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Combined blur and affine moment invariants and their use in pattern recognition

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Abstract

The paper is devoted to the recognition of objects and patterns deformed by imaging geometry as well as by unknown blurring. We introduce a new class of features invariant simultaneously to blurring with a centrosymmetric PSF and to affine transformation. As we prove in the paper, they can be constructed by combining affine moment invariants and blur invariants derived earlier. Combined invariants allow to recognize objects in the degraded scene without any restoration.

Keywords: Degraded image; Blur invariants; Affine moment invariants; Combined invariants; Object recognition

1. Introduction

Recognition of objects and patterns that are deformed in various ways has been a goal of much recent research. The degradations (geometric as well as graylevel degradations) are introduced during the image acquisition process by such factors as imaging geometry, illumination changes, wrong focus, lens aberration, systematic and random sensor errors, object occlusion, etc. Finding a set of invariant descriptors is a key step to recognizing degraded objects regardless of the particular deformations.

Many papers have been devoted to invariants with respect to spatial coordinate transforms, like rigid-body, affine, and projective transforms (see Refs. [1,2] for a survey and other references). Moment invariants [3–8], Fourier descriptors [9], differential invariants [10–12], and point sets invariants [13–15] belong to the most popular classes of geometric invariants.

Much less attention has been paid to invariants with respect to changes of the image intensity function (we call them radiometric invariants) and to combined radiometric–geometric invariants. In fact, just the invariants both to radiometric and geometric image degradations are necessary to resolve practical object recognition tasks because usually both types of degradations are present in input images.

Van Gool et al. introduced the so-called affine-photometric invariants of graylevel [16] and color [17] images. These features are invariant to the affine transform and to the change of contrast and brightness of the image simultaneously. Some other authors used various local features (mostly derivatives of the image function) to find invariants to rigid-body transform and contrast/brightness changes [18,19]. This technique has become popular namely in image retrieval. Numerous references can be found in Ref. [18].

An important class of radiometric degradations we are faced with often in practice is image blurring. Blurring can be caused by camera defocus, atmospheric turbulence, vibrations, sensor and/or scene motion, and/or by interpolation-based enlargement of the image. If the

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scene is flat and the imaging system is linear and space invariant, blurring can be described by a convolution $g(x, y) = (f * h)(x, y)$, where f is an original (ideal) image, g is an acquired image and h is a point spread function (PSF) of the imaging system. Since in most practical tasks the PSF is unknown, having the invariants to convolution is of prime importance when recognizing objects in a blurred scene. An alternative approach, that would not require convolution invariants, must include blind image deconvolution, which is an ill-posed and practically unsolvable problem.

A pioneer work on this field was done by Flusser and Suk [20] who derived invariants to convolution with an arbitrary centrosymmetric PSF. From the geometric point of view, their descriptors were invariant to translation only. Despite of this, the invariants have found successful applications in face recognition on out-of-focused photographs [21], in normalizing blurred images into the canonical forms [22,34], in template-to-scene matching of satellite images [20], in blurred digit and character recognition [23,24], in registration of images obtained by digital subtraction angiography [25] and in focus/defocus quantitative measurement [26]. Other sets of blur invariants (but still only shift-invariant) were proposed for some particular kinds of PSF—axisymmetric blur invariants [27] and motion blur invariants [28,29]. A significant improvement motivated by a problem of registration of blurred images was made by Flusser and Zitová. They introduced the so-called combined blur-rotation invariants [30] and reported their successful usage in satellite image registration [31] and in camera motion estimation [32]. Most recently, Flusser et al. proposed a group-theoretic approach to extension of the combined invariants into 3-D [33].

However, in a real world, the imaging geometry is projective rather than rigid-body. If the scene is flat and the camera is far from the object in comparison to its size, the projective transform can be well approximated by an affine transform. Thus, having combined affine-blur invariants is in great demand. The first attempt to find such invariants was published by Zhang et al. [34]. They did not derive combined invariants explicitly. They transformed the image into canonical form and then they calculated pure blur invariants [20] from this normalized image.

In this paper, we introduce explicit combined invariants to affine transform and to convolution with an arbitrary centrosymmetric PSF. They can be calculated directly from the degraded image; there is no need to perform any geometric normalization and/or deblurring of the image.

In Section 2, we briefly recall the basic terms and earlier results of the theory of moment invariants. Section 3 performs the core of this work—we present the theorem which allows to construct arbitrary number of combined invariants in explicit form. The proof of this theorem is a major result of the paper. Section 4 demonstrates the numerical properties and recognition power of the invariants.

2. Recalling the theory of the moment invariants

In this section we introduce some basic terms and briefly recall the theorems on the moment invariants which will be used later in this paper.

2.1. Basic terms

Definition 1. By *image function* (or *image*) we understand any real function $f(x, y)$ having a bounded support and a finite nonzero integral.

Definition 2. *Central moment* $\mu_{pq}^{(f)}$ of order $(p+q)$ of the image $f(x, y)$ is defined as

$$\mu_{pq}^{(f)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - x_c)^p(y - y_c)^q f(x, y)\,\mathrm{d}x\,\mathrm{d}y, \qquad (1)$$

where the coordinates (x_c, y_c) denote the centroid of $f(x, y)$.

Definition 3. *Affine transform* is a transformation of spatial coordinates (x, y) into new coordinates (u, v) defined by the equations

$$u = a_0 + a_1x + a_2y,$$

$$v = b_0 + b_1x + b_2y. \qquad (2)$$

Proposition 1. *Every PSF mentioned in this paper is assumed to be* centrosymmetric *and* energy-preserving, i.e.

$$h(x, y) = h(-x, -y),$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1.$$

The assumption of centrosymmetry is not a significant limitation of practical utilization of the method. Most real sensors and imaging systems, both optical and non-optical ones, have the PSF with certain degree of symmetry. In many cases they have even higher symmetry than the central one, such as axial or radial symmetry. Thus, the central symmetry is general enough to handle almost all practical situations.

Note that because of centrosymmetry all moments of odd orders of the PSF equal zero.

2.2. Blur invariants

In our earlier paper [20], the following theorem of blur invariants was proven.

Theorem 1. *Let $f(x, y)$ be an image function. Let us define the following function $C^{(f)}$: $\mathcal{N}_0 \times \mathcal{N}_0 \to \mathcal{R}$. If $(p+q)$ is even then*

$$C(p, q)^{(f)} = 0.$$

If $(p+q)$ is odd then

$$C(p,q)^{(f)} = \mu_{pq}^{(f)} - \frac{1}{\mu_{00}^{(f)}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} \times C(p-n,q-m)^{(f)} \cdot \mu_{nm}^{(f)}. \quad (3)$$

Then $C(p,q)$ is invariant to convolution with any centrosymmetric function $h(x,y)$, i.e.

$$C(p,q)^{(f)} = C(p,q)^{(f*h)}$$

for any p and q.

2.3. Affine moment invariants

Affine moment invariants (AMIs) were introduced independently by Reiss [35] and Flusser and Suk [36,37]. They originate from the classical theory of algebraic invariants [38,39]. The fundamental theorem, which was used for the derivation of the explicit forms of the invariants, can be formulated as follows.

Theorem 2. *If the binary form of order p has an algebraic invariant of weight w and order k*

$$I(a'_{p,0},\ldots,a'_{0,p}) = \triangle^{w} I(a_{p,0},\ldots,a_{0,p})$$

($\triangle$ denotes the determinant of the respective affine transform) then the moments of order p have the same invariant but with the additional factor $|J|^k$:

$$I(\mu'_{p0},\ldots,\mu'_{0p}) = \triangle^{w}\, |J|^{k} I(\mu_{p0},\ldots,\mu_{0p}),$$

where $|J|$ is the absolute value of the Jacobian of the affine transform.

We refer to [36,35] for the proof of Theorem 2 and for a deeper explanation of the theory of the AMIs.

3. Combined blur and affine invariants

By combined blur-affine invariants (CBAIs) we understand any functional defined on the set of image functions whose value does not change if the image function is convolved with a centrosymmetric PSF and transformed by an affine transform. Note, that the convolution and the affine transform are commutative here. The following theorem not only guarantees the existence of the CBAIs but also provides an explicit algorithm how to construct them.

Theorem 3. *Let $I(\mu_{00},\ldots,\mu_{PQ})$ be an affine moment invariant. Then $I(C(0,0),\ldots,C(P,Q))$, where $C(0,0)=\mu_{00}$ and all other blur invariants $C(p,q)$ are defined by Theorem* 1, *is a combined blur-affine invariant.*

Proof. Since $I(C(0,0),\ldots,C(P,Q))$ is a function of blur invariants $C(p,q)$ only, it is also a blur invariant. To prove its invariance to affine transform, it is sufficient to prove that

$$\frac{\partial I(C(0,0),\ldots,C(P,Q))}{\partial a} \equiv \sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{\partial I(C(0,0),\ldots,C(P,Q))}{\partial C(p,q)} \frac{\partial C(p,q)}{\partial a} = 0 \quad (4)$$

for each parameter $a \in \{a_0,a_1,a_2,b_0,b_1,b_2\}$ of the affine transform.

The affine invariance of $I(\mu_{00},\ldots,\mu_{PQ})$ implies that

$$\frac{\partial I(\mu_{00},\ldots,\mu_{PQ})}{\partial a} \equiv \sum_{p=0}^{P}\sum_{q=0}^{Q} \frac{\partial I(\mu_{00},\ldots,\mu_{PQ})}{\partial \mu_{pq}} \frac{\partial \mu_{pq}}{\partial a} = 0. \quad (5)$$

Thus, it is sufficient to prove that the partial derivatives of $C(p,q)$ are identical to the derivatives of moments μ_{pq}, when substituting $C(p,q)$ for μ_{pq}, i.e.

$$\frac{\partial C(p,q)}{\partial a} = \left.\frac{\partial \mu_{pq}}{\partial a}\right|_{\mu_{pq}=C(p,q)} \quad (6)$$

for each parameter a.

To prove this, we decompose the affine transform (2) into six one-parametric transformations:

Horizontal translation:

$$u = x+\alpha, \quad v = y. \quad (7)$$

Vertical translation:

$$u = x, \quad v = y+\beta. \quad (8)$$

Uniform scaling:

$$u = sx, \quad v = sy. \quad (9)$$

Stretching:

$$u = rx, \quad v = \frac{y}{r}. \quad (10)$$

Horizontal skewing:

$$u = x+ty, \quad v = y. \quad (11)$$

Vertical skewing:

$$u = x, \quad v = y+zx. \quad (12)$$

We prove that (6) holds for each of these simple transformations.

Constraint (6) holds trivially for translations (7) and (8). Without loss of generality, we assume in the rest of the proof that $(x_c, y_c) = (0, 0)$. For uniform scaling (9) we have

$$\begin{aligned}\frac{\partial \mu_{pq}}{\partial s} &= \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^p v^q f'(u, v)\, \mathrm{d}u\, \mathrm{d}v \\ &= \frac{\partial}{\partial s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} s^{p+q+2} x^p y^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p+q+2) s^{p+q+1} x^p y^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \frac{p+q+2}{s} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u^p v^q f'(u, v)\, \mathrm{d}u\, \mathrm{d}v \\ &= \frac{p+q+2}{s} \mu_{pq}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial C(p,q)}{\partial s} = \frac{p+q+2}{s} \mu_{pq} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \\ \times \binom{q}{m} \left[\frac{\mu_{nm}}{\mu_{00}} \cdot \frac{\partial C(p-n, q-m)}{\partial s} \right. \\ \left. + C(p-n, q-m) \cdot \frac{\partial}{\partial s} \frac{\mu_{nm}}{\mu_{00}} \right].\end{aligned}$$

At this moment we use the induction principle. Clearly, (6) holds for $p+q=3$ because in that case $C(p,q) = \mu_{pq}$ (if $p+q<3$ then (6) holds trivially). Let us assume the validity of (6) for all orders less than $p+q$. Using this assumption and performing the derivatives we obtain

$$\begin{aligned}&\frac{\partial C(p,q)}{\partial s} \\ &= \frac{p+q+2}{s} \mu_{pq} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} \\ &\quad \times \left[\frac{(p+q-n-m+2)\mu_{nm}}{s\mu_{00}} \cdot C(p-n, q-m) \right. \\ &\quad + \frac{(n+m+2)\mu_{nm}}{s\mu_{00}} \cdot C(p-n, q-m) \\ &\quad \left. - \frac{2\mu_{nm}}{s\mu_{00}} \cdot C(p-n, q-m) \right] \\ &= \frac{p+q+2}{s} \mu_{pq} - \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} \\ &\quad \times \frac{(p+q+2)\mu_{nm}}{s\mu_{00}} \cdot C(p-n, q-m) \\ &= \frac{p+q+2}{s} \cdot C(p,q),\end{aligned}$$

which proves the validity of (6) in case of scaling.

Similarly, for stretching (10) we get

$$\begin{aligned}\frac{\partial \mu_{pq}}{\partial r} &= \frac{\partial}{\partial r} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} r^{p-q} x^p y^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (p-q) r^{p-q-1} x^p y^q f(x, y)\, \mathrm{d}x \mathrm{d}y \\ &= \frac{p-q}{r} \mu_{pq}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial C(p,q)}{\partial r} = \frac{p-q}{r} \mu_{pq} - \frac{1}{\mu_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \\ \times \binom{q}{m} \left[\mu_{nm} \frac{\partial C(p-n, q-m)}{\partial r} \right. \\ \left. + \frac{n-m}{r} C(p-n, q-m) \mu_{nm} \right].\end{aligned}$$

By means of induction, similarly to the previous case, we get

$$\frac{\partial C(p,q)}{\partial r} = \frac{p-q}{r} C(p,q).$$

Finally, for horizontal skewing (11) we have

$$\begin{aligned}\frac{\partial \mu_{pq}}{\partial t} &= \frac{\partial}{\partial t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x+ty)^p y^q f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(x+ty)^{p-1} y^{q+1} f(x, y)\, \mathrm{d}x\, \mathrm{d}y \\ &= p\mu_{p-1,q+1}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial C(p,q)}{\partial t} = p\mu_{p-1,q+1} - \frac{1}{\mu_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} \times \\ \left[\mu_{nm} \frac{\partial C(p-n, q-m)}{\partial t} + C(p-n, q-m) \frac{\partial \mu_{nm}}{\partial t} \right].\end{aligned}$$

Employing the induction principle we get

$$\begin{aligned}&\frac{\partial C(p,q)}{\partial t} \\ &= p\mu_{p-1,q+1} - \frac{1}{\mu_{00}} \sum_{n=0}^{p} \sum_{\substack{m=0 \\ 0<n+m<p+q}}^{q} \binom{p}{n} \binom{q}{m} \\ &\quad \times [(p-n)\mu_{nm} C(p-n-1, q-m+1) \\ &\quad + nC(p-n, q-m)\mu_{n-1,m+1}].\end{aligned}$$

Shifting the indices and using the identities

$$(p-n)\binom{p}{n} = p\binom{p-1}{n} = (n+1)\binom{p}{n+1},$$

$$\binom{q}{m} + \binom{q}{m-1} = \binom{q+1}{m}$$

we obtain the final relation

$$\begin{aligned}\frac{\partial C(p,q)}{\partial t} &= p\mu_{p-1,q+1} - \frac{1}{\mu_{00}}\left[\sum_{n=0}^{p-1}\sum_{\substack{m=0\\0<n+m<p+q}}^{q} p\binom{p-1}{n}\right.\\ &\quad\times\binom{q}{m} C(p-n-1,q-m+1)\mu_{nm}\\ &\quad+\sum_{n=0}^{p-1}\sum_{\substack{m=1\\0<n+m<p+q}}^{q+1}(n+1)\binom{p}{n+1}\\ &\quad\left.\times\binom{q}{m-1} C(p-n-1,q-m+1)\mu_{nm}\right]\\ &= p\mu_{p-1,q+1} - \frac{1}{\mu_{00}}\sum_{n=0}^{p-1}\sum_{\substack{m=0\\0<n+m<p+q}}^{q+1} p\binom{p-1}{n}\\ &\quad\times\binom{q+1}{m} C(p-n-1,q-m+1)\mu_{nm}\\ &= pC(p-1,q+1).\end{aligned}$$

Because of symmetry, the same is true for vertical skewing (12). Thus, the constraint (6) holds for each affine parameter and, consequently, $I(C(0,0),\ldots,C(P,Q))$ is a combined invariant. □

To express the simplest CBAIs in explicit forms, we take six affine moment invariants of the third, fifth and seventh orders. The first two can be found in Ref. [2], the others have been derived newly.

- Third order only:

$$\begin{aligned}I_1 = (&\mu_{30}^2\mu_{03}^2 - 6\mu_{30}\mu_{21}\mu_{12}\mu_{03} + 4\mu_{30}\mu_{12}^3\\ &+4\mu_{21}^3\mu_{03} - 3\mu_{21}^2\mu_{12}^2)/\mu_{00}^{10}.\end{aligned}$$

- Fifth order only:

$$\begin{aligned}I_2 = (&\mu_{50}^2\mu_{05}^2 - 10\mu_{50}\mu_{41}\mu_{14}\mu_{05} + 4\mu_{50}\mu_{32}\mu_{23}\mu_{05}\\ &+16\mu_{50}\mu_{32}\mu_{14}^2 - 12\mu_{50}\mu_{23}^2\mu_{14} + 16\mu_{41}^2\mu_{23}\mu_{05}\\ &+9\mu_{41}^2\mu_{14}^2 - 12\mu_{41}\mu_{32}^2\mu_{05} - 76\mu_{41}\mu_{32}\mu_{23}\mu_{14}\\ &+48\mu_{41}\mu_{23}^3 + 48\mu_{32}^3\mu_{14} - 32\mu_{32}^2\mu_{23}^2)/\mu_{00}^{14}.\end{aligned}$$

- Third and fifth order:

$$\begin{aligned}I_3 = (&\mu_{30}^2\mu_{12}\mu_{05} - \mu_{30}^2\mu_{03}\mu_{14} - \mu_{30}\mu_{21}^2\mu_{05} - 2\mu_{30}\mu_{21}\mu_{12}\mu_{14}\\ &+4\mu_{30}\mu_{21}\mu_{03}\mu_{23} + 2\mu_{30}\mu_{12}^2\mu_{23} - 4\mu_{30}\mu_{12}\mu_{03}\mu_{32}\\ &+\mu_{30}\mu_{03}^2\mu_{41} + 3\mu_{21}^3\mu_{14} - 6\mu_{21}^2\mu_{12}\mu_{23} - 2\mu_{21}^2\mu_{03}\mu_{32}\\ &+6\mu_{21}\mu_{12}^2\mu_{32} + 2\mu_{21}\mu_{12}\mu_{03}\mu_{41} - \mu_{21}\mu_{03}^2\mu_{50} - 3\mu_{12}^3\mu_{41}\\ &+\mu_{12}^2\mu_{03}\mu_{50})/\mu_{00}^{11}.\end{aligned}$$

$$\begin{aligned}I_4 = (&2\mu_{30}\mu_{12}\mu_{41}\mu_{05} - 8\mu_{30}\mu_{12}\mu_{32}\mu_{14} + 6\mu_{30}\mu_{12}\mu_{23}^2\\ &-\mu_{30}\mu_{03}\mu_{50}\mu_{05} + 3\mu_{30}\mu_{03}\mu_{41}\mu_{14} - 2\mu_{30}\mu_{03}\mu_{32}\mu_{23}\\ &-2\mu_{21}^2\mu_{41}\mu_{05} + 8\mu_{21}^2\mu_{32}\mu_{14} - 6\mu_{21}^2\mu_{23}^2\\ &+\mu_{21}\mu_{12}\mu_{50}\mu_{05} - 3\mu_{21}\mu_{12}\mu_{41}\mu_{14} + 2\mu_{21}\mu_{12}\mu_{32}\mu_{23}\\ &+2\mu_{21}\mu_{03}\mu_{50}\mu_{14} - 8\mu_{21}\mu_{03}\mu_{41}\mu_{23} + 6\mu_{21}\mu_{03}\mu_{32}^2\\ &-2\mu_{12}^2\mu_{50}\mu_{14} + 8\mu_{12}^2\mu_{41}\mu_{23} - 6\mu_{12}^2\mu_{32}^2)/\mu_{00}^{12}.\end{aligned}$$

$$\begin{aligned}I_5 = (&\mu_{30}\mu_{41}\mu_{23}\mu_{05} - \mu_{30}\mu_{41}\mu_{14}^2 - \mu_{30}\mu_{32}^2\mu_{05} + 2\mu_{30}\mu_{32}\mu_{23}\mu_{14}\\ &-\mu_{30}\mu_{23}^3 - \mu_{21}\mu_{50}\mu_{23}\mu_{05} + \mu_{21}\mu_{50}\mu_{14}^2 + \mu_{21}\mu_{41}\mu_{32}\mu_{05}\\ &-\mu_{21}\mu_{41}\mu_{23}\mu_{14} - \mu_{21}\mu_{32}^2\mu_{14} + \mu_{21}\mu_{32}\mu_{23}^2 + \mu_{12}\mu_{50}\mu_{32}\mu_{05}\\ &-\mu_{12}\mu_{50}\mu_{23}\mu_{14} - \mu_{12}\mu_{41}^2\mu_{05} + \mu_{12}\mu_{41}\mu_{32}\mu_{14} + \mu_{12}\mu_{41}\mu_{23}^2\\ &-\mu_{12}\mu_{32}^2\mu_{23} - \mu_{03}\mu_{50}\mu_{32}\mu_{14} + \mu_{03}\mu_{50}\mu_{23}^2\\ &+\mu_{03}\mu_{41}^2\mu_{14} - 2\mu_{03}\mu_{41}\mu_{32}\mu_{23} + \mu_{03}\mu_{32}^3)/\mu_{00}^{13}.\end{aligned}$$

- Seventh order only:

$$\begin{aligned}I_6 = (&\mu_{70}^2\mu_{07}^2 - 14\mu_{70}\mu_{61}\mu_{16}\mu_{07} + 18\mu_{70}\mu_{52}\mu_{25}\mu_{07}\\ &+24\mu_{70}\mu_{52}\mu_{16}^2 - 10\mu_{70}\mu_{43}\mu_{34}\mu_{07} - 60\mu_{70}\mu_{43}\mu_{25}\mu_{16}\\ &+40\mu_{70}\mu_{34}^2\mu_{16} + 24\mu_{61}^2\mu_{25}\mu_{07} + 25\mu_{61}^2\mu_{16}^2 - 60\mu_{61}\mu_{52}\mu_{34}\mu_{07}\\ &-234\mu_{61}\mu_{52}\mu_{25}\mu_{16} + 40\mu_{61}\mu_{43}^2\mu_{07} + 50\mu_{61}\mu_{43}\mu_{34}\mu_{16}\\ &+360\mu_{61}\mu_{43}\mu_{25}^2 - 240\mu_{61}\mu_{34}^2\mu_{25} + 360\mu_{52}^2\mu_{34}\mu_{16}\\ &+81\mu_{52}^2\mu_{25}^2 - 240\mu_{52}\mu_{43}^2\mu_{16} - 990\mu_{52}\mu_{43}\mu_{34}\mu_{25}\\ &+600\mu_{52}\mu_{34}^3 + 600\mu_{43}^3\mu_{25} - 375\mu_{43}^2\mu_{34}^2)/\mu_{00}^{18}.\end{aligned}$$

Now we can substitute $C(p,q)$ for μ_{pq}. Since the even-order $C(p,q)$'s equal zero by definition (even-order blur invariants cannot exist in principle), only those AMIs composed from odd-order moments are meaningful for this purpose. The explicit forms of $C(p,q)$'s can be found in Ref. [20] or derived directly from Theorem 1. Those needed for substitution into the above AMIs are listed below.

- Third order:

$$C(3,0) = \mu_{30},$$

$$C(2,1) = \mu_{21},$$

$$C(1,2)=\mu_{12},$$

$$C(0,3)=\mu_{03}.$$

- Fifth order:

$$C(5,0)=\mu_{50}-\frac{10\mu_{30}\mu_{20}}{\mu_{00}},$$
$$C(4,1)=\mu_{41}-\frac{2}{\mu_{00}}(3\mu_{21}\mu_{20}+2\mu_{30}\mu_{11}),$$
$$C(3,2)=\mu_{32}-\frac{1}{\mu_{00}}(3\mu_{12}\mu_{20}+\mu_{30}\mu_{02}+6\mu_{21}\mu_{11}),$$
$$C(2,3)=\mu_{23}-\frac{1}{\mu_{00}}(3\mu_{21}\mu_{02}+\mu_{03}\mu_{20}+6\mu_{12}\mu_{11}),$$
$$C(1,4)=\mu_{14}-\frac{2}{\mu_{00}}(3\mu_{12}\mu_{02}+2\mu_{03}\mu_{11}),$$
$$C(0,5)=\mu_{05}-\frac{10\mu_{03}\mu_{02}}{\mu_{00}}.$$

- Seventh order:

$$C(7,0)=\mu_{70}-\frac{7}{\mu_{00}}(3\mu_{50}\mu_{20}+5\mu_{30}\mu_{40})+\frac{210\mu_{30}\mu_{20}^2}{\mu_{00}^2},$$
$$C(6,1)=\mu_{61}-\frac{1}{\mu_{00}}(6\mu_{50}\mu_{11}+15\mu_{41}\mu_{20}+15\mu_{40}\mu_{21}$$
$$+20\mu_{31}\mu_{30})+\frac{30}{\mu_{00}^2}(3\mu_{21}\mu_{20}^2+4\mu_{30}\mu_{20}\mu_{11}),$$
$$C(5,2)=\mu_{52}-\frac{1}{\mu_{00}}(\mu_{50}\mu_{02}+10\mu_{30}\mu_{22}+10\mu_{32}\mu_{20}$$
$$+20\mu_{31}\mu_{21}+10\mu_{41}\mu_{11}+5\mu_{40}\mu_{12})$$
$$+\frac{10}{\mu_{00}^2}(3\mu_{12}\mu_{20}^2+2\mu_{30}\mu_{20}\mu_{02}+4\mu_{30}\mu_{11}^2$$
$$+12\mu_{21}\mu_{20}\mu_{11}),$$
$$C(4,3)=\mu_{43}-\frac{1}{\mu_{00}}(\mu_{40}\mu_{03}+18\mu_{21}\mu_{22}+12\mu_{31}\mu_{12}$$
$$+4\mu_{30}\mu_{13}+3\mu_{41}\mu_{02}+12\mu_{32}\mu_{11}+6\mu_{23}\mu_{20})$$
$$+\frac{6}{\mu_{00}^2}(\mu_{03}\mu_{20}^2+4\mu_{30}\mu_{11}\mu_{02}+12\mu_{21}\mu_{11}^2$$
$$+12\mu_{12}\mu_{20}\mu_{11}+6\mu_{21}\mu_{02}\mu_{20}),$$
$$C(3,4)=\mu_{34}-\frac{1}{\mu_{00}}(\mu_{04}\mu_{30}+18\mu_{12}\mu_{22}+12\mu_{13}\mu_{21}$$
$$+4\mu_{03}\mu_{31}+3\mu_{14}\mu_{20}+12\mu_{23}\mu_{11}+6\mu_{32}\mu_{02}),$$
$$+\frac{6}{\mu_{00}^2}(\mu_{30}\mu_{02}^2+4\mu_{03}\mu_{11}\mu_{20}+12\mu_{12}\mu_{11}^2$$
$$+12\mu_{21}\mu_{02}\mu_{11}+6\mu_{12}\mu_{20}\mu_{02}),$$
$$C(2,5)=\mu_{25}-\frac{1}{\mu_{00}}(\mu_{05}\mu_{20}+10\mu_{03}\mu_{22}+10\mu_{23}\mu_{02}$$
$$+20\mu_{13}\mu_{12}+10\mu_{14}\mu_{11}+5\mu_{04}\mu_{21})$$
$$+\frac{10}{\mu_{00}^2}(3\mu_{21}\mu_{02}^2+2\mu_{03}\mu_{02}\mu_{20}+4\mu_{03}\mu_{11}^2$$
$$+12\mu_{12}\mu_{02}\mu_{11}),$$
$$C(1,6)=\mu_{16}-\frac{1}{\mu_{00}}(6\mu_{05}\mu_{11}+15\mu_{14}\mu_{02}+15\mu_{04}\mu_{12}$$
$$+20\mu_{13}\mu_{03})+\frac{30}{\mu_{00}^2}(3\mu_{12}\mu_{02}^2+4\mu_{03}\mu_{02}\mu_{11}),$$
$$C(0,7)=\mu_{07}-\frac{7}{\mu_{00}}(3\mu_{05}\mu_{02}+5\mu_{03}\mu_{04})+\frac{210\mu_{03}\mu_{02}^2}{\mu_{00}^2}.$$

It should be noted that some affine moment invariants themselves are also invariant to convolution, for instance I_1 (because $\mu_{03}=C(0,3)$, $\mu_{12}=C(1,2)$, etc.) and I_3 (because the additional terms introduced by substituting $C(p,q)$'s cancel each other). However, this is not a rule and the AMIs cannot be used directly when the image is blurred, as will be demonstrated in the next section.

4. Numerical experiments

In this section, we study numerical properties of the proposed invariants and the possibility of using them as features for recognition of objects on blurred and geometrically deformed images.

4.1. Simulated data

The aim of the first experiment is to demonstrate numerical behavior of the combined invariants when both affine transform and blurring are computer generated. The Lena subimage of the size 101×101 was used as the test image. It was gradually skewed with parameter t (see Eq. (11)) ranging from 0 to 1 by 0.1. This parameter expresses the tangent of the skew angle. Each image was then blurred by approximately circular mask with diameter from 1 to 15 pixels to simulate out-of-focus blur. Examples of the test images are shown in Fig. 1.

For each image the values of the combined invariants were calculated. Relative error (i.e. the relative distance between the original and deformed images in the space of the invariants) was taken as a measure of stability of the invariants. Relative errors of I_1 and I_2 are visualized in Fig. 2. It can be seen that the errors caused by blurring are negligible comparing to the errors caused by skewing. Higher skewing errors are consequences of discretization, because the images had to be interpolated and resampled during the computer-generated skewing. The error depends significantly on the interpolation method. For illustration, the maximum relative error of I_1 using the nearest-neighbor interpolation is 9%. Bilinear interpolation which is more realistic (and which was used in this experiment) produces much lower error—only 0.1% (see the graphs in Fig. 2), and bicubic interpolation which is almost ideal yields only 0.0003% relative error. This illustrates the perfect invariance of the proposed features under simulated degradations that exactly meet the assumptions.

To investigate numerical behavior of the invariants in the presence of noise, we repeated the previous experiment with

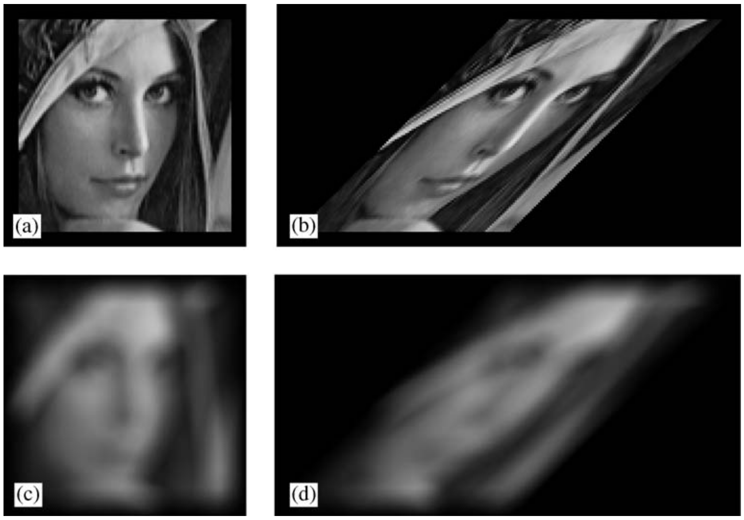

Fig. 1. The Lena subimage used in the first experiment: (a) original, (b) maximum skew, (c) maximum blurring, and (d) maximum skew and blurring.

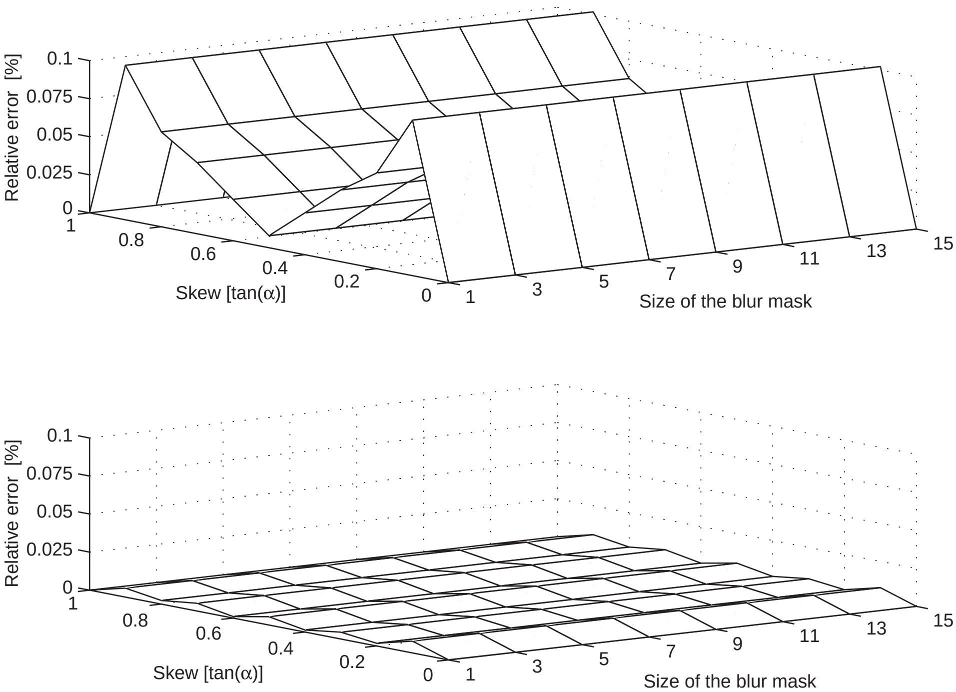

Fig. 2. The relative error of I_1 and I_2 of the skewed and blurred images.

the same data but each blurred image was corrupted also by additive white noise the SNR of which gradually increased from 26 to 62 dB. The results are shown in Fig. 3. (On each noise level, 20 realizations of noisy image were generated and their invariants were averaged to get the "representative" value.) One can observe that relative errors are much higher than in a noise-free case and that they depend almost exclusively on the noise level. The effect of image blurring is negligible thanks to the invariance property. The relative errors on "reasonable" noise levels (i.e. where SNR > 40 dB) is far below 10%, which illustrates good robustness of the invariants.

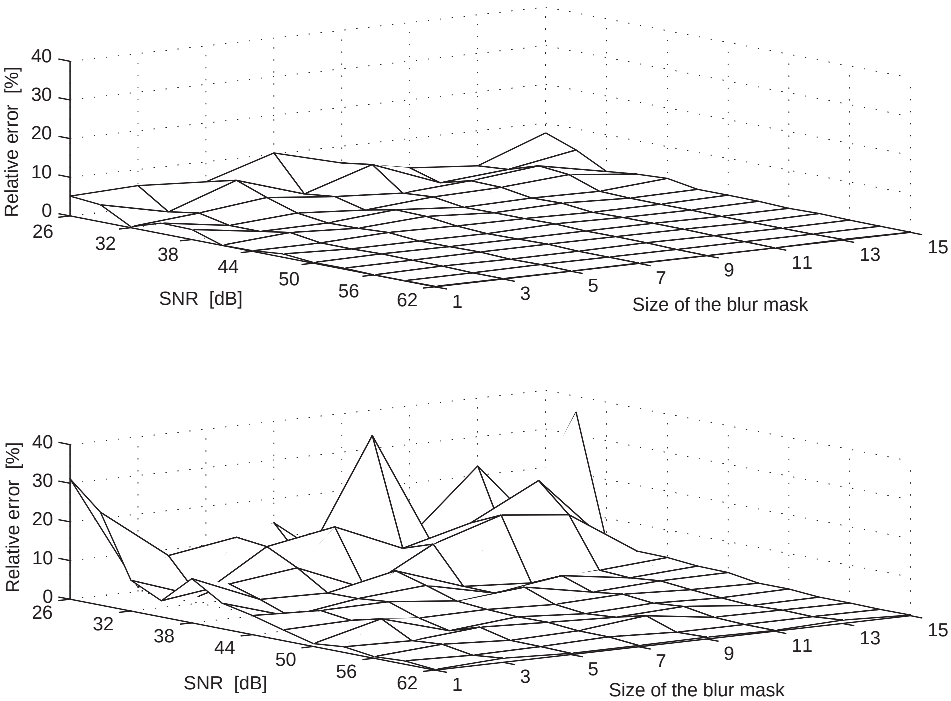

Fig. 3. The relative error of I_1 and I_2 of the blurred and noisy images.

4.2. Real data

Another experiment was carried out to test the performance of the invariants on images degraded by real out-of-focus blur.

A sequence of fifteen pictures of a comb laying on a black background was taken by digital camera Nikon 900. The images differ from each other by viewing angles and by the level of out-of-focus blur. We used five viewing angles ranking from orthogonal view (0°) to approximately 75°, that yielded perspective deformations of the image of various extent. From each angle the picture was captured three times from the same position but with different focus depth, that was set manually. The first exposure was focused on 50 cm, that corresponds to "ideal" focusing. In order to introduce out-of-focus blur, the second exposure was focused on 10 cm (weak defocus) and the third one on 0.2 cm (strong defocus). The images were normalized to the same contrast and the background was segmented and zeroed. All the test images are depicted in Fig. 4.

The values of five combined invariants from Section 3 were computed for each image (see Table 1). We can see that the values of the invariants are fairly stable with respect to image blurring. They change a bit more when changing the viewing angle. This is because the invariants are invariant to affine transform but not to perspective projection, which occurs in this experiment. It is well known that affine transform is a good approximation of perspective projection when the size of the object is small in comparison with the camera-object distance, but this was not exactly true in this experiment. Due to this fact we can observe the loss of invariance property when taking the picture from sharp angles. On the other hand, we may notice a reasonable stability in case of weak perspective projections.

4.3. Digit recognition

This experiment was carried out to demonstrate the discrimination power of the combined invariants in case of affinely deformed and blurred objects and to compare it to the discrimination power of the pure affine moment invariants.

Binary pictures of the size 48×32 of 10 digits $1, 2, \ldots, 9, 0$ were generated (see Fig. 5a). Each of them was deformed by ten affine transforms and every instance was blurred by ten different masks. The parameters of the affine transforms were generated as random values with Gaussian distribution, the "mean value" was identity transform, the standard deviation of the translation was 8 pixels and the standard deviation σ of the other parameters was set to 0.25. The convolution masks were also created randomly. The mask coefficients were generated as Gaussian-distributed random values with the means one for the central coefficient and zero for the other coefficients, respectively, and with the standard deviations $\sigma_m = 10\mathrm{e}^{-\lambda r}$, where r is the distance from the mask center. The parameter λ was chosen 0.2, the size of the

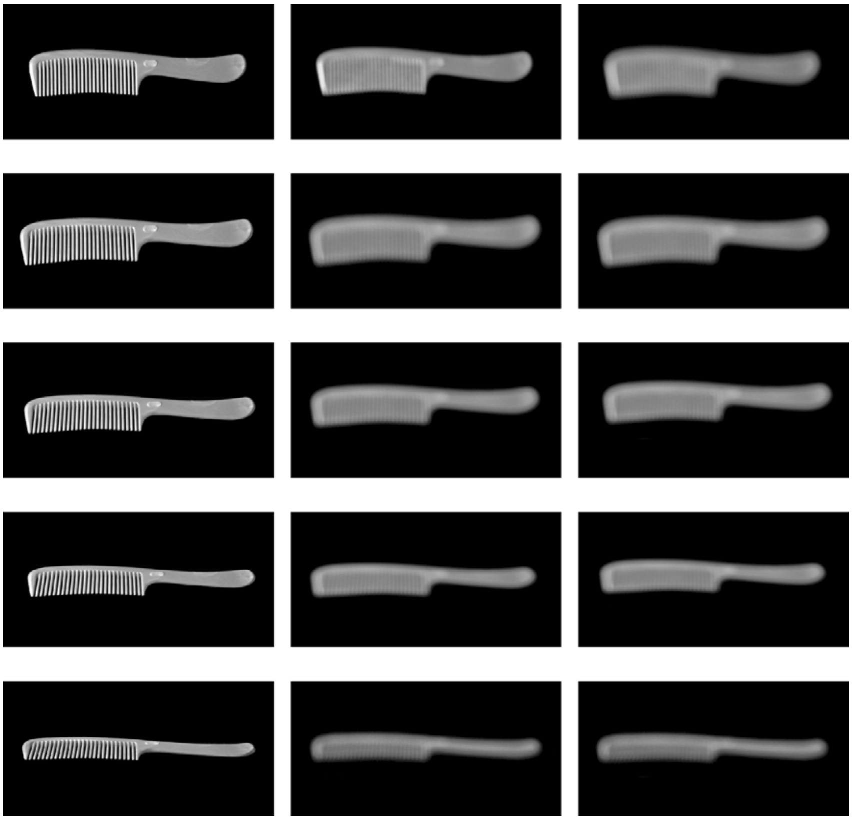

Fig. 4. The comb. The viewing angle increases from 0° (top row) to 75° (bottom row). The extent of out-of-focus blur increases from left to right.

Table 1
The values of the combined invariants of the comb

Viewing angle (deg)	Focus depth (cm)	$I_1[10^{-6}]$	$I_2[10^{-7}]$	$I_3[10^{-8}]$	$I_4[10^{-7}]$	$I_5[10^{-9}]$
0	50	−1.391	−1.488	2.084	4.775	−1.266
0	10	−1.291	−1.531	2.301	4.533	−1.748
0	0.2	−1.221	−1.514	2.228	4.377	−1.747
30	50	−1.175	−1.057	2.426	3.762	−1.586
30	10	−1.085	−1.174	2.565	3.715	−2.141
30	0.2	−1.067	−1.138	2.554	3.632	−2.154
45	50	−1.202	−1.120	2.671	3.889	−1.947
45	10	−1.118	−1.229	2.843	3.843	−2.479
45	0.2	−1.104	−1.199	2.853	3.776	−2.510
60	50	−1.260	−1.219	2.982	4.119	−2.424
60	10	−1.174	−1.309	3.121	4.043	−2.871
60	0.2	−1.166	−1.288	3.184	4.003	−2.945
75	50	−0.8072	−0.2247	3.426	1.925	−1.493
75	10	−0.8642	−0.2231	4.681	2.225	−1.516
75	0.2	−0.8068	−0.03673	6.182	2.001	−4.480

masks was 47×47. Only one half of the coefficients were generated randomly; the others were determined by the constraint of centrosymmetry. Examples of the deformed digits can be seen in Fig. 5b. All deformed digits were classified independently by two minimum-distance classifiers—the first one operates in the space of nine affine moment invariants while the second classifier operates in the space of nine corresponding combined invariants. The results are

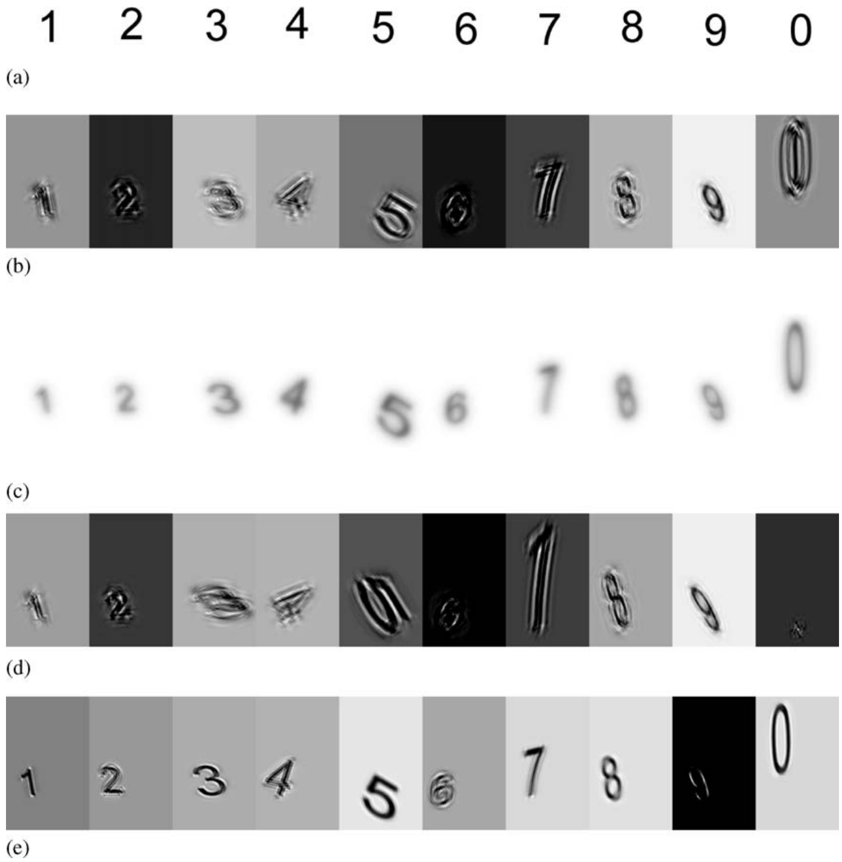

Fig. 5. (a) Original digits and (b–e) examples of the deformed digits used in the experiment.

summarized in Table 2a. Two important facts are clearly visible. First, the combined invariants yielded an excellent success rate 100%. Second, the affine moment invariants perform significantly worse. A part of the results is graphically visualized in Figs. 6 and 7, where one can see the distribution of digits 1, 2, 4, and 5 in the space of two combined invariants (Fig. 6) and in the space of two corresponding affine moment invariants (Fig. 7). In the space of the combined invariants all digits form compact clusters, well separated from each other. On the contrary, in the space of affine moment invariants all patterns form one bigger cluster and few outliers. The same situation occurs in case of the other digits and can be observed also other feature subspaces than in (I_2, I_6). This illustrates that the combined invariants are an actual step toward more robust object recognition and that they significantly improve the recognition rate in case of blurred images.

The above experiment was repeated many times with different parameter settings. In Fig. 5c, the absolute values of the convolution masks generated as indicated above were used. Thanks to this, each mask is in fact a low-pass smoothing filter. In Fig. 5d, the standard deviation σ of the parameters of the affine transforms was increased to 0.5. In Fig. 5e, the size of the blurring masks was decreased to 25×25 with parameter $\lambda = 0.4$. In the last two experiments, the robustness to the additive noise was examined. Digit deformations and blurring were generated in the same way as in the first experiment (i.e. blurring mask 47×47, $\lambda = 0.2$, STD of the affine transform parameters $\sigma = 0.25$). Then additive white noise of STD =0.025 (inducing signal-to-noise ratio 26 dB) and of STD =0.1 (inducing signal-to-noise ratio 14 dB), respectively, was added. (To eliminate the role of the background, the noise was added to the smallest rectangle circumscribed to the digits only.) The results of these experiments are summarized in Table 2b–f. They are very similar to the results of the previous experiment. In almost all cases, the recognition rate of the combined invariants approached 100% and was twice better than that of the affine moment invariants. Two exclusions are experiment in Fig. 5e, where the AMIs gave 80% because of small blurring masks, and the last experiment, where the success rate of the combined invariants decreased to 77% due to heavy noise.

5. Conclusion

The paper concerns the image features which are invariant simultaneously to blurring by a filter with centrally symmetric PSF and to affine transformation. The major theoretical

Table 2
The success rates of the affine moment invariants (AMI) and the combined affine and blur invariants (CBAI) in digit recognition

	1	2	3	4	5	6	7	8	9	0	Overall
(a) *Blurring mask* 47 × 47, $\lambda = 0.2$, *STD of the affine transform parameters* $\sigma = 0.25$ (*corresponds to Fig.* 5b).											
AMI [%]	12	55	39	80	13	13	85	74	25	100	50
CBAI [%]	100	100	100	100	100	100	100	100	100	100	100
(b) *Blurring mask* 47 × 47, $\lambda = 0.2$, *STD of the affine transform parameters* $\sigma = 0.25$, *positive blurring masks only* (*corresponds to Fig.* 5c)											
AMI [%]	10	45	10	91	10	10	100	46	10	100	43
CBAI [%]	100	100	100	100	100	100	100	100	100	100	100
(c) *Blurring mask* 47 × 47, $\lambda = 0.2$, *STD of the affine transform parameters* $\sigma = 0.5$ (*corresponds to Fig.* 5d)											
AMI [%]	14	47	31	63	14	12	77	61	24	80	42
CBAI [%]	90	80	90	90	90	100	90	90	90	80	89
(d) *Blurring mask* 25 × 25, $\lambda = 0.4$, *STD of the affine transform parameters* $\sigma = 0.25$ (*corresponds to Fig.* 5e)											
AMI [%]	56	78	87	92	75	60	94	90	66	100	80
CBAI [%]	100	100	100	100	100	100	100	100	100	100	100
(e) *Blurring mask* 47 × 47, $\lambda = 0.2$, *STD of the affine transform parameters* $\sigma = 0.25$. *Additive white noise with STD* =0.025 *was added*											
AMI [%]	14	50	36	72	15	24	84	74	22	100	49
CBAI [%]	95	100	99	100	100	100	100	100	99	100	99
(f) *Blurring mask* 47 × 47, $\lambda = 0.2$, *STD of the affine transform parameters* $\sigma = 0.25$. *Additive white noise with STD* =0.1 *was added*											
AMI [%]	13	29	36	55	14	37	85	62	27	96	45
CBAI [%]	26	75	75	100	100	85	67	90	56	100	77

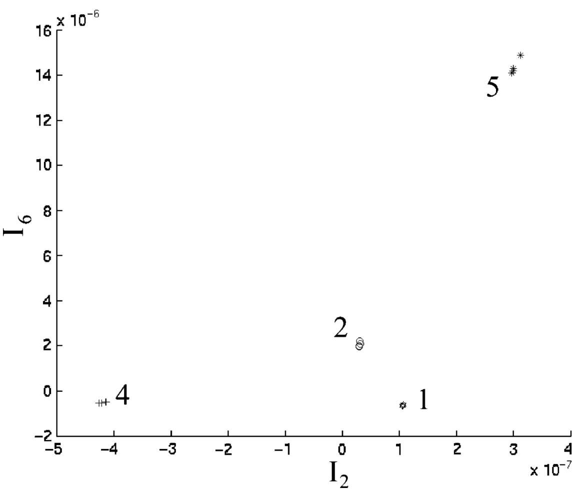

Fig. 6. The digits 1, 2, 4, and 5 in the feature space of the combined invariants I_2 and I_6.

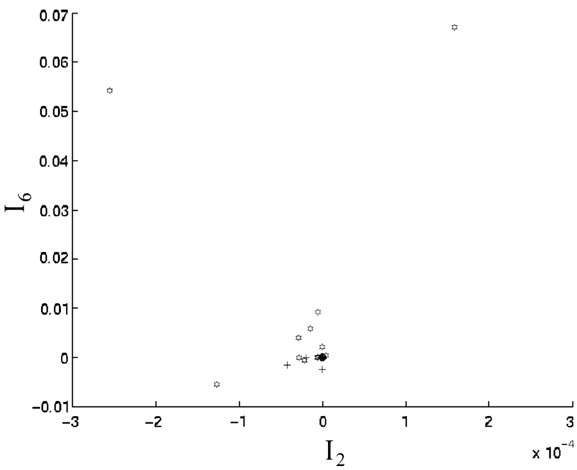

Fig. 7. The digits 1, 2, 4, and 5 in the feature space of the affine moment invariants I_2 and I_6.

result of this work is Theorem 3, showing that the combined invariants can be constructed by substituting blur invariants into the formulae of affine moment invariants. The numerical experiment verified the theoretical results and also illustrated the discrimination power of the invariants. We proved experimentally that the combined invariants form qualitatively new class of features, which clearly outperform earlier affine moment invariants when one wants to recognize objects on images filtered by an unknown PSF, such as out-of-focus blur, turbulence blur, and motion blur. The method does not require any prior knowledge of the PSF. The only assumption about its shape—centrosymmetry—is not a significant limitation for practical utilization of the method. Real imaging systems usually have the PSFs with central or higher symmetry such as axial or radial symmetry. Thus, the assumption of central symmetry is general enough to be fulfilled in most practical situations. It should be pointed out that although our primary motivation was to find invariants to low-pass blurring, the invariants described in this paper are applicable to any centrosymmetric PSF, even if it has negative values.

Although the theory presented in the paper is formulated in the continuous domain, its conversion to the discrete domain does not cause any serious problems. The invariance property might be slightly violated (like in case of all other discrete moment-based invariants) but it has no impact on applicability of the method. The complexity of the combined invariants calculation depends solely on the complexity of calculation of discrete moments themselves. Once the moments are calculated, all other calculations can be performed in a low constant time because the invariants are relatively simple functions of moments and does not directly depend on the image size. It is well-known that moment calculation requires in general $O(N^2)$ operations for an $N \times N$ image, which guarantees a reasonable speed. This complexity can be even substantially reduced for binary or piecewise constant images.

An important issue in recognition tasks is how many invariants should be involved. There is no general rule, the answer always depends on the data. If the objects to be recognized are very different, from three to five invariants might be enough. However, if the inter-class differences are only slight, one may need much more invariants to ensure sufficient discriminability. From theoretical point of view this is not a problem because there exist an infinite number of the invariants but in practice one should keep in mind that the applicability of those comprising high-order moments is questionable because they are more vulnerable to noise.

We envisage the applications mainly in object recognition in blurred and noisy environment, in template matching, and in registration of images taken by non-ideal sensors. There are many application areas where one has to deal with blurred images. Satellite images are often blurred due to the composite sensor PSF and atmospheric turbulence and astronomical images are also degraded by a low-pass filtering due to non-ideal observational conditions. In the area of video surveillance and person authentication, face recognition from out-of-focused photographs is often required. Traditional moment invariants which do not take into account the effect of blurring cannot resolve these tasks successfully.

However, our method has several limitations. When the invariants are calculated from a certain part (region of interest) of the image only, the gray values along the boundary of this part is influenced by the pixels from the outside and convolution is not well defined within the region of interest. The robustness to this so-called "boundary effect" depends on the size of the region of interest and on the size of the blurring filter. The robustness may be low when both sizes are comparable, which prevents from using the blur invariants in such cases. In face and character recognition tasks the limitation is induced by the fact that our invariants (as well as all other moment-based invariants) are intrinsically global, i.e. they are calculated from the whole image including background. Thus, the object must be segmented from the background prior calculating the invariants (which may be problematical in case of heavy blur) or, alternatively, the background must be the same for all objects entering the system. Another limitation appears when we want to distinguish among symmetric objects. It has a deep theoretical reason—any shape descriptor invariant to a certain class of transformations cannot in principle distinguish objects which differ from each other only by transformations from this class. Thus, any invariant (even different from those presented here) to convolution with a centrosymmetric PSF cannot distinguish different centrosymmetric objects because it must give a constant response on all centrosymmetric images (any centrosymmetric image can be considered as a blurring PSF acting on delta-function).

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