{epi} f_i), \ \ i = 1, ..., k-1,$$

$$\psi_k(x, \alpha, \alpha_1, ..., \alpha_{k-1}) := d\left(\left(x, \alpha - \sum_{i=1}^{k-1} \alpha_i\right), \text{epi} f_k\right),$$

and put

$$\varphi(x, \alpha) := \inf_{\alpha_1, ..., \alpha_{k-1}} \psi(x, \alpha, \alpha_1, ..., \alpha_{k-1}).$$

It is clear that φ satisfies the Lipschitz condition and $\varphi(x, \alpha) = 0$ when $\alpha \geqslant f(x)$. Therefore by Proposition 3.1

$$\partial d((x,\ f(x)), \text{epi} f) \subset \partial \varphi(x,\ f(x))$$

for any x from the domain of f.

We next observe that (as we are interested in the behavior near $\bar{x}$ and all functions are lower semicontinuous and finite at $\bar{x}$) we can harmlessly assume that all functions are bounded from below. This guarantees that $\psi(x, \alpha, \alpha_1, ... \alpha_{k-1}) \to \infty$ when $\max_i |\alpha_i| \to \infty$. Indeed, this is obvious if one of α_i goes to $-\infty$. If, on the other hand, all α_i remain bounded from below while their maximum goes to infinity, then clearly $\alpha - \sum_{i=1}^{k-1} \alpha_i$ goes to minus infinity. It follows that the infimum in the definition of φ is always attained. Moreover, it is clear that the mapping $(x, \alpha) \mapsto \text{argmin} \psi(x, \alpha, \cdot)$ is lower semicompact.[3] As follows from Theorem 3 of [15], this allows to conclude that

$$\partial \varphi(x, \alpha) \times \{(0, ..., 0)\} \subset \bigcup_{(\alpha_1, ..., \alpha_{k-1}) \in \text{argmin} \psi(x, \alpha, \cdot)} \partial \psi(x, \alpha, \alpha_1, ..., \alpha_{k-1}).$$

All ψ_i are Lipschitz continuous and the last one is, in addition, a composition of a linear operator and a Lipschitz function. Applying the corresponding rules of subdifferential calculus, we can for any $(x^*, \beta) \in \partial \varphi(\bar{x},\ f(\bar{x}))$ find $\alpha_1, ..., \alpha_{k-1}$ and $(x_i^*, \beta_{i0}, ..., \beta_{i(k-1)}) \in \partial \psi_i(\bar{x},\ f(\bar{x}), \alpha_1, ..., \alpha_{k-1})$, $i = 1, ..., k$, such that

$$\begin{aligned} &d((\bar{x}, \alpha_i), \text{epi}\, f_i) = 0, \ \ i = 1, ..., k, \ \ \left(\alpha_k = f_k(\bar{x}) - \sum_{i=1}^{k-1} \alpha_i\right) \\ &x_1^* + ... + x_k^* = x^*; \\ &\beta_{10} + ... + \beta_{k0} = \beta; \\ &\beta_{1j} + ... + \beta_{kj} = 0, \ \ j = 1, ..., k-1. \end{aligned} \tag{3.13}$$

[3] A set-valued mapping $F(x)$ is *lower semicompact* if for any sequence (x_m, y_m) such that $y_m \in F(x_m)$ and (x_m) converges to a certain x, there is a subsequence (y_{m_s}) converging to some $y \in F(x)$—see [28]. In finite dimensions this property holds whenever F is uniformly bounded.

The equalities in the first line of (3.13) imply that $\alpha_i = f_i(\bar{x}),\ i = 1, ..., k$. Furthermore, taking into account the special structure of ψ_k and the fact that the functions $\psi_i, i = 1, ..., k-1$, depend only on x and α_i, we conclude that

$$\beta_{i0} = 0,\ \text{ for } i = 1, ..., k-1;$$

$$\beta_{ij} = 0,\ \text{ if } k \neq i,\ i \neq j,\ j \neq 0;$$

$$\beta_{k1} = ... = \beta_{kk-1} = -\beta_{k0}.$$

Setting $\beta_i = \beta_{ii},\ i = 1, ..., k-1,\ \beta_k = \beta_{k0}$, we see that $\beta_1 = ... = \beta_k = \beta$ and get finally the existence of $x_1^*, ..., x_k^*$ such that

$$(x_i^*, \beta) \in \partial d((\bar{x}, f_i(\bar{x})), \text{epi} f_i),\ i = 1, ..., k, ; \quad x_1^* + ... + x_k^* = x^*.$$

This basically concludes the proof. Indeed, if $x^* \in \partial f(\bar{x})$, then there is a positive λ such that $(x^*, -1) \in \lambda \partial d((\bar{x}, f(\bar{x})), \text{epi} f)$, hence there are x_i^* such that $(x_i^*, -1) \in \lambda \partial d((\bar{x}, f_i(\bar{x})), \text{epi} f_i)$, that is $x_i^* \in \partial f_i(\bar{x})$, and $x^* = \sum x_i^*$. Likewise, if $x^* \in \partial^\infty f(\bar{x})$, then $(x^*, 0) \in \partial d(\cdot, \text{epi} f)(\bar{x}, f(\bar{x}))$ and the same argument leads to the conclusion that x^* is the sum of certain $x_i^* \in \partial^\infty f_i(\bar{x})$. □

The standard subdifferential qualification condition for inclusions (3.10) is

$$\left.\begin{array}{l} x_i^* \in \partial^\infty f_i(\bar{x}),\ i = 1, ..., k \\ x_1^* + ... + x_k^* = 0 \end{array}\right\} \Rightarrow\ x_1^* = ... = x_k^* = 0. \tag{3.14}$$

Proposition 3.7 *Condition* (3.14) *is equivalent to metric regularity of F near*

$$((\bar{x},\ f_1(\bar{x}), ...,\ f_k(\bar{x})), ((0, 0, ..., 0), ..., (0, 0, ...0))).$$

Proof Set $w = (x,\ f_1(x), ...,\ f_k(x)),\ \bar{w} = (\bar{x},\ f_1(\bar{x}), ...,\ f_k(\bar{x}))$. Then

$$F(w) = (C_1 - w) \times ... \times (C_k - w).$$

As mentioned in connection with $(\mathbf{S}_1)$, F is metrically regular near $(\bar{w}, 0, ..., 0)$ if and only if

$$w_i^* \in N(C_i, \bar{w}),\ w_1^* + ... + w_k^* = 0\ \Rightarrow\ w_1^* = ... = w_k^* = 0. \tag{3.15}$$

By definition $w_i^* = (x_i^*, \beta_{i1}, ..., \beta_{ik})$, where $x_i^* \in \mathbb{R}^n$ and $\beta_{ij} \in \mathbb{R}$. As C_i contains the entire copy of the jth real line for $j \neq i$, for any $w_i^* \in N(C_i, \bar{w})$ we have $\beta_{ij} = 0$ if $i \neq j$ and $(x_i^*, \beta_{ii}) \in N(\text{epi} f_i, (\bar{x},\ f_i(\bar{x})))$. Thus any collection of $w_i^*,\ i = 1, ..., k$, must satisfy $\beta_{ij} = 0$ for all i, j, so that $x_i^* \in \partial^\infty f_i(\bar{x})$ and $\sum x_i^* = 0$. Moreover, the condition on the right-hand side of (3.15) reduces to the condition on the right-hand side of (3.14). □

(F_2) Maximum of functions: $f(x) = \sup\limits_{1\leqslant i\leqslant k} f_i(x)$;

Assumptions: all functions are continuous near $\bar{x}$;

Calculus rule:

$$\partial f(x) \subset \bigcup_{J\subset I_0} \left(\left\{ \sum_{i\in J} \alpha_i \partial f_i(\bar{x}) : \ \alpha_i \geqslant 0, \ \sum_{i\in J} \alpha_i = 1 \right\} + \sum_{i\in I\setminus J} \partial^{\infty} f_i(\bar{x}) \right),$$

$$\partial^{\infty} f(\bar{x}) \subset \sum_{i\in I} \partial^{\infty} f_i(\bar{x}),$$

where

$$I = \{i\colon \ f_i(\bar{x}) = f(\bar{x})\}, \quad I_0 = \{i \in I : \ \partial f_i(\bar{x}) \neq \emptyset\};$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0, \ \delta > 0$ such that

$$d((x,\alpha), \mathrm{epi} f) \leqslant \gamma \sum_{i=1}^{k} d((x,\alpha), \mathrm{epi} f_i)$$

provided $\|x - \bar{x}\| < \delta, \ |\alpha - f(\bar{x})| < \delta$;
- *Calmness qualification condition*: the set-valued mapping

$$M\left[\mathbb{R}^{k(n+1)} \rightrightarrows \mathbb{R}^{n+1}\right] : \ M((x_1,\alpha_1), ..., (x_k,\alpha_k)) = \{(x,\alpha) : (x,\alpha) + (x_i,\alpha_i) \in \mathrm{epi} f_i\}$$

is calm at $(((0,0), ..., (0,0)), (\bar{x}, \ f(\bar{x})))$;
- *Subregularity qualification condition*: the set-valued mapping

$$F\left[\mathbb{R}^{n+1} \rightrightarrows \mathbb{R}^{k(n+1)}\right] : \ F(x,\alpha) = (\mathrm{epi} f_1 - (x,\alpha)) \times ... \times (\mathrm{epi} f_k - (x,\alpha))$$

is subregular at $((\bar{x}, \ f(\bar{x})), ((0,0), ..., (0,0)))$.

The standard subdifferential qualification condition is similar to that for the sum:

$$x_i^* \in \partial f_i(\bar{x}), \ i \in I, \quad \sum_{i\in I} x_i^* = 0 \ \Rightarrow \ x_i^* = 0, \ \forall\, i \in I.$$

Here epi$f$ is the intersection of epif_i, so all reduces to (S_1): the qualification conditions are equivalent, imply the rule and the standard qualification condition is necessary and sufficient for metric regularity of F.

(F_3) Composition: $f = g \circ \varphi$;

Assumptions: $\varphi[R^n \to \mathbb{R}^m]$ is continuous near $\bar{x} \in \mathbb{R}^n$; g is an extended-real-valued lower semicontinuous function on $\mathbb{R}^m$ which is finite at $\bar{y} = \varphi(\bar{x})$.

Calculus rule:

$$\partial f(\bar{x}) \subset \bigcup_{y^*\in\partial g(\bar{y})} D^*\varphi(\bar{x})(y^*), \qquad \partial^{\infty} f(x) \subset \bigcup_{y^*\in\partial^{\infty} g(\bar{y})} D^*\varphi(\bar{x})(y^*); \tag{3.16}$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0,\ \delta > 0$ such that

$$d((x,\alpha),\mathrm{epi}f) \leqslant \gamma[d((x,y),\mathrm{Graph}\varphi) + d((y,\alpha),\mathrm{epi}g)], \tag{3.17}$$

  provided $\|x-\bar{x}\| < \delta,\ |\alpha - f(\bar{x})| < \delta,\ \|y-\bar{y}\| < \delta$;
- *Calmness qualification condition*: the set-valued mapping $M[\mathbb{R}^n \times (\mathbb{R}^m)^2 \times \mathbb{R} \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}]$ defined by

$$M(x,y_1,y_2,\alpha) = \{(u,z,\beta):\ (u,z) \in \mathrm{Graph}\varphi - (x,y_1),\ (z,\beta) \in \mathrm{epi}g - (y_2,\alpha)\}$$

  is calm at $((0,...,0),(\bar{x},\bar{y},f(\bar{x})))$;
- *Subregularity qualification condition*: the set-valued mapping $F[\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R} \rightrightarrows \mathbb{R}^n \times (\mathbb{R}^m)^2 \times \mathbb{R}]$ defined by

$$F(x,y,\alpha) = (\mathrm{Graph}\varphi - (x,y)) \times (\mathrm{epi}g - (y,\alpha))$$

  is subregular at $((\bar{x},\bar{y},f(\bar{x})),(0,...,0))$.

As in the two previous cases, we can justify the introduction of the set-valued mappings M, F by reference to $(\mathbf{S}_1)$. Indeed, consider the sets

$$C_1 = \{(x,y,\alpha):\ (x,y) \in \mathrm{Graph}\varphi\} \quad \text{and} \quad C_2 = \{(x,y,\alpha):\ (y,\alpha) \in \mathrm{epi}g\}.$$

Then the epigraph of f is the projection of $C = C_1 \cap C_2$ onto the (x,α)-space. Applications of the constructions of $(\mathbf{S}_1)$ to this intersection gives precisely the above mappings M and F.

We see further that $M(0,...,0) = C$, so calmness of M at $(0,...,0)$ reduces to

$$d((x,y,\alpha),C) \leqslant K[d((x,y,\alpha),C_1) + d((x,y,\alpha),C_2)].$$

The right-hand side of this inequality is identical to that in inequality (3.17). On the other hand, it is obvious that

$$d((x,\alpha),\mathrm{epi}f) \leqslant d((x,y,\alpha),C),$$

actually even

$$d((x,\alpha),\mathrm{epi}f) = \inf_y d((x,y,\alpha),C), \tag{3.18}$$

which again suggests a possibility that the metric qualification condition is weaker. For the moment, however, it is not clear whether this is the case or not.

Proposition 3.8 *Inequality* (3.17) $\Rightarrow$ *inclusions* (3.16).

Proof The proof is basically the same as the proof of Proposition 3.6. We may assume that $g(y) \to \infty$ as $\|y\| \to \infty$, which would guarantee that the set-valued mapping

$$(x,\alpha) \mapsto \mathrm{argmin}(d((x,\cdot),\mathrm{Graph}\varphi) + d((\cdot,\alpha),\mathrm{epi}g))$$

is lower semicompact. Application of Proposition 3.1 and Proposition 3.3 of [15] then gives

$$(x^*, \beta) \in \partial d((\bar{x}, f(\bar{x})), \text{epi}f) \Rightarrow (x^*, 0, \beta)$$
$$\in \gamma\big[\partial d((\bar{x}, \varphi(\bar{x})), \text{Graph}\varphi) + \partial d((\bar{y}, g(\bar{y})), \text{epi}g)\big].$$

It follows that there is a y^* such that

$$(x^*, -y^*) \in N(\text{Graph}\varphi, (\bar{x}, \varphi(\bar{x}))), \qquad (y^*, \beta) \in N(\text{epi}g, (\bar{y}, g(\bar{y}))),$$

which proves the claim. □

Remark

1. Applying this result to g being the indicator of a closed set $Q \subset \mathbb{R}^m$, we conclude that the inclusion

$$N(\varphi^{-1}(Q), \bar{x}) \subset \bigcup_{y^* \in N(Q, \varphi(\bar{x}))} D^*\varphi(\bar{x})(y^*)$$

holds also for a continuous mapping φ under the qualification condition: there are $\gamma > 0$ and $\delta > 0$ such that

$$d\left(x, \varphi^{-1}(Q)\right) \leqslant \gamma(d((x, y), \text{Graph}\varphi) + d(y, Q)),$$

whenever $\|x - \bar{x}\| < \delta$, $\|y - \bar{y}\| < \delta$ (cf. Proposition 3.4).
2. More can be said in the case when φ satisfies the Lipschitz condition. In this case $d((x, y), \text{Graph}\varphi) = \|y - \varphi(x)\|$ if we take a suitable norm in $\mathbb{R}^n \times \mathbb{R}^m$ (e.g. $\|(x, y) - (u, v)\| = L\|x - u\| + \|y - v\|$ with L being a Lipschitz constant of φ). Moreover,

$$\inf_y\{\|y - \varphi(x)\| + d((y, \alpha), \text{epi}g)\} \leqslant d((\varphi(x), \alpha), \text{epi}g),$$

which suggests the following (clearly weaker) qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0$, $\delta > 0$ such that

$$d((x, \alpha), \text{epi}f) \leqslant \gamma d((\varphi(x), \alpha), \text{epi}g), \tag{3.19}$$

provided $\|x - \bar{x}\| < \delta$, $|\alpha - f(\bar{x})| < \delta$;
- *Calmness qualification condition*: the set-valued mapping

$$M\big[\mathbb{R}^m \times \mathbb{R} \rightrightarrows \mathbb{R}^n \times \mathbb{R}\big]: \; M(y, \alpha) = \{(u, \beta) : (\varphi(u), \beta) \in \text{epi}g - (y, \alpha)\}$$

is calm at $((0, 0), (\bar{x}, f(\bar{x})))$;
- *Subregularity qualification condition*: the set-valued mapping

$$F[\mathbb{R}^n \times \mathbb{R} \rightrightarrows \mathbb{R}^m \times \mathbb{R}]: \; F(x, \alpha) = \text{epi}g - (\varphi(x), \alpha)$$

is subregular at $((\bar{x}, f(\bar{x})), (0, 0))$.

The fact that these conditions imply inclusions (3.16) follows easily from Proposition 3.4 if we observe that $\text{epi}f = \Psi^{-1}(\text{epi}g)$, where

$$\Psi(x, \alpha) = \begin{bmatrix} \varphi(x) \\ \alpha \end{bmatrix}. \tag{3.20}$$

Moreover, in this case we can be sure that the metric condition (3.19) is equivalent to the subsequent calmness/subregularity condition. Indeed, $M(0,0) = \{(x,\alpha) : (\varphi(x),\alpha) \in \mathrm{epi} g\}$, so that (3.19) is precisely the calmness property of M at zero.

Let us return to the general case. The standard subdifferential qualification condition for inclusions (3.16) is (see [26, Theorem 3.38])

$$\left.\begin{array}{r} y^* \in \partial^\infty g(\varphi(\bar{x})) \\ 0 \in D^*\varphi(\bar{x})(y^*) \end{array}\right\} \Rightarrow \; y^* = 0. \tag{3.21}$$

Proposition 3.9 *Condition* (3.21) *implies metric regularity of* F *near* $((\bar{x}, f(\bar{x})), (0,0))$.

Proof Indeed, condition (3.21) amounts exactly to the condition (3.9) with $P = \mathbb{R}^n \times \mathbb{R}$, $Q = \mathrm{epi} g$ and φ replaced by the mapping Ψ given in (3.20). □

We mentioned in the Introduction that different representations of the same operation may lead to different qualification conditions. Now we can give an illustration of this statement.

Example We can represent the sum of functions $f_1, ..., f_k$ on $\mathbb{R}^n$ as the composition $f = g \circ \varphi$, where $\varphi(x) = (x, ..., x)$ is the diagonal mapping $\mathbb{R}^n \to \mathbb{R}^{nk}$ and $g(x_1, ..., x_k) = f_1(x_1) + ... + f_k(x_k)$. The calculus rule (3.16) then reduces to (3.10) (which is obvious as the adjoint map to φ is $(x_1^*, ..., x_k^*) \mapsto x_1^* + ... + x_k^*$ and $\partial g(x_1, ..., x_k) = \partial f_1(x_1) \times ... \times \partial f_k(x_k)$, due to the separability of g). But the qualification condition which results from (3.17) differs from the metric qualification condition of $(\mathbf{F}_1)$. What we get is: there are $\gamma > 0$, $\delta > 0$ such that

$$d((x,\alpha), \mathrm{epi} f) \leqslant \gamma d((x, x, \ldots, x, \alpha), \mathrm{epi} g),$$

provided $\|x - \bar{x}\| < \delta$, $|\alpha - f(\bar{x})| < \delta$.

Clearly, this condition is different from inequality (3.11). Moreover it is almost obvious that the two conditions are not comparable in the sense that none of them is weaker than the other one.

3.3 Coderivatives of Multifunctions

We shall consider two operations with set-valued mappings in this subsection: composition and summation. The discussions of this subsection can be viewed as elaboration upon the results of [15]. Though differing in some details, the results presented below are very similar to that proved and used in the previous subsections. Therefore we can afford to be more sketchy here.

$(\mathbf{M}_1)$ **Composition**: $S = S_2 \circ S_1$, $S_1[\mathbb{R}^n \rightrightarrows \mathbb{R}^p]$ and $S_2[\mathbb{R}^p \rightrightarrows \mathbb{R}^m]$ (defined by $S(x) := \{\bigcup S_2(w) : w \in S_1(x)\}$);

Assumptions:

(1) the graphs of S_1 and S_2 are closed; $(\bar{x}, \bar{y}) \in \mathrm{Graph} S$;
(2) the map Γ: $(x, y) \mapsto S_1(x) \cap S_2^{-1}(y)$ is locally bounded around $(\bar{x}, \bar{y})$;

Calculus rule:

$$D^*S(\bar{x},\bar{y})(y^*) \subset \bigcup_{\tilde{w}\in\Gamma(\bar{x},\bar{y})} D^*S_1(\bar{x},\tilde{w}) \circ D^*S_2(\tilde{w},\bar{y})(y^*). \tag{3.22}$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0, \delta > 0$ such that

$$d((x,y),\mathrm{Graph}S) \leqslant \gamma(d((x,w),\mathrm{Graph}S_1) + d((w,y),\mathrm{Graph}S_2)) \tag{3.23}$$

  for all x, w, y satisfying $\|x-\bar{x}\| < \delta,\ d(w, S_1(\bar{x}) \cap S_2^{-1}(\bar{y})) < \delta,\ \|y-\bar{y}\| < \delta$;

- *Calmness qualification condition*: the set-valued mapping

$$M\big[\mathbb{R}^n \times (\mathbb{R}^p)^2 \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m\big]:$$
$$M(x,w_1,w_2,y) = \big\{(u,v,z):\ (u,v) \in \mathrm{Graph}S_1 - (x,w_1),$$
$$(v,z) \in \mathrm{Graph}S_2 - (w_2,y)\big\}$$

  is calm at $((0,0,0,0),(\bar{x},w,\bar{y}))$, for any $w \in \Gamma(\bar{x},\bar{y})$;
- *Subregularity qualification condition*: the set-valued mapping

$$F\big[\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times (\mathbb{R}^p)^2 \times \mathbb{R}^m\big]:$$
$$F(x,w,y) = (\mathrm{Graph}S_1 - (x,w)) \times (\mathrm{Graph}S_2 - (w,y))$$

  is subregular at $((\bar{x},w,\bar{y}),(0,0,0,0))$, for any $w \in \Gamma(\bar{x},\bar{y})$.

We have $M(0,\ldots,0) = \{(u,v,z) : (u,v) \in \mathrm{GraphS}_1, (v,z) \in \mathrm{GraphS}_2\}$ so that, similarly as in $(\mathbf{F}_3)$,

$$d((x,y),\mathrm{Graph}S) = \inf_w d((x,w,y),M(0,\ldots,0)).$$

It follows that the metric qualification condition is at least not stronger than the calmness/subregularity condition.

Proposition 3.10 *Inequality* (3.23) $\Rightarrow$ *inclusion* (3.22).

Proof As Γ is locally bounded by (2), it is lower semicompact. The result now follows from Proposition 7.1 of [15]. □

The standard subdifferential qualification condition for (3.22) is (see [23, Theorem 5.1])

$$\left.\begin{aligned} 0 &\in D^*S_1(\bar{x},w)(w^*) \\ w^* &\in D^*S_2(w,\bar{y})(0) \end{aligned}\right\} \Rightarrow w^* = 0 \tag{3.24}$$

for any $w \in \Gamma(\bar{x},\bar{y})$.

Proposition 3.11 *Condition* (3.24) *is equivalent to metric regularity of* F *near* $((\bar{x},w,\bar{y}),(0,0,0))$ *for any* $w \in \Gamma(\bar{x},\bar{y})$.

Proof Clearly, the set-valued mapping F from the subregularity qualification condition has the structure (2.2) with

$$Q = \text{Graph} S_1 \times \text{Graph} S_2,\ \varphi(x, v, y) = \begin{bmatrix} (x, v) \\ (v, y) \end{bmatrix}.$$

Condition (3.24) is exactly the respective MFCQ (2.4). □

If S_1 is single-valued and Lipschitz around $\bar{x}$, then assumption (2) is automatically fulfilled and the requirements of the respective qualification conditions can be replaced by the following:

- *Metric qualification condition*: there are $\gamma > 0, \delta > 0$ such that

$$d((x, y), \text{Graph} S) \leqslant \gamma d((S_1(x), y), \text{Graph} S_2)$$

  for all x, y satisfying $\|x - \bar{x}\| < \delta$, $d_{S_2^{-1}(\bar{y})}(S_1(x)) < \delta$, $\|y - \bar{y}\| < \delta$;
- *Calmness qualification condition*: the set-valued mapping

$$M\big[\mathbb{R}^p \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m\big] :\ M(w, y) = \{(u, z) : (S_1(u), z) \in (\text{Graph} S_2 - (w, y))$$

  is calm at $((0, 0), (\bar{x}, \bar{y}))$;
- *Subregularity qualification condition*: the set-valued mapping

$$F\big[\mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^p \times \mathbb{R}^m\big] :\ F(x, y) = \text{Graph} S_2 - (S_1(x), y)$$

  is subregular at $((\bar{x}, \bar{y}), (0, 0))$.

As in $(\mathbf{F}_3)$, we can easily verify that these conditions (with single-valued Lipschitz map S_1) are equivalent.

$(\mathbf{M}_2)$ **Sum of multifunctions**: $S = S_1 + \ldots + S_k, \quad S_i[\mathbb{R}^n \rightrightarrows \mathbb{R}^m],\ i = 1, 2, \ldots, k.$

Assumptions:

(1) The graphs of S_i are closed , $(\bar{x}, \bar{y}) \in \text{Graph} S$;
(2) There exist neighborhoods $\mathcal{U}$ of $\bar{x}$ and $\mathcal{V}$ of $\bar{y}$ and a positive real ϱ such that

$$\left.\begin{array}{l} x \in \mathcal{U} \\ y_i \in S_i(x), i = 1, 2, \ldots, k \\ \sum_{i=1}^{k} y_i \in \mathcal{V} \end{array}\right\} \Rightarrow |y_i| < \varrho \text{ for } i = 1, 2, \ldots, k;$$

Calculus rule: For all $y^* \in \mathbb{R}^m$

$$D^* S(\bar{x}, \bar{y})(y^*) \subset \bigcup_{\sum_{i=1}^{k} \tilde{y}_i = \bar{y}, \tilde{y}_i \in S_i(\bar{x})} D^* S_1(\bar{x}, \tilde{y}_1)(y^*) + \ldots + D^* S_k(\bar{x}, \tilde{y}_k)(y^*); \tag{3.25}$$

Qualification conditions:

- *Metric qualification condition*: there are $\gamma > 0, \delta > 0$ such that

$$d((x, y), \mathrm{Graph} S) \leqslant \gamma\big(d((x, y_1), \mathrm{Graph} S_1) + \ldots + d((x, y_k), \mathrm{Graph} S_k)\big) \quad (3.26)$$

  for all $x, y, y_i \in S_i(x)$ such that $\|x - \bar{x}\| < \delta$, $\|y - \bar{y}\| < \delta$ and $y_1 + \ldots + y_k = y$;
- *Calmness qualification condition*: the set-valued mapping

$$M[(\mathbb{R}^n \times \mathbb{R}^m)^k \rightrightarrows \mathbb{R}^n \times (\mathbb{R}^m)^k] :$$
$$M((x_1, v_1), \ldots, (x_k, v_k)) = \{(x, y_1, \ldots, y_k) : (x, y_i) \in \mathrm{Graph} S_i - (x_i, v_i),\, i = 1, \ldots, k\}$$

  is calm at points $(((0, 0), \ldots, (0, 0)), (\bar{x}, \tilde{y}_1, \ldots, \tilde{y}_k))$, such that $\tilde{y}_i \in S_i(\bar{x})$, $\sum_{i=1}^{k} \tilde{y}_i = \bar{y}$;
- *Subregularity qualification condition*: the set-valued mapping

$$F[\mathbb{R}^n \times (\mathbb{R}^m)^k \rightrightarrows (\mathbb{R}^n \times \mathbb{R}^m)^k] :$$
$$F(x, y_1, \ldots, y_k) = (\mathrm{Graph} S_1 - (x, y_1)) \times \ldots \times (\mathrm{Graph} S_k - (x, y_k))$$

  is subregular at points $((\bar{x}, \tilde{y}_1, \ldots, \tilde{y}_k), ((0, 0), \ldots, (0, 0)))$, such that $\tilde{y}_i \in S_i(\bar{x})$, $\sum_{i=1}^{k} \tilde{y}_i = \bar{y}$.

Proposition 3.12 *Inequality* (3.26) $\Rightarrow$ *inclusion* (3.25).

Proof The set-valued mapping

$$H(x, y) = \{(y_1, \ldots, y_k) : \; y_i \in S_i(x), \; y_1 + \ldots + y_k = y\}$$

is lower semicompact by (2). The result now follows from Proposition 7.2 of [15]. □

The standard subdifferential qualification condition for sums of set-valued mappings is (see [23, Theorem 4.1]):

$$\left.\begin{array}{l} y_i \in S_i(\bar{x}) \\ y_1 + \ldots + y_k = \bar{y} \\ x_i^* \in D^* S_i(\bar{x}, y_i)(0) \\ x_1^* + \ldots + x_k^* = 0 \end{array}\right\} \Rightarrow x_1^* = \ldots = x_k^* = 0. \quad (3.27)$$

As in all previous cases one has

Proposition 3.13 *Condition* (3.27) *implies metric regularity of* F *near every point* $((\bar{x}, \tilde{y}_1, \ldots, \tilde{y}_k), ((0, 0), \ldots, (0, 0)))$ *such that* $\tilde{y}_i \in S_i(\bar{x})$, $i = 1, 2, \ldots, k$, *and* $\sum \tilde{y}_i = \bar{y}$.

Proof Again, to prove it, it suffices to note that (3.27) is the MFCQ (2.4) for F if we write it as

$$F(x, y_1, \ldots, y_k) = (\mathrm{Graph} S_1 \times \ldots \times \mathrm{Graph} S_k) - ((x, y_1), \ldots, (x, y_k)).$$

□

A remark concerning second-order calculus In second-order analysis we often compute (outer estimates of) coderivatives of composite mappings containing subdifferentials or normal cones. In the respective calculus rules we have then to do with first- and second-order qualification conditions, cf. e.g. [27, Theorem 3.1]. The above theory enables us to weaken the second-order conditions. To illustrate it, consider e.g. the composition investigated in ($\mathbf{F}_3$) and assume additionally that φ is twice continuously differentiable on a neighborhood of $\bar{x}$ and g is convex. It follows that under the standard subdifferential qualification condition (3.21) one has

$$\partial f(x) = (\nabla\varphi(x))^* \partial g(\varphi(x)) \tag{3.28}$$

for all x from a neighborhood of $\bar{x}$. Let us fix a certain $\bar{v} \in \partial f(\bar{x})$ and compute $D^*(\partial f)(\bar{x}, \bar{v})$, which is the so-called second-order subdifferential of f at $\bar{x}$ relative to $\bar{v}$ ([23]).

Proposition 3.14 *Let $f = g \circ \varphi$, where $\varphi[\mathbb{R}^n \to \mathbb{R}^m]$ is twice continuously differentiable and $g[\mathbb{R}^m \to \bar{\mathbb{R}}]$ is proper convex and lower semicontinuous. Let $\bar{x} \in \mathbb{R}^n$ be given such that $\varphi(\bar{x}) \in \operatorname{dom} g$ and condition (3.21) is fulfilled. Further assume that*

(1) *For a given $\bar{v} \in \partial f(\bar{x})$, the mapping $S[\mathbb{R}^n \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m]$ with the values*

$$S(x, v) := \{y \in \mathbb{R}^m : y \in \partial g(\varphi(x)),\ (\nabla\varphi(x))^* y = v\}$$

is lower semicompact at $(\bar{x}, \bar{v})$;

(2) *The mapping*

$$M[\mathbb{R}^m \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n \times \mathbb{R}^m] : M(w, y) = \{(u, z) : z + y \in \partial g(\varphi(u) + w)\}$$

is calm at $(0, 0, \bar{x}, y)$ for all $y \in S(\bar{x}, \bar{v})$.

Then one has

$$D^*(\partial f)(\bar{x}, \bar{v})(v^*) \subset \bigcup_{y \in S(\bar{x},\bar{v})} \left[\nabla^2\langle y, \varphi\rangle(\bar{x})v^* + (\nabla\varphi(\bar{x}))^* D^*(\partial g)(\varphi(\bar{x}), y)(\nabla\varphi(\bar{x})v^*)\right] \tag{3.29}$$

for all $v^* \in \mathbb{R}^n$.

Proof The first part of the proof is identical with the proof of [25, Corollary 4.3](reduced to the finite-dimensional setting). In this way one verifies the inclusion

$$D^*(\partial f)(\bar{x}, \bar{v})(v^*) \subset \bigcup_{y \in S(\bar{x},\bar{v})} \left[\nabla^2\langle y, \varphi\rangle(\bar{x})v^* + D^*(\partial g \circ \varphi)(\bar{x}, y)(\nabla\varphi(\bar{x})v^*)\right]$$

for all $v^* \in \mathbb{R}^n$. It remains to compute the coderivative of the composition $\partial g \circ \varphi$. To do it, we apply the rule ($\mathbf{M}_1$), where $S_1 = \partial g$, and $S_2 = \varphi$ is single-valued. Notice that the graph of ∂g is closed by convexity of g and the respective calmness qualification condition amounts exactly to assumption (2). It follows that for each $y \in S(\bar{x}, \bar{v})$

$$D^*(\partial g \circ \varphi)(\bar{x}, y)(\nabla\varphi(\bar{x})v^*) \subset (\nabla\varphi(\bar{x}))^* D^*(\partial g)(\varphi(\bar{x}), y)(\nabla\varphi(\bar{x})v^*)$$

and we are done. □

Unfortunately, in this context condition (3.21) cannot be replaced by some qualification condition from ($\mathbf{F}_3$). The reason is that we need to ensure the validity of Eq. 3.28 on a whole neighborhood of $\bar{x}$, which cannot be enforced by any from the "weak" qualification conditions. This phenomenon can be observed in connection with other rules of second-order analysis as well.

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